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Arakelov Geometry

Organised by
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Abstract. The workshop on Arakelov geometry was attended by 45 participants, many of them young researchers. In 19 talks an overview of recent developments in Arakelov geometry was given.

Mathematics Subject Classification (2000): 14G40.

Introduction by the Organisers

Arakelov geometry studies the geometry and arithmetic of schemes of finite type over Spec **Z**, i.e. systems of polynomial equations with integer coefficients. It combines methods from algebraic geometry, number theory, and hermitian differential geometry.

The workshop was organized by Jean-Benoît Bost (Orsay), Klaus Künnemann (Regensburg) and Damian Roessler (Paris). It brought together internationally leading experts in the area as well as a considerable number of young researchers. The talks covered various aspects of Arakelov geometry from analytic torsion over adelic and non-archimedean analytic spaces to modular forms and diophantine geometry.

A non-mathematical complement was a piano recital by Harry Tamvakis on Thursday night featuring Bach, Beethoven and Chopin.

Workshop: Arakelov Geometry

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Abstracts

Semi-stable extensions an arithmetic surfaces

Christophe Soulé

Let F be a number field, $\mathcal{O}_F$ its ring of integers and $S = \mathrm{Spec}(\mathcal{O}_F)$. Consider a semi-stable curve X over S such that X is regular and its generic fiber X_F is geometrically irreducible of genus $g \geq 0$. Let $\bar{L} = (L, h)$ be an hermitian line bundle over X, *i.e.* a line bundle L on X together with an hermitian metric h on the restriction $L_{\mathbb{C}}$ of L to $X(\mathbb{C})$ which is invariant under complex conjugation. The cohomology group

$$\Lambda = H^1(X, L^{-1})$$

is a finitely generated module over $\mathcal{O}_F$. For every complex embedding $\sigma : F \to \mathbb{C}$, let $X_\sigma = X \otimes \mathbb{C}$ be the corresponding surface and $\Lambda_\sigma = \Lambda \otimes \mathbb{C}$. This cohomology group

$$\Lambda_\sigma = H^1(X_\sigma, L_{\mathbb{C}}^{-1})$$

is canonically isomorphic to the complex vector space $\Omega^1(X_\sigma, L_{\mathbb{C}}^{-1})$ of holomorphic differential forms with coefficients in the restriction $L_{\mathbb{C}}^{-1}$ of the line bundle L^{-1} to $X(\mathbb{C}) = \coprod_\sigma X_\sigma$. Given $\alpha \in \Omega^1(X_\sigma, L_{\mathbb{C}}^{-1})$, we let α^* be its transposed conjugate (the definition of which uses the metric h), and we define

$$\|\alpha\|_{L^2}^2 = \frac{i}{2\pi} \int_{X_\sigma} \alpha^* \, \alpha \, .$$

Given $e \in \Lambda$, we let

$$\|e\| = \operatorname*{Sup}_\sigma \|\sigma(e)\|_{L^2} \, ,$$

where σ runs over all complex embeddings of F.

We are interested in (the logarithm of) the successive minima of Λ. Namely, for any positive integer $k \leq rk(\Lambda)$, we let μ_k be the infimum of all real numbers μ such that there exist k elements $e_1, \ldots, e_k$ in Λ which are linearly independent in

$$\Lambda \otimes F = H^1(X_F, L^{-1})$$

and such that

$$\|e_i\| \leq \exp(\mu) \qquad \text{for all } i = 1, \ldots, k \, . \tag{4}$$

Let d be the degree of the restriction of L to X_F.

Theorem : Assume that d is even and $k \geq d/2 + g$. Then

$$\mu_k \geq \frac{\hat{c}_1(\bar{L})^2}{2\, d\, [F : \mathbb{Q}]} - A \, ,$$

where

$$A = 1 + 2 \log(d + g - 1)$$

and $\hat{c}_1(\bar{L})^2 \in \mathbb{R}$ denotes the arithmetic self-intersection of the arithmetic Chern class $\hat{c}_1(\bar{L}) \in \widehat{\mathrm{CH}}^1(X)$.

We refer to [3] for the proof of that theorem. Previous results on other successive minima of Λ were proved in [1] and [2].

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Direct images in Arakelov geometry

José Ignacio Burgos Gil

As a special case of the formalism of cohomological arithmetic Chow groups developed in joint work with U. Kuehn and J. Kramer one can construct a variant of arithmetic Chow groups that are covariant for arbitrary proper morphisms and a module over the usual arithmetic Chow groups. Related covariant arithmetic Chow groups were introduced by the author in his thesis and independently by Moriwaki and Kawaguchi.

Similarly, one can introduce a theory of covariant arithmetic K groups that admit direct images for arbitrary projective morphism between regular arithmetic varieties. In order to be functorial these direct images depend on the choice of a metric in the tangent spaces of the source and the target arithmetic varieties (and not only on the relative tangent bundle or the normal bundle).

With this formalism, one can interpret results of Bismut, Gillet and Soulé on complex immersions as an arithmetic Riemann Roch theorem for closed immersions and a result of Bismut on higher analytic torsion forms as the compatibility between direct images for closed immersions and direct images for morphisms that are smooth on the generic fiber.

Moreover, combining the arithmetic Riemann-Roch theorem for closed immersions with the arithmetic Riemann-Roch theorem for projective spaces, one obtains formally an arithmetic Riemann-Roch theorem for arbitrary projective morphisms.

p-adic equidistribution of points of small height

Antoine Chambert-Loir

This talk is devoted to the investigation of equidistribution properties of algebraic points of small height on projective varieties defined over number fields.

Let $F \subset \mathbf{C}$ be a number field and let X be a projective variety over F. For any algebraic point $x \in X(\bar{\mathbf{Q}})$, denote by μ_x the probability measure on $X(\mathbf{C})$

which is supported by the orbit of x under $\mathrm{Gal}(\bar{\mathbf{Q}}/F)$ and which is invariant under this group : it is the average of the Dirac measures δ_{x_j}, where $x_1, \ldots, x_d$ are the conjugates of x and $d = [F(x) : F]$.

Let $\bar{L}$ be an ample line bundle on X, endowed with a semipositive adelic metric in the sense of Zhang [7]. For any closed subvariety Y of X, we denote by $h_{\bar{L}}(Y)$ the height of Y, defined by following formula mixing usual intersection theory and Zhang's extension of arithmetic intersection theory to such line bundles endowed with integrable adelic metrics :

$$h_{\bar{L}}(Y) = \frac{(\widehat{c}_1(\bar{L})^{1+\dim Y}|Y)}{(c_1(L)^{\dim Y}|Y)}.$$

($\widehat{c}_1(\bar{L})$ is the first arithmetic Chern class of $\bar{L}$ and and $c_1(L)$ is the first usual Chern classes of $\bar{L}$.) With these notations, Szpiro, Ullmo and Zhang have shown in [5] the following equidistribution theorem.

Theorem. *Assume the metric at infinity is smooth with a positive Chern form $c_1(\bar{L})$. Let $(x_n)_n$ be a sequence of algebraic points on X such that $h_{\bar{L}}(x_n)$ converges to $(\widehat{c}_1(\bar{L})^{1+\dim X}|X)/(1+\dim X)(c_1(L)^{\dim X}|X)$. If no proper subvariety of X contains a subsequence of $(x_n)_n$, then the sequence of measures $(\mu_{x_n})_n$ converges vaguely to the measure $c_1(\bar{L})^{\dim X}/(c_1(L)^{\dim X}|X)$.*

This theorem played an crucial role in Ullmo and Zhang's proof of Bogomolov's conjecture (see [6] and [8]). It has been extend to other situations, notably when the hypothesis of positivity of $c_1(\bar{L})$ is relaxed: let us quote toric varieties (Bilu, [3]), curves (Autissier, [1]), this last example including the especially important case of heights normalized by a dynamical system of the projective line.

The talk was devoted to the analogous question when the complex topology is replaced by the p-adic one. Indeed, the measures μ_x defined above, when x is an algebraic point of X, make sense as measures on the p-adic spaces $X(\mathbf{C}_p)$. However, as is well known, these spaces are badly behaved topological spaces (totally disconnected and not locally compact). A proper study of this question of equidistribution requires therefore the introduction of analytic spaces in the sense of Berkovich (see [2]).

If v is an ultrametric place of F, Berkovich defines from the algebraic variety X_{F_v} a topological space, which we will denote by X_v^{an}. This space is defined by glueing local models, defined from an affinoid algebra $\mathscr{A}$ by taking its "spectrum", namely the space $\mathscr{M}(\mathscr{A})$ of all bounded multiplicative seminorms on $\mathscr{A}$ which extend the norm of F_v.

The space X_v^{an} contains the closed points of X_{F_v} (that is the rigid analytic points of X_{F_v}) as a dense subset, but also many other points. For any normal model $\mathscr{X}$, flat over the ring of integers R_v of F_v, one has a reduction map from X_v^{an} to the scheme $\mathscr{X} \otimes k_v$, where k_v is the residue field of R_v. The generic points of the special fibre have a unique preimage in X_v^{an}. This space is compact, locally connected, and locally contractible if X is smooth.

To give a flavor of our theorems, let us just quote for the moment the analogue of Bilu's theorem. It states as follows:

Theorem 1. *Let F be a number field, let v be an ultrametric place of F and let (x_n) be a sequence of points in $\mathbf{P}^1(\bar{F})$ such that no strict subvariety contains a subsequence. If the Weil heights of x_n go to zero, then the sequence of measures $(\mu_{x_n})_n$ on X_v^{an} converges to the Dirac measure at the unique point of X_v^{an} which reduces to the generic point of $\mathbf{P}^1_{k_v}$.*

The statement of a proof of a general theorem involves two steps:

- construction of a measures $\mu_{\bar{L}}$ on X_v^{an} attached to a metrized line bundle $\bar{L}$;
- proof of the equidistribution theorem under suitable hypotheses.

The measures $\mu_{\bar{L}}$ are itself defined in two steps. First, when $\bar{L}$ is a metrized line bundle given by a model $(\mathscr{X}, \mathscr{L})$ of (X, L) over the ring of integers of F, the measure $\mu_{\bar{L}}$ is defined by taking a linear combination of Dirac measures at the points of X_v^{an} which reduce to the generic points of the components of the special fibre of $\mathscr{X}$, the coefficients being the various degrees of these components with respect to $\mathscr{L}$. One then proves that if $\bar{L}$ is what Zhang calls semi-positive and if the metric of $\bar{L}$ is well approximated by such metrics given by models where $\mathscr{L}$ is nef, the corresponding measures converge to a well-defined measure on X_v^{an}. This measure is the analogue of the measure $c_1(\bar{L})^{\dim X}$ on $X(\mathbf{C})$ which one can define even if the hermitian metric at infinity is not smooth but only continuous and plurisubharmonic. The arguments are close to those of Zhang in [7]; at that point, it is convenient to make use of constructions of Gubler in [4].

The proof of the equidistribution theorem then follows the strategy of Szpiro, Ullmo and Zhang. It relies ultimately on the arithmetic analogue of Hilbert-Samuel formula which requires ampleness hypotheses on the metric when the dimension is at least 2. However, in the case of curves, Autissier proved such a theorem without any other hypothesis than the ampleness on the generic fibre.

Our final theorem can be stated as follows:

Theorem 2. *Fix a place v of F and assume the metric at the place v is given by a model $(\mathscr{L}, \mathscr{X})$, where $\mathscr{L}$ is relatively ample at the place v, or that X is a curve. Let $(x_n)_n$ be a sequence of algebraic points on X such that $h_{\bar{L}}(x_n)$ converges to $(\widehat{c}_1(\bar{L})^{1+\dim X}|X)/(1+\dim X)(c_1(L)^{\dim X}|X)$. If no proper subvariety of X contains a subsequence of $(x_n)_n$, then the sequence of measures $(\mu_{x_n})_n$ on X_v^{an} converges vaguely to the measure $\mu_{\bar{L}}/(c_1(L)^{\dim X}|X)$.*

The proofs of these results are available on the arXiv, (`math.NT/0304023`) and will appear in the Journal für die reine und angewandte Mathematik.

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On the Arakelov-Green's function of a hyperelliptic Riemann surface

Robin de Jong

Let X be a hyperelliptic Riemann surface of genus $g \geq 2$, and let $G : X \times X \to \mathbf{R}_{\geq 0}$ be its Arakelov-Green's function as in [1] and [3]. In this talk we prove two formulas for $\prod_{(W,W')} G(W,W')$, the product running over all pairs of Weierstrass points of X. In the case that $g = 2$, our formulas are certainly implied by the results of [2].

For the first formula, let $\|J\|$ be the jacobian function on $\mathrm{Sym}^g X$ as defined in [2] (for $g = 2$) or [4] (for general g). Let $\delta(X)$ be Faltings' delta-invariant on X, and let $\|\varphi_g\|(X)$ be the normalised modular discriminant of X, which is the Petersson norm of the modular form φ_g defined in [6]. Then we have

$$\prod_{(W,W')} G(W,W')^{n(g-1)}$$
$$= e^{-m(g+2)\delta(X)/4} \|\varphi_g\|(X)^{(g^2-1)/2} \prod_{\{i_1,\ldots,i_g\}} \|J\|(W_{i_1},\ldots,W_{i_g})^{-(2g+4)} .$$

Here $n = \binom{2g}{g+1}, m = \binom{2g+2}{g}$ and the product on the right hand side is running over all sets $\{W_{i_1},\ldots,W_{i_g}\}$ of g Weierstrass points on X. This formula is a consequence of the main result of [4].

The second formula is

$$\prod_{(W,W')} G(W,W')^{n(g-1)} = e^{-m(g+2)\delta(X)/4} \pi^{-2g(g+2)m} \|\varphi_g\|(X)^{-3(g+1)/2} .$$

This formula can be obtained by working out an explicit version of the Mumford isomorphism for hyperelliptic curves. When combined, both formulas yield

$$\prod_{\{i_1,\ldots,i_g\}} \|J\|(W_{i_1},\ldots,W_{i_g}) = \pi^{gm} \|\varphi_g\|(X)^{(g+1)/4} ,$$

which is a symmetric version of a classical identity due to Thomae. Our arguments provide a geometric explanation of this identity, since they are based on exhibiting certain correspondences between canonical sections of canonical line bundles under canonical isomorphisms. It would be interesting to modify the arguments to make them work entirely in the holomorphic category, which is of course the context in which Thomae was working. Probably this requires a replacement of the Arakelov-Green's function by the so-called Riemann prime form. For detailed proofs of the formulas we refer to [5].

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Arithmetic Intersection Theory on Stacks

Henri Gillet

Because of the importance of moduli stacks in arithmetic geometry, it is natural to ask whether the arithmetic intersection theory introduced in [3] can be extended to stacks.

In the following, all stacks will be regular Deligne-Mumford stacks flat and proper over $\mathrm{Spec}(\mathbb{Z})$.

We shall assume that all Chow groups *etc.* have rational coefficients.

Before discussing arithmetic intersection theory on stacks, let us briefly review the situation for the usual Chow groups.

Definition 1.
1. If $\mathfrak{X}$ is a stack, the group of codimension p cycles $Z^p(\mathfrak{X})$ is the free abelian group on the set of codimension p reduced irreducible substacks of $\mathfrak{X}$. This is isomorphic to the group of codimension p cycles on the coarse space $|\mathfrak{X}|$.
2. If $\mathfrak{W}$ is a integral stack, write $\mathbf{k}(\mathfrak{W})$ for its function field. This may be defined as étale H^0 of the sheaf of total quotient rings on $\mathfrak{W}$; and if $f \in \mathbf{k}(\mathfrak{W})$, we may define $\mathrm{div}(f)$ *locally in the étale topology. Therefore, if $\mathfrak{W} \subset \mathfrak{X}$ is a codimension p integral substack, and $f \in \mathbf{k}(\mathfrak{W})$, we have* $\mathrm{div}(f) \in Z^{p+1}(\mathfrak{X})$.
3. We define $\mathrm{CH}^p(\mathfrak{X})$ *to be the quotient of $Z^p(\mathfrak{X})$ by the subgroup consisting of divisors of rational functions on integral subschemes of codimension $p-1$.*

In the 1980's two different approaches to intersection theory on stacks over fields were introduced; the first was via Bloch's formula:

Theorem 2 ([2])**.** *If $\mathfrak{X}$ is a regular Deligne-Mumford of finite type over a field, then there is an isomorphism:*

$$H^p_{\text{ét}}(\mathfrak{X}, K_p(\mathcal{O}_{\mathfrak{X}}))_{\mathbb{Q}} \simeq \mathrm{CH}^p(\mathfrak{X})_{\mathbb{Q}}$$

Sketch of Proof We know that locally on the big Zariski site, there is a quasi-isomorphism of sheaves

$$K_p(\mathcal{O}_{\mathfrak{X}})) \simeq \mathcal{R}_{p.\mathfrak{X}}$$

where $\mathcal{R}_{p.\mathfrak{X}}$ is the (Zarisksi) sheaf of Gersten complexes. However one can show, using the existence of transfer maps for K-theory, that the étale cohomology of the K-theory sheaves on the étale site of the spectrum of a field is torsion. It follows that

$$\mathrm{CH}^p(\mathfrak{X})_{\mathbb{Q}} \simeq H^p(R_p^*(\mathfrak{X}))_{\mathbb{Q}} \simeq \mathbb{H}^p(\mathfrak{X}, \mathcal{R}_{p.\mathfrak{X}})_{\mathbb{Q}} \simeq H^p_{\text{ét}}(\mathfrak{X}, K_p(\mathcal{O}_{\mathfrak{X}}))_{\mathbb{Q}}$$

The Chow groups then inherit a product from the product structure on Higher K-theory, which one can show is compatible with the intersection product on the Chow groups of schemes.

The other construction of intersection theory on stacks was by Vistoli [5], using "Fulton style" intersection theory.

For the arithmetic Chow groups, the problem is both how to extend the product constructed for stacks over fields to stacks over $\mathrm{Spec}(\mathbb{Z})$ and how to add the "archimedean" data. Elsewhere I shall describe an approach via sheaf theory which is not, to date, completely successful because we do not know whether Gersten's conjecture holds for arithmetic varieties. Here I shall describe an "extrinsic" approach to constructing an intersection product.

The idea is to use the existence of hypercovers by regular schemes together with the fact that we know how do arithmetic intersection theory on any regular scheme which is projective over $\mathrm{Spec}(\mathbb{Z})$.

Combining Théoréme 16.5 of [4], and DeJong's theorem [1], we have:

Lemma 3 (Existence of Regular Hypercovers)**.** *Given a regular stack $\mathfrak{X}$, and any point $\xi \in \mathfrak{X}$ there is a proper, representable morphism $p : V \to \mathfrak{X}$ with V a regular scheme, which is étale in a neighborhood of ξ*

For the construction of the arithmetic Chow groups, we start by observing that the sheaf $\mathcal{A}^{p,q}_{\mathfrak{X}}$ of differential forms of type (p, q) on the stack $\mathfrak{X}$ makes sense, since the differential forms are local in the étale topology, as do the ∂ and $\overline{\partial}$ operators. The total deRham complex is a resolution of the constant sheaf $\mathbb{C}$, and so we have the usual Hodge spectral sequence:

$$E_1^{p,q}(\mathfrak{X}) = H^q(\mathfrak{X}, \Omega^p) \Rightarrow H^{p+q}(\mathfrak{X}, \mathbb{C}) \ .$$

Lemma 4. *The Hodge spectral sequence degenerates at E_1.*

The proof is similar to the case of algebraic spaces: show that given $p : V \to \mathfrak{X}$ proper and surjective, with V regular, then there is an injective map of spectral sequences:

$$E_2^{p,q}(\mathfrak{X}) \to E_2^{p,q}(\mathfrak{V})$$

and since $E_*^{p,q}(V)$ degenerates, so must $E_*^{p,q}(\mathfrak{X})$.

Corollary 5. *The $\partial\overline{\partial}$-lemma holds for $\mathfrak{X}(\mathbb{C})$.*

Similarily we can talk about the sheaf $\mathcal{D}^{p,q}_{\mathfrak{X}}$ of currents on $\mathfrak{X}$ since this is local in the étale topology, and the cohomology of the sheaf of currents will be the same as that of the sheaf of forms.

It therefore makes sense to talk about Green currents for cycles on a stack, and therefore we can define the groups $\widehat{CH}^*(\mathfrak{X})$, in the usual fashion, so that we have the usual exact sequence:

$$\begin{aligned} \mathrm{CH}^{p,p-1}(\mathfrak{X}) \to H^{2p-1}_{\mathcal{D}}(\mathfrak{X}(\mathbb{C}), \mathbb{R}(p)) \to \widehat{\mathrm{CH}}^p(\mathfrak{X}) \to \\ \to \mathrm{CH}^p(\mathfrak{X}) \oplus Z^{p,p}(\mathfrak{X}(\mathbb{C})) \to \mathrm{H}^{2p}(\mathfrak{X}(\mathbb{C}), \mathbb{R}(p)) \end{aligned}$$

Suppose for a moment that these groups are contravariant and have products. Then for each $p : V \to \mathfrak{X}$, with p proper and surjective, and V a regular scheme we will have a natural homomorphism $p^* : \widehat{\mathrm{CH}}^*(\mathfrak{X}) \to \widehat{\mathrm{CH}}^*(V)$. and hence a homomorphism

$$\widehat{\mathrm{CH}}^*(\mathfrak{X}) \to \varprojlim_{p:V\to\mathfrak{X}} \widehat{\mathrm{CH}}^*(V)$$

Since we already have well defined functorial products on the groups $\widehat{\mathrm{CH}}^*(V)$, it follows that $\lim_{\leftarrow} \widehat{\mathrm{CH}}^*(V)$ has a natural product structure, which is contravariant with respect to $\mathfrak{X}$. If $\overline{E} = (E, h)$ is a Hermitian vector bundle on $\mathfrak{X}$ since the bundle pulls back to any V over $\mathfrak{X}$, it has Chern classes in $\lim_{\leftarrow} \widehat{\mathrm{CH}}^*(V)$.

Now the key point is that, even though we do not have products and pull-backs on $\widehat{\mathrm{CH}}^*(\mathfrak{X})$, we have:

Theorem 6 (Main Theorem)**.** *There is a canonical isomorphism*

$$\varprojlim_{p:V\to\mathfrak{X}} \widehat{\mathrm{CH}}^*(V) \to \widehat{\mathrm{CH}}^*(\mathfrak{X})$$

The idea of the proof is to construct for each $p : V \to \mathfrak{X}$ which is proper, surjective and generically finite, with V regular, a push forward map $p_* : \widehat{\mathrm{CH}}^*(V) \to \widetilde{\mathrm{CH}}^*(\mathfrak{X})$, where $\widetilde{\mathrm{CH}}^*(\mathfrak{X})$ is group which contains $\widehat{\mathrm{CH}}^*(\mathfrak{X})$, and in which one replaces $Z^{p,p}(\mathfrak{X}(\mathbb{C}))$ by a space of forms with singularities. This direct image map then exhibits $\widehat{\mathrm{CH}}^*(\mathfrak{X})$ as a direct summand of $\widehat{\mathrm{CH}}^*(V)$

Corollary 7. *There is a product structure on* $\widehat{\mathrm{CH}}^*(\mathfrak{X})$ *which is functorial in* $\mathfrak{X}$*.*

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Estimates for Faltings's delta function

Jürg Kramer
(joint work with Jay Jorgenson)

1. Let X be a compact Riemann surface of genus $g_X \geq 1$. In [6], G. Faltings attached to X a new invariant $\delta_{\mathrm{Fal}}(X)$, which we call Faltings's delta function. The function $\delta_{\mathrm{Fal}}(X)$ is defined as a rather complicated function in terms of classical Riemann theta functions and Arakelov's Green's functions. For $g_X = 1$, resp. $g_X = 2$, G. Faltings, resp. J.-B. Bost gave explicit formulas (see [6], [3]). For arbitrary genus $g_X \geq 1$, J. Jorgenson and R. Wentworth were able to give expressions for $\delta_{\mathrm{Fal}}(X)$ in terms of Riemann theta functions and abelian integrals (see [7], [18]).

The aim of this talk is to give explicit bounds for $\delta_{\mathrm{Fal}}(X)$ in terms of differential geometric invariants arising from X, when $g_X > 1$. In the end this will lead to bounds for Faltings's delta function for the modular curves $X_0(N)$, which is of particular interest (see [17]).

2. To achieve our goal, we start from an alternative definition of $\delta_{\mathrm{Fal}}(X)$, which is based on the work [2], [5], [16]. Before recalling this definition, we need to introduce some notation.

By the uniformization theorem, we have $X = \Gamma\backslash\mathbb{H}$, where Γ is a Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$. We let μ_{hyp}, resp. μ_{can} denote the hyperbolic, resp. canonical metric form on X. By means of the Green's function $g_{\mathrm{can}}(z,w)$ attached to the canonical metric, one obtains the residual canonical metric on the canonical line bundle of X, which is nothing but the Arakelov metric $\|\cdot\|_{\mathrm{Ar}}$. The difference between the hyperbolic and the Arakelov metric is measured by the C^∞-function ϕ_{Ar} defined by $\mu_{\mathrm{Ar}} = \exp(\phi_{\mathrm{Ar}})\mu_{\mathrm{hyp}}$; here μ_{Ar} is the $(1,1)$-form attached to $\|\cdot\|_{\mathrm{Ar}}$. It is convenient to also introduce the scaled hyperbolic metric μ_{shyp}, which measures the volume of X to be 1.

If now $\det^*(\Delta_{\mathrm{Ar}})$, resp. $\mathrm{vol}_{\mathrm{Ar}}(X)$ denotes the regularized determinant of the Laplacian, resp. the volume of X with respect to the Arakelov metric, Faltings's delta function is given by the formula

$$\delta_{\mathrm{Fal}}(X) = -6\log\left(\frac{\det^*(\Delta_{\mathrm{Ar}})}{\mathrm{vol}_{\mathrm{Ar}}(X)}\right) + a(g_X) \tag{1}$$

with an explicitly given constant $a(g_X)$, which is of order $O(g_X)$.

3. Using results of [14] and [15], formula (1) can be rewritten as

$$\delta_{\mathrm{Fal}}(X) = -6\log(Z'_X(1)) - (g_X - 1)\int_X \phi_{\mathrm{Ar}}(z)(\mu_{\mathrm{shyp}}(z) + \mu_{\mathrm{can}}(z)) + c(g_X)$$

with $Z_X(s)$ denoting the Selberg zeta function associated to X, and $c(g_X) = O(g_X)$. Introducing the hyperbolic heat kernel $K_{\mathrm{hyp}}(t;z,w)$ ($t \in \mathbb{R}_{>0}$; $z,w \in X$),

$HK_{\mathrm{hyp}}(t;z) = K_{\mathrm{hyp}}(t;z,z) - K_{\mathbb{H}}(t;0)$, and the function

$$F(z) = \int_0^\infty \left(HK_{\mathrm{hyp}}(t;z) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) \mathrm{d}t\,,$$

we arrive at the following expression for $\delta_{\mathrm{Fal}}(X)$, solely in hyperbolic terms,

$$\delta_{\mathrm{Fal}}(X) = 2\pi \left(1 - \frac{1}{g_X}\right) \int_X F(z)\Delta_{\mathrm{hyp}}F(z)\mu_{\mathrm{hyp}}(z) - 6\log(Z'_X(1)) + 2c_X + C(g_X)\,,$$

where $c_X = \lim_{s\to 1}\left(Z'_X/Z_X(s) - 1/(s-1)\right)$ and $C(g_X) = O(g_X)$ (see [11], [12]).

4. Using the techniques developed in [8], we are now able to estimate $\delta_{\mathrm{Fal}}(X)$ working from the last formula. To state the result, put

$$h(X) = g_X + \frac{2}{\lambda_{X,1}} \left(g_X(d_{\mathrm{sup},X}+1)^2 + C_{\mathrm{Hub},X} + N^{[0,1/4)}_{\mathrm{ev},X} \right) + \frac{1}{\ell_X} N^{(0,5)}_{\mathrm{geo},X}\,,$$

with $\lambda_{X,1}$ the smallest non-zero eigenvalue for the hyperbolic Laplacian, $d_{\mathrm{sup},X} = \sup_{z\in X}(\mu_{\mathrm{can}}/\mu_{\mathrm{hyp}}(z))$, $C_{\mathrm{Hub},X}$ the implied constant in the error term of the prime geodesic theorem, $N^{[0,1/4)}_{\mathrm{ev},X}$ the number of eigenvalues less than $1/4$, $N^{(0,5)}_{\mathrm{geo},X}$ the number of primitive geodesics with length in the interval $(0,5)$, and ℓ_X the length of the shortest geodesic on X. In [12], we then show $\delta_{\mathrm{Fal}}(X) = O(h(X))$.

5. In the case that $X \to X_0$ is a covering of finite degree over the fixed base Riemann surface X_0 of genus $g_{X_0} > 1$, the results of [8], [9], and [10] allow us to bound the complicated invariants occuring in $h(X)$ in order to obtain the estimate

$$\delta_{\mathrm{Fal}}(X) = O(g_X(1+\lambda_{X,1}^{-1})).$$

By suitably refining the arguments used to arrive at the previous bound, and using the uniform bound for the smallest non-zero eigenvalue on the modular curves $X(N)$ given in [4], we find for the modular curves $X_0(N)$

$$\delta_{\mathrm{Fal}}(X_0(N)) = O\left(g_{X_0(N)}\right). \tag{2}$$

Using the results obtained in [1] and [13], the bound (2) shows that the Faltings height $h_{\mathrm{Fal}}(J_0(N))$ of the Jacobian $J_0(N)$ of $X_0(N)$ has the asymptotics $h_{\mathrm{Fal}}(J_0(N)) \sim 4g_{X_0(N)}$ as $N \to \infty$.

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Bergman kernels and symplectic reduction

Xiaonan Ma
(joint work with Weiping Zhang)

Let (X, ω) be a compact Kähler manifold, and let (L, h^L) be a holomorphic Hermitian line bundle with its holomorphic Hermitian connection ∇^L such that $\frac{\sqrt{-1}}{2\pi}(\nabla^L)^2 = \omega$. Let (E, h^E) be a holomorphic Hermitian vector bundle on X. Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. We suppose that G acts holomorphically on X and its action lifts on L, E and preserves h^L, h^E, then we have the moment map $\mu : X \to \mathfrak{g}^*$ defined by $\mu(K) = \frac{\sqrt{-1}}{2\pi}(L_K - \nabla^L_{K^X})$ for $K \in \mathfrak{g}$, here K^X is the vector field on X induced by K. We suppose that G acts freely on $\mu^{-1}(0)$, then the symplectic reduction of (X, ω) is the Kähler manifold $(X_G = \mu^{-1}(0)/G, \omega_G)$.

It is important to understand the relations on the invariants of X and the corresponding ones of the fixed point set of $g \in G$ or of X_G, and in this spirit, we have varies localization formulas for the equivariant closed differential forms or the cohomology group. In this talk, we try to understand the relations for the spectrum invariants such as the important geometric invariant, analytic torsion. We present some results in this direction, especially, let $H^0(X, L^p \otimes E)^G$ be the G-invariant holomorphic sections of $L^p \otimes E$, we show that the isomorphism

$$\sigma_p : H^0(X, L^p \otimes E)^G \to H^0(X_G, L^p_G \otimes E_G)$$

induced by the restriction is asymptotically isometry up to a factor $(2p)^{-\dim G/4}$. The key point is to study the full asymptotic expansion of the G-invariant Bergman kernel $P_p^G(x,x')(x,x'\in X)$ which is the smooth kernel of the orthogonal projection $P_p^G : C^\infty(X, L^p\otimes E)\to H^0(X,L^p\otimes E)^G$. Another consequence is that we can see the scalar curvature of X_G from $P_p^G(x,x')$ which should have applications in Donaldson's program on the Kähler metrics with constant scalar curvature.

This is a joint work with Weiping Zhang (Nankai Institute of Mathematics, China).

Traces of CM values of modular functions

Jan Hendrik Bruinier
(joint work with J. Funke)

The classical j-function on the complex upper half plane $\mathbb{H}$ is defined by

$$j(\tau)=\frac{E_4(\tau)^3}{\eta(\tau)^{24}}=q^{-1}+744+196884q+21493760q^2+\dots.$$

In this formula $\eta=q^{1/24}\prod_{n=1}^\infty(1-q^n)$ denotes the Dedekind eta function, $E_4=1+240\sum_{n=1}^\infty\sum_{m|n}m^3q^n$ is the normalized Eisenstein series of weight 4 for the group $\Gamma(1)=\mathrm{PSL}_2(\mathbb{Z})$, and $q=e(\tau)=e^{2\pi i\tau}$ for $\tau\in\mathbb{H}$. The j-function is a Hauptmodul for the group $\Gamma(1)$, i.e., it generates the field of all meromorphic modular functions for this group.

The values of $j(\tau)$ at CM points are known as *singular moduli.* They are algebraic integers generating Hilbert class fields of imaginary quadratic fields. In our talk we consider the traces of singular moduli and more generally the traces of CM values of modular functions on modular curves of arbitrary genus.

Let D be a positive integer and write $\mathcal{Q}_D$ for the set of positive definite integral binary quadratic forms $[a,b,c]$ of discriminant $-D=b^2-4ac$. The group $\Gamma(1)$ acts on $\mathcal{Q}_D$. If $Q=[a,b,c]\in\mathcal{Q}_D$ we write $\Gamma(1)_Q$ for the stabilizer of Q in $\Gamma(1)$ and $\alpha_Q=\frac{-b+i\sqrt{D}}{2a}$ for the corresponding CM point in $\mathbb{H}$. By the theory of complex multiplication, the values of j at such points α_Q are algebraic integers whose degree is equal to the class number of $K=\mathbb{Q}(\sqrt{-D})$. Moreover, $K(j(\alpha_Q))$ is the Hilbert class field of K. In [3], Gross and Zagier derived a closed formula for the norm to $\mathbb{Z}$ of $j(\alpha_Q)$ as a special case of their work on the Gross-Zagier formula. In a later paper [8], Zagier studies the trace of $j(\alpha_Q)$. We briefly recall his result.

To this end it is convenient to consider the normalized Hauptmodul $J(\tau)=j(\tau)-744$ for $\Gamma(1)$ instead of $j(\tau)$ itself. The modular trace of J of index D is defined as

$$\mathbf{t}_J(D)=\sum_{Q\in\mathcal{Q}_D/\Gamma(1)}\frac{1}{|\Gamma(1)_Q|}J(\alpha_Q). \tag{1}$$

Zagier discovered that the generating series

$$-q^{-1}+2+\sum_{D=1}^\infty\mathbf{t}_J(D)q^D=-q^{-1}+2-248q^3+492q^4-4119q^7+7256q^8+\dots \tag{2}$$

is a meromorphic modular form of weight 3/2 for the Hecke group $\Gamma_0(4)$ whose poles are supported at the cusps. More precisely, it is equal to the weight 3/2 form

$$g(\tau) = \frac{\eta(\tau)^2 E_4(4\tau)}{\eta(2\tau)\eta(4\tau)^6}. \tag{3}$$

Zagier gives two different proofs of this result. The first uses certain recursion relations for the $\mathbf{t}_J(D)$, the second uses Borcherds products on $\mathrm{SL}_2(\mathbb{Z})$ and an application of Serre duality. Both proofs rely on the fact that (the compactification of) $\Gamma(1)\backslash\mathbb{H}$ has genus zero. In [4, 5], Kim extends Zagier's results to other modular curves of genus zero using similar methods.

The above connection between the weight 3/2 form g for $\Gamma_0(4)$ and the weight 0 form J for $\Gamma(1)$ reminds us of (a special case of) the Shimura lift which is a linear map from holomorphic modular forms of weight $k + 1/2$ for $\Gamma_0(4)$ in the Kohnen plus space to holomorphic modular forms of weight $2k$ for $\Gamma(1)$. Moreover, it reminds of the theta lift from weight 0 Maass wave forms to weight 1/2 Maass forms first considered by Maass and later reconsidered by Duke and Katok and Sarnak. However, there are two obvious differences: First, in our case the half integral weight form has weight 3/2 rather than 1/2; and second, neither J nor g is holomorphic at the cusps. The first difference should be not so serious, since there is often a duality between weight k and weight $2-k$ forms on modular curves as a consequence of Serre duality. If we ignore the second difference for a moment, in view of the work of Shintani and Niwa realizing the Shimura lift as a theta lift, it is natural to ask, whether Zagier's result can also be interpreted in the light of the theta correspondence?

In other words, one might ask if there is a suitable theta function $\theta(\tau, z, \varphi)$ which transforms like a modular form of weight 3/2 in τ and is invariant under $\Gamma(1)$ in z such that $g(\tau)$ is equal to the theta integral

$$I(\tau, J) = \int_{\Gamma(1)\backslash\mathbb{H}} J(z)\theta(\tau, z, \varphi)\,\frac{dx\,dy}{y^2}. \tag{4}$$

Clearly one has to be very careful with the convergence of the integral because of the pole of J at the cusp. We show that it is possible to obtain such a description by considering the theta kernel corresponding to a particular Schwartz function φ constructed by Kudla and Millson. This generalizes [2] where the lifting $I(\tau, 1)$ of the constant function 1 was studied. A very nice feature of the theta kernel is its very rapid decay at the cusps which leads to absolute convergence of the integral.

The theta lift description of the correspondence between J and g can now be used to generalize Zagier's result to modular functions (with poles supported at the cusps) on modular curves of arbitrary genus. Moreover, one can study the lifting for other automorphic functions. It turns out that already the lifting of the non-holomorphic Eisenstein series $E_0(z, s)$ of weight 0 for $\Gamma(1)$ provides interesting geometric and arithmetic insights. Combined with the Kronecker limit formula it can be used to realize a certain generating series of arithmetic intersection numbers as the derivative of Zagier's Eisenstein series of weight 3/2. This recovers a result of Kudla, Rapoport and Yang [7].

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Rigidity of log morphisms

Atsushi Moriwaki
(joint work with Isamu Iwanari)

In this talk, I explained the rigidity of log morphisms and its applications. For details, see [2] and [4]. You can also find the pdf file of my talk at [6].

Let S be a locally noetherian scheme and let X and Y be schemes of finite type over S. We assume that X and Y are semistable over S, that is, $X \to S$ and $Y \to S$ are flat and all geometric fibers of $X \to S$ and $Y \to S$ are semistable varieties. Let M_S, M_X and M_Y be fine log structures on S, X and Y respectively in the sense of [3]. We assume that the scheme morphisms $X \to S$ and $Y \to S$ extend to smooth and integral log morphisms $(X, M_X) \to (S, M_S)$ and $(Y, M_Y) \to (S, M_S)$. Let $\phi : X \to Y$ be a scheme morphism over S such that ϕ is admissible with respect to M_Y/M_S, i.e., every irreducible component of the geometric fibers of $X \to S$ does not map to the boundary of the log morphism $(Y, M_Y) \to (S, M_S)$. Then, the rigidity theorem of log morphisms is the following:

Rigidity theorem : The number of extensions $(\phi, h) : (X, M_X) \to (Y, M_Y)$ over (S, M_S) of $\phi : X \to Y$ is at most one, that is, if there are log morphisms $(\phi, h), (\phi, h') : (X, M_X) \to (Y, M_Y)$ over (S, M_S), then $h = h'$.

The first application is a generalization of Kobayashi-Ochiai's theorem in the category of log schemes, which was conjectured by Kazuya Kato. Kobayashi-Ochiai's theorem says that for a connected compact complex manifold X of general type and a connected compact complex manifold Y, the number of dominant meromorphic maps $Y \dashrightarrow X$ is finite. From the viewpoint of Diophantine geometry, it means that a variety of general type has only finitely many rational points for a big function field, which gives an evidence of Lang's conjecture. Their theorem was generalized to the case over a field of positive characteristic by

Dechamps and Menegaux [1]. Furthermore, Tsushima [5] established finiteness for open varieties over a field of characteristic zero. Let k be an algebraically closed field and M_k a fine log structure of $\mathrm{Spec}(k)$. Let X and Y be proper semistable varieties over k, and let M_X and M_Y be fine log structures of X and Y over M_k respectively such that (X, M_X) and (Y, M_Y) are smooth and integral over $(\mathrm{Spec}(k), M_k)$. We assume that (Y, M_Y) is of log general type over $(\mathrm{Spec}(k), M_k)$, that is, $\det(\Omega^1_{Y/k}(\log(M_Y/M_k)))$ is a big line bundle on Y. Then, our generalization is the finiteness of log rational maps

$$(\phi, h) : (X, M_X) \dashrightarrow (Y, M_Y)$$

over $(\mathrm{Spec}(k), M_k)$ such that (1) $\phi : X \dashrightarrow Y$ is a rational map defined over a dense open set U with $\mathrm{codim}(X \setminus U) \geq 2$, and $(\phi, h) : (U, M_X|_U) \to (Y, M_Y)$ is a log morphism over $(\mathrm{Spec}(k), M_k)$, and that (2) for any irreducible component X' of X, there is an irreducible component Y' of Y such that $\phi(X') \subseteq Y'$ and the induced rational map $\phi' : X' \dashrightarrow Y'$ is dominant and separable.

The second application is the descent theorem of log morphisms. Let (S, M_S), (X, M_X), (Y, M_Y) and $\phi : X \to Y$ be the same as in the second paragraph. Let $\pi : S' \to S$ be a faithfully flat and quasi-compact morphism of locally noetherian schemes. Let $X' = X \times_S S'$, $Y' = Y \times_S S'$ and $\phi' = \phi \times_S \mathrm{id}_{S'}$, and let us denote the induced morphisms $X' \to X$ and $Y' \to Y$ by π_X and π_Y respectively. Then, the descent theorem guarantees that a log morphism

$$(\phi', h') : (X', \pi_X^*(M_X)) \to (Y', \pi_Y^*(M_Y))$$

over $(S', \pi^*(M_S))$ descends to a log morphism

$$(\phi, h) : (X, M_X) \to (Y, M_Y)$$

over (S, M_S).

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On heights and intersection numbers on arithmetic surfaces

Ulf Kühn

It is expected that there are certain upper bounds for the canonical height function $\mathrm{ht}_{\overline{\omega}}(\cdot)$ and for the arithmetic self intersection number $\overline{\omega}^2$ associated to an arithmetic surface (see e.g. [9], [10]). In this talk I presented recent results [6] and [7] which support these expectations. More precisely there is strong believe in

Conjecture I.(effective Mordell, Vojta's conjecture) *Let K be a number field and $X_{/K}$ a smooth, projective curve defined over K. Then for any $\varepsilon > 0$ there exists a constant $\kappa(\varepsilon)$ such that for all $P \in X(\overline{K})$ we have*

$$\mathrm{ht}_{\overline{\omega}}(P) \leq (1+\varepsilon)\mathrm{d}(P) + \kappa(\varepsilon),$$

where $\mathrm{ht}_{\overline{\omega}}(P)$ is the normalised height with respect to the metrized line bundle $\overline{\omega}$ and $\mathrm{d}(P) = \frac{1}{[K(P):\mathbb{Q}]} \log |\Delta_{K(P)|\mathbb{Q}}$ is the logarithmic discriminant of the number field $K(P)$.

This conjecture is equivalent to the uniform *abc*-conjecture for number fields [3], which in turn has lots of different applications and consequences in number theory.

We say a subset $\mathcal{V} \subseteq X(\overline{K})$ is *unbounded* if for all $u, v > 0$ there are infinitely many $P \in \mathcal{V}$ with $[K(P) : \mathbb{Q}] > u$ and $(d) > v$. With this notion we have

Theorem I. *Let $X_{/K}$ and $\varepsilon > 0$ be as before. Assume all the Dirichlet series $L(\chi_D, s)$ for $\chi_D = \left(\frac{D}{\cdot}\right)$ with $D < 0$ a negative prime satisfy the generalized Riemann hypothesis. Then there exists an unbounded subset $\mathcal{V} \subseteq X(\overline{K})$ and a constant $\kappa(\varepsilon, \mathcal{V})$ so that for all $P \in \mathcal{V}$ it holds*

$$\mathrm{ht}_{\overline{\omega}}(P) \leq \varepsilon \mathrm{d}(P) + \kappa(\varepsilon, \mathcal{V}).$$

The main idea of proof is to identify the projective line $\mathbb{P}^1$ with the modular curve $X(1)$ and then define $\mathcal{V}$ to be the preimage of the set of Heegner points on $X(1)$ with respect to a morphism $f : X \to X(1)$. Now general properties of height functions combined with the explicit knowledge of the modular heights and logarithmic discriminants of Heegner points eventually complete the proof.

We also considered the following conjecture:

Conjecture II. *Let $\mathcal{X}$ be a regular model of X over $\mathrm{Spec}\,\mathcal{O}_K$ and let $\overline{\omega}$ be the dualizing sheaf equipped with the Arakelov metric. Then the arithmetic self intersection number $\overline{\omega}^2$ satisfies*

$$\overline{\omega}^2 = a_1(2g-2)\log|\Delta_{K|\mathbb{Q}}| + a_2 \left(\sum_{x \in \mathcal{X}^{\mathrm{sing}}} \log(\#k(x)) + \sum_{\sigma: K \to \mathbb{C}} \delta_{\mathrm{Fal}}(\mathcal{X}_\sigma(\mathbb{C})) \right) + a_3;$$

here a_1, a_2, $a_3 \in \mathbb{R}$, $\Delta_{K|\mathbb{Q}}$ is the absolute discriminant of the number field K and δ_{Fal} is a function on the moduli space of compact Riemann surfaces.

Conjecture II implies via the Kodaira-Parshin construction an effective version of Mordell's conjecture (cf. [10], [9]).

We approach conjecture II by means of a Belyi uniformization $X(\Gamma)$ of an algebraic curve X. Recall that a complex curve X is defined over a number field, if and only if there exist a morphism $\boldsymbol{\beta} : X \to \mathbb{P}^1$ with only three ramification points, if and only $X(\mathbb{C})$ is isomorphic to a modular curve $X(\Gamma)$ associated with a finite index subgroup of $\Gamma(1) = \mathrm{Sl}_2(\mathbb{Z})$.

A regular model $\mathcal{X}$ of X associated with a Belyi uniformization $\boldsymbol{\beta}$ is an arithmetic surface together with an morphism $\boldsymbol{\beta} : \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ which extends $\boldsymbol{\beta} : X \to \mathbb{P}^1$. In many cases the arithmetic surface $\mathcal{X} \to \mathcal{X}(1)$ can be chosen to be the minimal regular model, but we have to stress the fact that in general the arithmetic surface $\mathcal{X}$ is not the minimal regular model.

Theorem II. *Let $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{X}(1)$ be an arithmetic surface associated with a Belyi uniformization $X(\Gamma)$ of a curve X defined over a number field K. Assume that all cusps are K-rational points and that all cuspidal divisors (= divisors on X with support in the cusps of degree zero) are torsion. Then there exists an absolute constant $\kappa \in \mathbb{R}$ independent of X such that the arithmetic self-intersection number of the dualizing sheaf on $\mathcal{X}$ satisfies the inequality*

$$\overline{\omega}^2_{\mathrm{Ar}} \leq (4g-4)\left(\log|\Delta_{K|\mathbb{Q}}| + [K:\mathbb{Q}]\kappa\right) + \sum_{\mathfrak{p}\ \mathrm{bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}), \tag{1}$$

where a prime $\mathfrak{p}$ is said to be bad if the fiber of $\mathcal{X}(\Gamma)$ above $\mathfrak{p}$ is reducible. The coefficients $a_{\mathfrak{p}}$ are rational numbers, which can be calculated explicitly. In particular, if the fiber of $\mathcal{X}$ above $\mathfrak{p}$ has $r_{\mathfrak{p}}$ irreducible components $C_j^{(\mathfrak{p})}$, then

$$a_{\mathfrak{p}} \leq 4g(g-1)(r_{\mathfrak{p}}-1)^2 \max_{j,k} |C_j^{(\mathfrak{p})}.C_k^{(\mathfrak{p})}|.$$

Moreover if $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{X}(1)$ is a Galois cover, then $a_{\mathfrak{p}} \leq 0$.

If $\mathcal{X}$ is not the minimal model $\mathcal{X}_{\min}$ of X_K, then our formula (1) will become additional contributions coming from those primes of $\mathcal{O}_K$ that give rise to fibers of $\mathcal{X}$ which contain a (-1)-curve. Notice that, the coefficients $a_{\mathfrak{p}}$, which could be seen as a measure of how complicated $\mathcal{X}$ is, may be arbitrarily large compared to the number of singular points.

We can apply our result whenever Γ is a congruence subgroup, this is because of the Manin-Drinfeld theorem (see e.g. [2]). In particular if Γ is of certain kind, then the coefficients $a_{\mathfrak{p}}$ in (1) could be calculated explicitly by using the descriptions of models for $X(\Gamma)$ (see e.g. [4]). We illustrated this with the following theorems.

Theorem III. *Let $\mathcal{X}_0(N)$ be the minimal regular model of the modular curve $X_0(N)$, where N is a square free integer having at least two different prime factors and $(N,6) = 1$. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}^2_{\mathrm{Ar}} \leq (4g-4)\kappa + (3g+1)\sum_{p|N} \frac{p+1}{p-1} \log p,$$

where $\kappa \in \mathbb{R}$ is an absolute constant independent of N.

The modular curves $X_0(N)$, with square free N and $(6,N)=1$, are defined over $\mathbb{Q}$. We point to the fact that the completely different methods in [1],[8],[5], which depend strongly on the specific arithmetic of $\Gamma_0(N)$, give the slightly better estimate $\overline{\omega}^2_{\mathcal{X}_0(N),\mathrm{Ar}} = 3g\log(N)(1+O(\log\log(N)/\log(N))$, which is the best possible one.

Theorem IV. *Let $\mathcal{X}(N)$ be the minimal regular model of the modular curve $X(N)$, where N has at least two different prime divisors. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}^2_{\mathrm{Ar}} \leq (4g-4)\left(\log|\Delta_{\mathbb{Q}(\zeta_N)|\mathbb{Q}}| + [\mathbb{Q}(\zeta_N):\mathbb{Q}]\kappa\right)$$

where $\kappa \in \mathbb{R}$ is an absolute constant independent of N.

Other examples of curves where our result could be applied are the Fermat curves $x^n + y^n = z^n$. In [7] we give formulas the Fermat curves with prime exponents.

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Non-Archimedean potential theory and Arakelov geometry

Amaury Thuillier

Let X be a proper and smooth curve over $\mathbb{Q}_p$. In [9], R. Rumely elaborated a non-Archimedean potential theory allowing him to define a notion of capacity for subsets of $X(\mathbb{C}_p)$. One basically encounters two difficulties when dealing with the topological space $X(\mathbb{C}_p)$: its lack of local connectedness (harmonicity should be a local notion) and its lack of local compactness (measure theory is much well-behaved on locally compact spaces). Rigid geometry is an answer to the first point and it was indeed used by E. Kani to define a potential theory, in which only affinoid subsets are allowed [8]. This last article also brought to light the potential theoretic interpretation of the non-Archimedean part of Arakelov geometry on curves.

The point of view introduced by V.G. Berkovich in non-Archimedean analytic geometry (see [1]) provides the good framework to elaborate a potential theory on curves displaying all usual properties known for Riemann surfaces. From now, k is a non-Archimedean field with non trivial absolute value $|.|$, ring of integers k° and $S := \mathrm{Spf}(k^\circ)$; all analytic spaces will be strict (see [2], 1.2).

We first define, in any dimension, the notion of a "smooth function" and a dd^c operator at the level of germs. These definitions come from an analogy between affinoids/analytic spaces and polytopes/polyhedra. Let X be a geometrically reduced k-analytic space; a real function φ on X is $\mathbb{Z}$-*smooth* (resp. *smooth*) if each point x has a neighborhood $V = V_1 \cup \ldots \cup V_n$, $V_i \subset X$ affinoid, such that $\varphi_{|V_i}$ belongs to the subgroup $\log|\mathcal{O}_X(V_i)^\times|$ of $\mathrm{C}^0(V_i, \mathbb{R})$ (resp. if it is locally a real combination of $\mathbb{Z}$-smooth functions). Smooth functions are the sections of a sheaf $\mathcal{A}^0_X$ on X. Given an admissible S-formal scheme $\mathcal{X}$ with generic fibre $\mathcal{X}_\eta$, one defines a natural map

$$\mathrm{Div}(\mathcal{X})' := \Gamma\big(\mathcal{X}, \big(\mathcal{O}_\mathcal{X} \otimes_{k^\circ} k\big)^\times / \mathcal{O}_\mathcal{X}^\times\big) \to \mathrm{A}^0(X) := \Gamma(X, \mathcal{A}^0_X)$$

by sending a divisor D, represented by the cocycle $\{(U, f_U)\}$, to the $\mathbb{Z}$-smooth function φ_D whose restriction to U_η is $-\log|f_U|$. From this follows an interpretation of smooth functions in terms of Cartier divisors on S-formal schemes.

Theorem. (W. Gubler, [7], Theorem 7.12) *The space of smooth functions with compact support is dense in* $\mathrm{C}^0_c(X, \mathbb{R})$.

Let $\mathrm{I}(X)$ be the set of points $x \in X$ whose residue field $\mathcal{H}(x)$ satisfies the property that the extension $\widetilde{\mathcal{H}(x)}/\widetilde{k}$ has transcendence degree $\dim_x(X)$. Any x in $\mathrm{I}(X)$ has a fundamental system of neighborhoods in the shape of $\mathcal{X}_\eta$, where $\mathcal{X}$ is an admissible S-formal scheme such that the reduction map $\mathcal{X}_\eta \to \widetilde{\mathcal{X}}$ maps x to a generic point, with corresponding irreducible component $\widetilde{\mathcal{X}}[x]$. Admissible blow-ups give rise to a direct system of groups $\mathrm{Div}(\widetilde{\mathcal{X}}[x])$ and let $\mathrm{Div}(X, x)_\mathbb{R} := \varinjlim_\mathcal{X} \mathrm{Div}(\widetilde{\mathcal{X}}[x]) \otimes_\mathbb{Z} \mathbb{R}$. One then defines a linear operator

$$\mathrm{dd}^c_x : \mathcal{A}^0_{X,x} \to \mathrm{Div}(X, x)_\mathbb{R}$$

by requiring that, for any divisor $D \in \mathrm{Div}(\mathcal{X})'$, $D = [\{(U, f_U)\}]$,

$$\mathrm{dd}^c_x\ \varphi_D = \varepsilon(x)\ [\{(U \cap \widetilde{\mathcal{X}}[x], \widetilde{\alpha^{-1} f_U^N})\}],$$

where the positive integer N and the element α of $k^\times$ are chosen such that $|f_U(x)|^N = |\alpha|$, $\widetilde{\alpha^{-1} f_U^N}$ is the reduction in $\widetilde{\mathcal{H}(x)}$ and $\varepsilon(x)$ is a positive rational number (multiplicity), the general definition of which requires the Grauert-Remmert finitness theorem ([4], Theorem 1.3). The following convexity property suits with the analogy mentioned above.

Proposition. *A smooth function φ has a local minimum at a point $x \in \mathrm{I}(X)$ if and only if* $\mathrm{cyc}(\mathrm{dd}^c_x\ \varphi) \geq 0$.

One should mention that the constructions just described provide a concrete interpretation of a (very) small part of what is done in [3].

Let us now restrict to curves. For any smooth k-analytic curve X, we get a global dd^c operator with values in the space $\mathrm{A}^1(X)$ of measures on X whose support is a locally finite subset of $\mathrm{I}(X)$:

$$\mathrm{dd}^c\ \varphi = \sum_{x \in \mathrm{I}(X)} \deg_{\widetilde{k}}(\mathrm{dd}^c_x\ \varphi)\delta_x.$$

A function h on an open subset Ω of X is *harmonic* if it is smooth and satisfies the equation $\mathrm{dd}^c\ h = 0$. Subharmonic functions are defined in the usual way with respect to harmonic functions.

The following two results are basic.

Theorem. (Maximum principle) *A point x is a local maximum of a subharmonic function u if and only if u is constant in a neighborhood of x.*

This is an easy consequence of the convexity property above.

Theorem. *Every point of X has a fundamental system of relatively compact neighborhoods on which the Dirichlet problem can be solved.*

The proof relies on the semistable reduction theorem and Berkovich's definition of skeletons for semistable S-formal schemes.

Spaces of currents are defined by (algebraic) duality :

$$\mathrm{D}^0(X) = \mathrm{A}^1_c(X)^\vee \quad \text{and} \quad \mathrm{D}^1(X) = \mathrm{A}^0_c(X)^\vee,$$

where the subscript c means compact support; $\mathrm{A}^i(X) \subset \mathrm{D}^i(X)$ and the dd^c operator can be extended to $\mathrm{D}^0(X)$ by duality. A current of degree 0 being nothing but a real valued function on $\mathrm{I}(X)$, dd^c can be applied to any function on X.

Theorem. *Assume that X is proper. Given $S \in \mathrm{D}^1(X)$, the equation $\mathrm{dd}^c T = S$ has a solution T in $\mathrm{D}^0(X)$ if and only if $\langle S, 1\rangle = 1$. Moreover, T belongs to $\mathrm{A}^0(X)$ if and only if S belongs to $\mathrm{A}^1(X)$.*

Theorem. *Assume that X is (the analytification of) an algebraic curve and let $\mathcal{SH}(X)$ (resp. $\mathrm{H}(X)$) be the space of subharmonic (resp. harmonic) functions on X. The map $\mathrm{dd}^c : \mathrm{D}^0(X) \to \mathrm{D}^1(X)$ induces a bijection between $\mathcal{SH}(X)/\mathrm{H}(X)$ and the set of positive Radon measures on the locally compact space X.*

It should be noticed that this potential theory is very similar to the classical one in the real dimension one case.

Let us conclude by a few words on applications to Arakelov geometry on curves. Given a smooth algebraic curve X over $\mathbb{Q}$, the choice of a normal model $\mathcal{X}$ over $\mathbb{Z}$ defines for each prime p a finite subset $\mathrm{S}_0(\mathcal{X})_p$ of $(X \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\mathrm{an}}$, corresponding to the set of generic points of $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p$. As Kani showed in [8], the p-adic contribution to classical Arakelov theory on $\mathcal{X}$ can be understood in terms of equilibrium potentials with respect to $\mathrm{S}_0(\mathcal{X})_p$; in particular, intersection numbers can be expressed as a star-product. We can now define generalized Arakelov divisors on X, going from smooth Green functions to Green functions with L^1_2-regularity as shown in [5] on the archimedean side, and reformulate previous generalizations.

Theorem. *At any place, Zhang's integral metrics are exactly the continuous metrics with measure curvature.*

This generalized Arakelov geometry can be applied to prove a refined equidistribution theorem for points of small height. It is also possible to give another proof of the main theorem of [10], based on the arithmetic Hilbert-Samuel theorem and an approximation result for subharmonic functions.

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Harmonic analysis on adelic spaces and local fields

A. N. Parshin

If X is a scheme of dimension n and of finite type over $\mathbb{Z}$ and $X_n \subset X_{n-1} \subset \ldots X_1 \subset X_0 = X$ is a flag of irreducible subschemess ($\mathrm{codim}(X_i) = i$), then one can define a ring $K_{X_0,\ldots,X_{n-1}}$ associated to the flag. In case everything is regularly embedded the ring is an n-dimensional local field. Then one can form an adelic object

$$\mathbb{A}_X = \prod{}' K_{X_0,\ldots,X_{n-1}}$$

where the product is taken over all the flags with respect to certain restrictions on components of adeles [**Hu**]. For a scheme over a finite field $\mathbb{F}_q$ this is the right definition of the adelic space attached to X. In general case, one has to extend it to $\mathbb{A}_X \oplus \mathbb{A}_{X\otimes\mathbb{R}}$.

In dimension 1 the adelic groups $\mathbb{A}_X$ and $\mathbb{A}_X^*$ are locally compact groups and thus we can apply the classical harmonic analysis. The starting point for that is the measure theory on locally compact local fields attached to the points on schemes X of dimension 1.

J. Tate and independently K. Iwasawa have introduced an analytically defined L-function $L(s,\chi,f) = \int_{\mathbb{A}^*} f(a)\chi'(a)|a|^s d^*a$, where d^* is a Haar measure on $\mathbb{A}^*$,the function f belongs to the Bruhat-Schwartz space of functions on $\mathbb{A}_X$ and χ is an abelian character, coming from the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ by the reciprocity map. For $L(s,\chi,f)$ they have proved the analytical continuation to the whole s-plane and the functional equation $L(s,\chi,f) = L(1-s,\chi^{-1},\hat{f})$, using Fourier transform for functions f on the space $\mathbb{A}_X$.

For a long time the author has advocated the following

Problem. *Extend Tate–Iwasawa's analytic method to higher dimensions.* (see, in particular, [**P2**]).

The higher adeles have been introduced exactly for this purpose. So we have the following underlying

Problem. *Develop a measure theory and harmonic analysis on n-dimensional local fields.*

Note that the n-dimensional local fields are not locally compact topological spaces for $n > 1$ and by Weil's theorem the existence of the Haar measure (in the usual sense) on a topological group implies its locally compactness.

In the talk, we show how to construct the harmonic analysis and the measure theory in the first non-trivial case of algebraic surface X over a finite field k. Let P be a closed point of X, $C \subset X$ be an irreducible curve such that $P \in C$.

If X and C are smooth at P then we let $t \in \mathcal{O}_{X,P}$ be a local equation of C at P and $u \in \mathcal{O}_{X,P}$ be such that $u|_C \in \mathcal{O}_{C,P}$ is a local parameter at P. Denote by $\wp$ the ideal defining the curve C near P. Now we can introduce the two-dimensional local field attached to the pair P, C by the following procedure including completions and localizations: $K_{P,C} = \mathrm{Frac}(\widehat{(\hat{\mathcal{O}}_{X,P})_\wp}) = k(P)((u))((t))$. Let K_P be the minimal subring of $K_{P,C}$ which contains $K = k(X)$ and $\hat{\mathcal{O}}_P$. Then $K \subset K_P \subset K_{P,C}$ and there is another intermediate subring $K_C = Frac(\mathcal{O}_C) \subset K_{P,C}$. Now we can compare the structure of adelic components in dimension 1 and 2. In dimension 1 the adelic complex reads as $K \oplus \prod\limits_{x\in C} \hat{\mathcal{O}}_x \to \prod'_{x\in C} K_x$ and we shorten it to $\mathbb{A}_0 \oplus \mathbb{A}_1 \to \mathbb{A}_{01}$. In dimension 2, we have to start from the complex $\mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \to \mathbb{A}_{01} \oplus \mathbb{A}_{02} \oplus \mathbb{A}_{12} \to \mathbb{A}_{012}$, where $\mathbb{A}_0 = K = k(X), \mathbb{A}_1 = \prod_{C\subset X} \widehat{\mathcal{O}}_C, \mathbb{A}_2 = \prod_{x\in X} \hat{\mathcal{O}}_x, \mathbb{A}_{01} = \prod'_{C\subset X} K_C, \mathbb{A}_{02} = \prod'_{x\in X} K_x, \mathbb{A}_{12} = \prod'_{x\in C} \widehat{\mathcal{O}}_{x,C}, \mathbb{A}_{012} = \mathbb{A}_X = \prod' K_{x,C}$.

In dimension 1, the group $\mathbb{A}_C$ is a locally compact group and the analysis starts with a definition of the functional spaces. Let V be a finite dimensional vector space over adelic ring $\mathbb{A}_C$ (or over an one-dimensional local field K with finite residue field $\mathbb{F}_q$). We put

$$\begin{aligned}\mathcal{D}(V) &= \{\text{ locally constant functions with compact support}\}\\ \mathcal{E}(V) &= \{\text{locally constant functions}\}\\ \mathcal{D}(V)' &= \{\text{dual to } \mathcal{D} \ = \ \text{all distributions}\}\\ \mathcal{E}(V)' &= \{\text{dual to } \mathcal{E} \ = \ \text{distributions with compact support}\}\end{aligned}$$

The harmonic analysis includes definitions of direct and inverse images in some category C_1 of spaces like V, definition of the Fourier transform F as a map from $\mathcal{D}'(V)\otimes\mu(V)$ to $\mathcal{D}'(\check{V})$ as well for other types of spaces. Here $\mu(V)$ is a space of the Haar measures on V and $\check{V}$ is the dual space. The main result is the following Poisson formula

$$F(\delta_{W,\mu_0}\otimes\mu)=\delta_{W^\perp,\mu^{-1}/\mu_0^{-1}}$$

for any closed subgroup $i: W\to V$. Here $\mu_0\in\mu(W), \mu\in\mu(V), \delta_{W,\mu_0}=i_*(1_W\otimes\mu_0)$ and $W^\perp$ is the annulator of W in $\check{V}$. This general formula combines the following facts of analysis on the self-dual group $\mathbb{A}_C$: $F(\delta_{\mathbb{A}_1(D)})=\mathrm{vol}(\mathbb{A}_1(D))\delta_{\mathbb{A}_1((\omega)-D)}$, and $F(\delta_K)=\mathrm{vol}(\mathbb{A}/K)^{-1}\delta_K$ for the standard subgroups in $\mathbb{A}_C$(correspondingly, attached to a divisor D and to the principal adeles). This easily implies Riemann-Roch and Serre duality for divisors on the curve C (see[**P2**]).

For dimension 2, the space $\mathbb{A}_X$ has a filtration by the subspaces $\mathbb{A}_{12}(D)$ where D runs through the Cartier divisors on X. The quotients $\mathbb{A}_{12}(D)/\mathbb{A}_{12}(D')$ will be the spaces of the type considered above in the one-dimensional situation. This allows us to introduce the following spaces of functions(distributions),using a trick suggested by M. Kapranov [**K**]:

$$\mathcal{D}_{P_0}(V) \ = \ \lim_{\substack{\leftarrow\\ j^*}} \ \lim_{\substack{\leftarrow\\ i_*}} \ \mathcal{D}(P/Q)\otimes\mu(P_0/Q,)$$

$$\mathcal{D}'_{P_0}(V) \ = \ \lim_{\substack{\rightarrow\\ j_*}} \ \lim_{\substack{\rightarrow\\ i^*}} \ \mathcal{D}'(P/Q)\otimes\mu(P_0/Q),$$

$$\mathcal{E}(V) \ = \ \lim_{\substack{\leftarrow\\ j^*}} \ \lim_{\substack{\rightarrow\\ i^*}} \ \mathcal{E}(P/Q), \quad \mathcal{E}'(V) \ = \ \lim_{\substack{\rightarrow\\ j_*}} \ \lim_{\substack{\leftarrow\\ i_*}} \ \mathcal{E}'(P/Q),$$

where $P\supset Q\supset R$ are some elements of the filtration in $\mathbb{A}_X$(or more generally, in a reasonable filtered space V with locally compact quotients), P_0 is a fixed subspace from the filtration and $j: Q/R\to P/R, i: P/R\to P/Q$ are the canonical maps. Note that both spaces $\mathcal{D}_{P_0}(V), \mathcal{D}'_{P_0}(V)$ are $\mathcal{E}(V)$-modules.

Just as in the case of dimension 1, one can define direct and inverse images in some category C_2 (see [**O**]) of filtered spaces like V and including all components of the adelic complex, the Fourier transform F that preserves the spaces $\mathcal{D}$ and $\mathcal{D}'$ but interchanges the spaces $\mathcal{E}$ and $\mathcal{E}'$, characteristic functions δ_W of subspaces and then prove a generalization of the Poisson formula. It is important that for a class of spaces V (but not for $\mathbb{A}_X$ itself) there exists an invariant measure, defined up to a constant, as an element of $\mathcal{D}'$. There is also an analytical expression for the intersection number of two divisors based on an adelic approach to the intersection theory [**P1**]. As a corollary, we get an analytical proof of the (easy part of) Riemann-Roch theorem for divisors on X.

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Dyson Theorem for curves

Carlo Gasbarri

§*1 Introduction.* In this talk I will describe an analogue of Dyson theorem for curves, which in particular gives a proof of Siegel theorem independent on the other big theorems. The proof is much easier of, for instance, Roth theorem and it is in the same spirit of the classical theorem by Dyson.

§*2 Qualitative Statement.* Let K be a number field, $L_1, \ldots, L_r$ be finite extensions of K and define $d := \max\{[L_i \cdot L_j : K]\}$. We also pose $A := \oplus L_i$. Let C_1 and C_2 be smooth projective curves over K and

$$D_i = Spec(A) \to C_i,$$

be effective geometrically reduced divisors of degree $\sum [L_i : K]$ on C_i. Let L_i be line bundle of degree one over C_i and $h_{L_i}(\cdot)$ height functions associated to L_i. Finally, let S be e a finite set of places of K and $\lambda_{D_i,S}$ be Weil functions associated to D_i and S.

Theorem 1. *Let ϑ_1, ϑ_2 and ϵ be three rational numbers such that*

$$\vartheta_1 \cdot \vartheta_2 \geq 2d + \epsilon$$

then the set of rational points $(P, Q) \in C_1(K) \times C_2(K)$ such that

$$\lambda_{D_1,S}(P) > \vartheta_1 \cdot h_{L_1}(P)$$

and

$$\lambda_{D_2,S}(Q) > \vartheta_2 \cdot h_{L_2}(Q)$$

is contained in a proper *closed subset whose irreducible components are either fibers or points.*

One easily sees that if we apply this to $C_1 = C_2$, $D_1 = D_2$ and $\vartheta_i = \sqrt{2d} + \epsilon$ we obtain:

Corollary 1. *Let C be a curve and D a divisor of degree d on C then*

$$\lambda_{D,S} \leq (\sqrt{2d} + \epsilon)h_L + O(1).$$

If $\deg(D) \geq 3$ the corollary is enough to imply Siegel theorem. To obtain the general case of it one take a suitable covering of the curve and conclude.

Remark. Classically one can prove that, if the genus of C is at leat one, then $\lambda_{D,S} \leq \epsilon h_L + O(1)$.; but the the proof needs Roth and weak Mordell–Weil. The proof of the theorem above is independent on these "big" theorems.

§*3 Effective statement.* Let K be a number field, O_K its ring of integers; let

$$f_i : \mathcal{X}_i \longrightarrow Spec(O_K)$$

$(i = 1, 2)$ be two regular arithmetic surfaces. We fix symmetric metrics on the line bundle $\mathcal{O}(\Delta_i)$, where $\Delta_i \hookrightarrow \mathcal{X}_i \times \mathcal{X}_i$ are the diagonals; if $P\mathcal{X}_i(O_K)$ is a section then we define a metric on $\mathcal{O}(P)$ by imposing that the isomorphism $\mathcal{O}_{\mathcal{X}_i}(P) \simeq \iota_P^*(\mathcal{O}(\Delta))$, where $\iota_P \colon \mathcal{X}_i \to \mathcal{X}_i \times \mathcal{X}_i$ is the embedding $x \mapsto (x, P)$, is an isometry. This gives a well defined metric and a well defined Weil function for every divisor D on $\mathcal{X}_i$. We also fix hermitian line bundles M_i of degree one on $\mathcal{X}_i$.

We fix a positive integer n, a rational positive number ϵ and two rational numbers ϑ_i such that

$$\vartheta_1 \cdot \vartheta_2 \geq 2n + \epsilon.$$

Finally let S be a finite set of places of K.

Theorem 2. *There exist two effectively computable constants R_1 and R_2 having the following property: Let $L_1, \dots, L_r$ be extensions of K, $n = \max\{[L_i \cdot L_j : K]\}$ and $A := Spec(\oplus O_{L_i})$. Let*

$$D_i \colon A \longrightarrow \mathcal{X}_i$$

be effective divisors; Denote by $D_1 := \sum P_i$ and by $D_2 = \sum Q_j$; we then pose

$$A(D_1, D_2) := \max\{h_{M_1}(P_i), h_{M_2}(Q_j)\},$$

$$B(D_1, D_2) := \max\{-\frac{(\mathcal{O}(P_i), \mathcal{O}(P_i))}{[K(P_i) : K]}, 1\} \cdot \max\{-\frac{(\mathcal{O}(Q_j), \mathcal{O}(Q_j))}{[K(Q_j) : K]}, 1\}$$

and

$$C(D_1, D_2) := \max_{v \in S}\{d_v(P_i, P_j), d_v(Q_i, Q_j)\}.$$

If $(P, Q) \in \mathcal{X}_1(K) \times \mathcal{X}_2(K)$ is a couple of points such that

a) $h_{M_1}(P) \geq \frac{N_1}{\epsilon} \cdot A(D_1, D_2) \cdot B(D_1, D_2) \cdot C(D_1, D_2)$;

b) $\lambda_{D_1,S}(P) \geq \vartheta_1 h_{M_1}(P_1)$ and $\lambda_{D_1,S}(Q) \geq \vartheta_1 h_{M_1}(Q)$; Then

$$h_{M_2}(Q) \leq N_2(A \cdot B \cdot C) \cdot h_{M_1}(P).$$

Remark that if you could replace in the theorem the condition $\vartheta_1 \cdot \vartheta_2 \geq 2n + \epsilon$ by the condition $\vartheta_1 \cdot \vartheta_2 \geq 2n - \epsilon$, then the statement above would give as corollary effective theorem and of Siegel Theorem:

Indeed you take as $\mathcal{X}_1$ the projective line and $\vartheta_1 = 2$; You suppose that D_i is irreducible over K and, and by Dirichlet theorem you can find *infinitely many rational points* such that

$$\lambda_{D_1,S}(P) \geq 2h_{M_1}(P)$$

(and these can be *explicitly* constructed, for instance by using continued fractions) consequently you can effectively bound the height of points of $\mathcal{X}_2$ which do not satisfy

$$\lambda_{D_2,S}(Q) \leq (n-\epsilon)h_{M_2}(Q)$$

in particular the height of (D,S)–integral points.

Corollary 2. *Fix an integer n and suppose that you have a curve $\mathcal{X}$ (not of general type) and an* effective *sequence of couples* $(D_m, P_m) \in Div^n(\mathcal{X}) \times \mathcal{X}(K)$ *such that*

$$\lambda_{D_m,S}(P_m) \geq (2+\epsilon)h_M(P)$$

and

$$R_m := \frac{h_M(P_m)}{\max\{-\frac{(\mathcal{O}(D_m),\mathcal{O}(D_m))}{[K(D_m):K]}, 1\} \cdot \max_{v\in S}\{d_v(D_m, D_m)\}} \to \infty$$

then we can prove an effective version of Siegel Theorem.

By *effective sequence* we mean that given a positive number W we can explicitly construct elements (D_m, P_m) in the sequence for which $R_m \geq W$.

The corollary can be restated, informally in the following way: If we can prove an effective version of Roth theorem then we can prove an effective version of Siegel theorem; but if we we can disprove *effectively* Roth theorem then we can prove again an effective version of Siegel Theorem.

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Around Chowla-Selberg formula

Vincent Maillot
(joint work with Damian Rössler)

The first part of this talk is mostly historical and devoted to the computation by Fagnano of the length of Bernoulli's Lemniscate (1750). We uncover the algebraic geometric construction underlying Fagnano's result. Some generalizations by Legendre (1811) and Lerch (1892) are presented as well, all of them encompassed by the Lerch (1896) & Chowla-Selberg (1949) formula. In the second part, we outline further developments by Gross & Deligne (1978) , Anderson (1982), Colmez (1993) et al. We end by presenting new results and conjectures in those directions due to Rössler and ourself (cf. [1], [2] and [3]) and relying mostly on a marvelous Lefschetz fixed point formula in Arakelov Geometry proven by Koehler and Rössler.

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Analytic torsion for Calabi-Yau threefolds

Ken-Ichi Yoshikawa
(joint work with Hao Fang, Zhiqin Lu)

Let X be a Calabi-Yau threefold. Let γ be a Kähler form on X and set $\overline{X} = (X, \gamma)$. Let $\tau(\overline{X}, \overline{\Omega}^p_X)$ be the Ray-Singer analytic torsion of $\Omega^p_X = \wedge^p T^*X$ with respect to γ. Bershadsky-Cecotti-Ooguri-Vafa [2] introduced the following combination of analytic torsions, which we call the *BCOV torsion*

$$\mathcal{T}_{\mathrm{BCOV}}(\overline{X}) = \prod_{p\geq 0} \tau(\overline{X}, \overline{\Omega}^p_X)^{(-1)^p p}.$$

Let $\mathrm{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma])$ be the covolume of the lattice $H^2(X, \mathbf{Z})/\text{Torsion}$ with respect to the L^2-metric induced from the Kähler class $[\gamma]$. Let η be a nowhere vanishing holomorphic 3-form on X. Let $c_3(\overline{X})$ be the top Chern form of $\overline{X}$. Let $\chi(X)$ be the topological Euler number of X. We define

$$\mathcal{A}(\overline{X}) = \mathrm{Vol}(\overline{X})^{\frac{\chi(X)}{12}} \exp\left[-\frac{1}{12}\int_X \log\left(\frac{\sqrt{-1}\,\eta\wedge\bar{\eta}}{\gamma^3/3!}\cdot\frac{\mathrm{Vol}(\overline{X})}{\|\eta\|^2_{L^2}}\right) c_3(\overline{X})\right],$$

which is independent of the choice of η. We define the real number $\tau_{\mathrm{BCOV}}(X)$ as

$$\tau_{\mathrm{BCOV}}(X) = \mathrm{Vol}(\overline{X})^{-3}\,\mathrm{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma])^{-1}\,\mathcal{A}(\overline{X})\,\mathcal{T}_{\mathrm{BCOV}}(\overline{X}).$$

Then $\tau_{\mathrm{BCOV}}(X)$ is independent of the choice of γ, so that $\tau_{\mathrm{BCOV}}(X)$ is an invariant of X. We regard τ_{BCOV} as a function on the moduli space of Calabi-Yau threefolds.

Let $\mathcal{X}$ be a smooth projective fourfold, and let $\pi\colon \mathcal{X} \to \mathbf{P}^1$ be a surjective flat morphism with discriminant locus $\mathcal{D}$. Let ψ be the inhomogeneous coordinate of $\mathbf{P}^1$, and set $X_\psi := \pi^{-1}(\psi)$ for $\psi \in \mathbf{P}^1$. We assume the following:

- $\infty \in \mathcal{D}$ and X_ψ is a Calabi-Yau threefold with $h^2(\Omega^1_{X_\psi}) = 1$ for $\psi \in \mathbf{P}^1 \setminus \mathcal{D}$;
- $\operatorname{Sing} X_\psi$ consists of a unique ordinary double point for $\psi \in \mathcal{D} \setminus \{\infty\}$;
- The Kodaira-Spencer map for $p\colon \mathcal{X} \to \mathbf{P}^1$ is an isomorphism at $\forall\, \psi \in \mathbf{P}^1 \setminus \{\infty\}$.

Outside $\mathcal{D}$, $T\mathbf{P}^1$ is equipped with the Weil-Petersson metric. Let $\|\cdot\|$ be the singular Hermitian metric on $(\pi_* K_{\mathcal{X}/\mathbf{P}^1})^{\otimes(48+\chi)} \otimes (T\mathbf{P}^1)^{\otimes 12}$ induced from the L^2-metric on $\pi_* K_{\mathcal{X}/\mathbf{P}^1}$ and from the Weil-Petersson metric on $T\mathbf{P}^1$.

Theorem 1. [5] *Let Ξ be a meromorphic section of $\pi_* K_{\mathcal{X}/\mathbf{P}^1}$ with divisor*

$$\operatorname{div}(\Xi) = \sum_{i\in I} m_i\, P_i + m_\infty\, P_\infty, \qquad P_i \neq P_\infty\, (i \in I).$$

Identify the points P_i, D_k with their coordinates $\psi(P_i), \psi(D_k) \in \mathbf{C}$, respectively. Set $\chi = \chi(X_\psi)$, $\psi \in \mathbf{P}^1 \setminus \mathcal{D}$. Then

$$\tau_{\mathrm{BCOV}}(X_\psi) = \mathrm{Const.} \left\| \frac{\prod_{k\in K}(\psi - D_k)^2}{\prod_{i\in I}(\psi - P_i)^{(48+\chi)m_i}}\, \Xi_\psi^{48+\chi} \otimes \left(\frac{\partial}{\partial\psi}\right)^{12} \right\|^{\frac{1}{6}}.$$

Theorem 1 applied to a family of quintic mirror threefolds yields a partial answer to the conjecture of Bershadsky-Cecotti-Ooguri-Vafa [1], [2].

Let $p\colon \mathcal{X} \to \mathbf{P}^1$ be the pencil of quintic threefolds of $\mathbf{P}^4$ defined by

$$\mathcal{X} = \{([z], \psi) \in \mathbf{P}^4 \times \mathbf{P}^1;\ F_\psi(z) = 0\}, \quad F_\psi(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi\, z_0 z_1 z_2 z_3 z_4.$$

On $\mathcal{X}$, a group G of order 125 acts and preserves the fibers of p. We have the induced family $p\colon \mathcal{X}/G \to \mathbf{P}^1$. There exist a resolution $r\colon \mathcal{W} \to \mathcal{X}/G$ satisfying:
(1) Set $r_\psi = r|_{W_\psi}$. Then $r_\psi\colon W_\psi \to X_\psi/G$ is a crepant resolution for $\psi \in \mathbf{P}^1 \setminus \mathcal{D}$;
(2) $\operatorname{Sing} W_\psi$ consists of a unique ordinary double point if $\psi^5 = 1$.

Set $\pi = r \circ q$. A family $\pi\colon \mathcal{W} \to \mathbf{P}^1$ satisfying (1), (2) is called a *family of quintic mirror threefolds*, whose general fiber is a smooth Calabi-Yau threefold.

Theorem 2. [1], [2], [5] *The following identity holds*

$$\tau_{\mathrm{BCOV}}(W_\psi) = \mathrm{Const.} \left\| \psi^{-62}(\psi^5 - 1)^{1/2}\, (\Xi_\psi)^{62} \otimes \left(\frac{d}{d\psi}\right)^3 \right\|^{2/3}.$$

Here Ξ_ψ is the 3-form on W_ψ induced from the following 3-form on X_ψ/G by r_ψ

$$\Omega_\psi = \left(\frac{2\pi i}{5}\right)^{-3} 5\psi \frac{dz_0 \wedge dz_1 \wedge dz_2}{\partial F_\psi(z)/\partial z_3}, \qquad \Xi_\psi = r_\psi^* \Omega_\psi.$$

Bershadsky-Cecotti-Ooguri-Vafa [1], [2] conjectured that the genus-g Gromov-Witten invariants $\{N_g(d)\}_{g\geq 0, d\geq 1}$ of a general quintic hypersurface of $\mathbf{P}^4$ and the

BCOV invariant of the quintic mirror threefolds satisfy the identity

$$\tau_{\mathrm{BCOV}}(W_\psi)=\left\|\left\{q^{\frac{25}{12}}\prod_{d=1}^{\infty}\widetilde{\eta}(q^d)^{N_1(d)}(1-q^d)^{\frac{N_0(d)}{12}}\right\}^6\left(\frac{\Xi_\psi}{y_0(\psi)}\right)^{62}\otimes\left(q\frac{d}{dq}\right)^3\right\|^{\frac{2}{3}}$$

up to a constant. Here

$$\widetilde{\eta}(q)=\prod_{n=1}^{\infty}(1-q^n),\qquad y_0(\psi)=\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5(5\psi)^{5n}},$$

and the parameters q and ψ^5 are identified via the "mirror map":

$$q=(5\psi)^{-5}\exp\left(\frac{5}{y_0(\psi)}\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\left\{\sum_{j=n+1}^{5n}\frac{1}{j}\right\}\frac{1}{(5\psi)^{5n}}\right),\qquad |\psi|\gg 1.$$

Define the function $F_{1,B}^{\mathrm{top}}(\psi)$ by the following formula:

$$\psi^{-62}(\psi^5-1)^{\frac{1}{2}}\,(\Xi_\psi)^{62}\otimes\left(\frac{d}{d\psi}\right)^3=\frac{1}{F_{1,B}^{\mathrm{top}}(\psi)^3}\left(\frac{\Xi_\psi}{y_0(\psi)}\right)^{62}\otimes\left(2\pi i\,q\frac{d}{dq}\right)^3,$$

and set $F_{1,A}^{\mathrm{top}}(q)=F_{1,B}^{\mathrm{top}}(\psi(q))$. By Theorem 2, the conjecture of BCOV is reduced to the following identity, which is studied by Li-Zinger [6]:

$$q\frac{d}{dq}\log F_{1,A}^{\mathrm{top}}(q)=\frac{50}{12}-\sum_{n,d=1}^{\infty}N_1(d)\frac{2nd\,q^{nd}}{1-q^{nd}}-\sum_{d=1}^{\infty}N_0(d)\frac{2d\,q^d}{12(1-q^d)}.$$

Since the choice of a crepant resolution of a Calabi-Yau orbifold is *not* unique in general, it is worth asking the following:

Question 1. *If Calabi-Yau threefolds X and X' are birationally equivalent, then*

$$\tau_{\mathrm{BCOV}}(X)=\tau_{\mathrm{BCOV}}(X')\quad ?$$

Notice that under the same assumption, the Hodge numbers of X and X' are equal, i.e., $h^{p,q}(X)=h^{p,q}(X')$ for $p,q\geq 0$.

To prove Theorem 1, we use the curvature formula and the anomaly formula for Quillen metrics [4] as well as the following results on Quillen metrics.

Let $\pi\colon Y\to S$ be a surjective holomorphic map from a compact Kähler manifold (Y,g_Y) to a compact Riemann surface S. Let Σ_π be the critical locus of π and set $\Delta=\pi(\Sigma_\pi)$. Let (ξ,h_ξ) be a holomorphic Hermitian vector bundle on Y. Set $\lambda(\xi)=\det R\pi_*\xi$. Let $\|\cdot\|_{Q,\lambda(\xi)}$ be the Quillen metric on $\lambda(\xi)$ with respect to $g_Y|_{TY/S}$ and h_ξ. Let σ be a local, nowhere vanishing holomorphic section of $\lambda(\xi)$ near Δ. Let $0\in\Delta$.

Let $\mu\colon Y\setminus\Sigma_\pi\to\mathbf{P}(TY)^\vee$ be the *Gauss map* that assigns $y\in Y\setminus\Sigma_\pi$ the hyperplane $\ker(\pi_*)_y\in\mathbf{P}(T_yY)^\vee$. Since μ is meromorphic, there is a resolution $q\colon(\widetilde{Y},E)\to(Y,\Sigma_\pi)$ of the indeterminacy of μ, so that $\widetilde{\mu}=\mu\circ q$ is a morphism from $\widetilde{Y}$ to $\mathbf{P}(TY)^\vee$. Let U be the universal hyperplane bundle on $\mathbf{P}(TY)^\vee$ and let $H=\mathcal{O}_{\mathbf{P}(TY)^\vee}(1)$.

Theorem 3. [3], [7] *Let s be a local parameter of S with* $\mathrm{div}(s) = \Delta$. *Then*

$$\log \|\sigma(s)\|^2_{Q,\lambda(\xi)} = \left(\int_{E \cap q^{-1}(Y_0)} \widetilde{\mu}^* \left\{ \mathrm{Td}(U) \, \frac{\mathrm{Td}(H) - 1}{c_1(H)} \right\} q^* \mathrm{ch}(\xi) \right) \log |s|^2 + O(1).$$

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Equidistribution and generalized Mahler measure

Lucien Szpiro

The Mahler measure formula expresses the height of an algebraic number as the integral of the log of the absolute value of its minimal polynomial on the unit circle. The height is in fact the canonical height associated to the monomial maps x^n. We show in a paper with J.Pineiro and T.Tucker that for any rational map $\varphi(x)$ the canonical height of an algebraic number with respect to φ can be expressed as the integral of the log of its equation against the invariant Brolin-Lyubich measure associated to φ, with additional adelic terms at finite places of bad reduction:

$$deg(F) h_\varphi(\alpha) = \sum_v \int_{P^1(C_v)} \log |F|_v d\mu_{v,\varphi}$$

We give a complete proof of this theorem using integral models for each iterate of φ. In a subsequent paper with T.Tucker on equidistribution and Julia sets, we give a survey of results obtained by P. Autissier, M. Baker, R. Rumely, and ourselves. In particular, our results, when combined with technics of diophantine approximation, will allow us to compute the integrals in the generalized Mahler formula by averaging on periodic points: For any $\alpha \in K$ and any nonzero irreducible $F \in K[t]$ such that $F(\alpha) = 0$ we have

$$[K(\alpha) : Q](h_\varphi(\alpha) - h_\varphi(\infty)) = \sum_v \lim_{k \to \infty} \frac{1}{d^k} \sum_{\varphi^k([w:1]) = [w:1], F(w) \neq 0} \log |F(w)|_v .$$

Papers are available on the authors web pages.

Arakelov theory on symplectic and orthogonal flag varieties

Harry Tamvakis

The goal of this talk is to describe the multiplicative structure of the arithmetic Chow ring $\widehat{CH}(G/P)$, where G is a classical Lie group and P is a parabolic subgroup of G. For simplicity we will assume here that $P = B$ is a Borel subgroup. We regard the homogeneous space $X = G/B$ as a smooth scheme over the ring of integers. When $G = GL_n$, the question was studied in the author's thesis, and published in [7]. Here we will extend these results to the symplectic and orthogonal Lie groups. We use the notation of Gillet and Soulé's arithmetic intersection theory, following the exposition of [6]. In each case, there is a short exact sequence

$$0 \to \widetilde{A}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}(X) \to CH(X) \to 0, \tag{1}$$

and one has to choose a good splitting map $\epsilon : CH(X) \to \widehat{CH}(X)$ for (1).

When G is the symplectic group Sp_{2n}, the situation is as follows. There is a tautological hermitian filtration of the trivial vector bundle E of rank $2n$ over X,

$$\overline{\mathcal{E}} \;:\quad 0 = \overline{E}_0 \subset \overline{E}_1 \subset \overline{E}_2 \subset \cdots \subset \overline{E}_{2n} = \overline{E}.$$

Here $\mathrm{rank}(E_i) = i$ for each i and the middle bundle $\overline{E}_n$ is a Lagrangian subbundle of $\overline{E}$. Furthermore, all the hermitian metrics are induced from the trivial metric on $E(\mathbb{C})$, which is compatible with the symplectic form. For $1 \leq i \leq n$ let $\overline{L}_i$ denote the quotient line bundle E_i/E_{i-1} equipped with the quotient metric and set $\widehat{x}_i = -\widehat{c}_1(\overline{L}_i)$.

The Chow ring $CH(X)$ may be presented as a quotient of the polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$ modulo the ideal I_n generated by the relations

$$e_i(X_1^2, \ldots, X_n^2) = 0 \tag{2}$$

for $1 \leq i \leq n$, where the e_i are the elementary symmetric polynomials. Let S_n denote the symmetric group, Π_n the set of partitions λ with $\lambda_1 \leq n$, and D_n the set of 2^n strict partitions in Π_n. The polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$ has a natural product basis $\{\mathfrak{C}_{w,\lambda}(X_1, \ldots, X_n)\}$ for $w \in S_n$ and $\lambda \in \Pi_n$, which was studied in [2, 3]. We have that $\mathfrak{C}_{w,\lambda} = \mathfrak{C}_w \mathfrak{C}_\lambda$, where the $\mathfrak{C}_w$ are essentially type A Schubert polynomials $\mathfrak{S}_w$, defined by Lascoux and Schützenberger [4], and the $\mathfrak{C}_\lambda = \widetilde{Q}_\lambda$ are the $\widetilde{Q}$-polynomials of Pragacz and Ratajski [5]. There are integer structure constants $e_{uvw}^{\lambda\mu\nu}$ defined by the equation

$$\mathfrak{C}_{u,\lambda} \cdot \mathfrak{C}_{v,\mu} = \sum e_{uvw}^{\lambda\mu\nu}\, \mathfrak{C}_{w,\nu}.$$

Moreover, the set $\{\mathfrak{C}_{w,\lambda} \mid w \in S_n, \lambda \in D_n\}$ forms a $\mathbb{Z}$-basis for the quotient ring $\mathbb{Z}[X_1, \ldots, X_n]/I_n \cong CH(X)$. This basis of $CH(X)$ corresponds in geometry to the fibration of X over the Lagrangian Grassmannian $LG(n, 2n)$, with fiber the type A flag variety GL_n/B. However, it should be emphasized that the polynomials $\mathfrak{C}_{w,\lambda}$ do not represent the Schubert classes in $CH(X)$.

The map ϵ we use to split the sequence (1) is defined by

$$\epsilon(\mathfrak{C}_{w,\lambda}(X_1, \ldots, X_n)) = \mathfrak{C}_{w,\lambda}(\widehat{x}_1, \ldots, \widehat{x}_n) =: \widehat{\mathfrak{C}}_{w,\lambda}.$$

The following basic equation then holds in $\widehat{CH}(X)$.

$$\widehat{\mathfrak{C}}_{u,\lambda} \cdot \widehat{\mathfrak{C}}_{v,\mu} = \sum_{\nu \in D_n} e_{uvw}^{\lambda\mu\nu}\, \widehat{\mathfrak{C}}_{w,\nu} + \sum_{\nu \notin D_n} e_{uvw}^{\lambda\mu\nu}\, \widetilde{\mathfrak{C}}_{w,\nu}.$$

Here, for $\nu \in \Pi_n \setminus D_n$, the element $\widetilde{\mathfrak{C}}_{w,\nu} := \widehat{\mathfrak{C}}_{w,\nu}(\widehat{x}_1, \ldots, \widehat{x}_n)$ is the image under the map a in (1) of an $Sp(2n)$-invariant differential form on $X(\mathbb{C})$. This form may be computed by an explicit algorithm, which uses the arithmetic analogue of the relations (2), namely

$$e_i(\widehat{x}_1^2, \ldots, \widehat{x}_n^2) = (-1)^i a(\widetilde{c}_i(\overline{\mathcal{E}})).$$

The elements $\widetilde{c}_i(\overline{\mathcal{E}})$ are the Bott-Chern forms of the hermitian filtration $\overline{\mathcal{E}}$. The point is that these forms are given by complex transgression as polynomials in the entries of the curvature matrices of the vector bundles involved, and these entries may be expressed using the Maurer-Cartan forms of $Sp(2n)$ (see for example [1]). As a consequence, one proves that all the natural arithmetic intersection numbers on X are rational numbers.

For the case of the even orthogonal group SO_{2n}, an extra ingredient is needed in the above analysis. This is because the classical Chow ring of SO_{2n}/B carries the additional relation $X_1 \cdots X_n = 0$. In the arithmetic Chow ring, this becomes

$$\widehat{x}_1 \cdots \widehat{x}_n = \widehat{c}_n(\overline{E}_n) + a(\widetilde{c}_n(\overline{\mathcal{E}}')),$$

where $\overline{\mathcal{E}}'$ denotes the hermitian filtration

$$0 = \overline{E}_0 \subset \overline{E}_1 \subset \overline{E}_2 \subset \cdots \subset \overline{E}_n.$$

According to [10], we have an equation

$$\widehat{c}_n(\overline{E}_n) = -\frac{1}{2}\mathcal{H}_{n-1} a(c_{n-1}(\overline{E}_n)),$$

with $\mathcal{H}_{n-1}$ a harmonic number. Using this, one completes the story as in the symplectic case.

In conclusion, we note that when $X = G/P$ is a hermitian symmetric space, there is a canonical choice of invariant hermitian metric on $X(\mathbb{C})$, and an Arakelov Chow ring $CH(\overline{X})$ which is a subring of $\widehat{CH}(X)$, containing all the above products. In this case, there are combinatorially explicit formulas available, which lead to an 'arithmetic Schubert calculus', extending the classical one. The corresponding theory has been studied in the author's papers [8, 9, 10].

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Subquotient metrics and arithmetic Hilbert-Samuel theorem

Hugues Randriam

The main aim of this talk was the construction of an arithmetic Hilbert-Samuel function for a coherent sheaf on an arithmetic variety, and the proof that such a function satisfies an analogue of the arithmetic Hilbert-Samuel theorem. In order to do that, one needs some notion of metrics on this coherent sheaf, or at least, on its spaces of global sections. Two cases are of particular interest: the case where the coherent sheaf is a subsheaf of a locally free hermitian sheaf (for example, the subsheaf of sections satisfying certain vanishing conditions), and the case where the coherent sheaf is a quotient of a locally free hermitian sheaf (for example, the structural sheaf of a closed subscheme). One can generalize both situations by considering a coherent sheaf that is given as a subquotient of some locally free hermitian sheaf, as follows.

Let K be a number field, $\mathcal{O}_K$ its ring of integers, $\mathfrak{X}$ a scheme projective over $\mathrm{Spec}\mathcal{O}_K$ with generic fiber $\mathfrak{X}_K$ reduced, and $\overline{\mathcal{L}}$ an ample invertible $\mathcal{O}_{\mathfrak{X}}$-module whose complex fibers are equipped with a complex conjugation invariant continuous metric with semi-positive curvature. Let also $\mathcal{C}$ be a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Then a *subquotient metric* on $\mathcal{C}$ consists in:

- a locally free hermitian $\mathcal{O}_{\mathfrak{X}}$-module $\overline{\mathcal{E}}$,
- a coherent $\mathcal{O}_{\mathfrak{X}}$-submodule $\mathcal{F}$ of $\mathcal{E}$, and
- an epimorphism of $\mathcal{O}_{\mathfrak{X}}$-modules from $\mathcal{F}$ onto $\mathcal{C}$.

Let us denote by $\overline{\mathcal{C}}$ the coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{C}$ equipped with this subquotient metric.

Then, for any integer n, the $\mathcal{O}_K$-module $\Gamma(\mathfrak{X}, \mathcal{E}\otimes\mathcal{L}^{\otimes n})$ can be equipped with the $L^\infty(\mathfrak{X}_\sigma)$ metrics (with σ ranging over embeddings of K into $\mathbb{C}$), and $\Gamma(\mathfrak{X}, \mathcal{F}\otimes\mathcal{L}^{\otimes n})$, which is an $\mathcal{O}_K$-submodule, with the induced metrics. Now, if n is big enough, $\Gamma(\mathfrak{X}, \mathcal{C}\otimes\mathcal{L}^{\otimes n})$ is a quotient of the latter, and can then be equipped with the quotient metrics. We denote by $\overline{\Gamma(\mathfrak{X}, \mathcal{C}\otimes\mathcal{L}^{\otimes n})}$ the metrized $\mathcal{O}_K$-module just constructed. Then the function that to n big enough associates the arithmetic Euler-Poincaré characteristic of this metrized $\mathcal{O}_K$-module can be considered as an arithmetic

Hilbert-Samuel function for $\overline{\mathcal{C}}$:

$$h(\overline{\mathcal{C}};n) = \widehat{\chi}\left(\overline{\Gamma(\mathfrak{X}, \mathcal{C}\otimes\mathcal{L}^{\otimes n})}\right).$$

An important fact is then that, if $[\mathcal{C}]$ denotes the cycle of dimension $d = \dim|\mathcal{C}|$ associated to $\mathcal{C}$, one has the following asymptotic estimate for $h(\overline{\mathcal{C}};.)$, which is a generalization to this setting of the so-called arithmetic Hilbert-Samuel theorem:

Theorem ([1], Th. A)**.** *The arithmetic Hilbert-Samuel function of $\overline{\mathcal{C}}$ satisfies*

$$h(\overline{\mathcal{C}};n) = \frac{n^d}{d!}(\widehat{c_1}(\overline{\mathcal{L}})^d.[\mathcal{C}]) + o(n^d)$$

when n tends to infinity.

It should be noted that (the leading term of) this estimate does not depend on the choice of the metric on $\mathcal{C}$.

To prove this theorem, since subquotient metrics are compatible with dévissage, one is easily reduced to the case where $\mathcal{C}$ is of the form $\mathcal{C} = i_*\mathcal{M}$ where i is the inclusion morphism of a closed integral subscheme $\mathfrak{Y}$ of $\mathfrak{X}$ and where $\mathcal{M}$ is an invertible $\mathcal{O}_{\mathfrak{Y}}$-module. In this setting, if $\mathcal{M}$ is equipped with any hermitian metric, one can estimate $\widehat{\chi}_{L^\infty}\left(\overline{\Gamma(\mathfrak{Y}, \mathcal{M}\otimes\mathcal{L}^{\otimes n})}\right)$ with (Zhang's version [2] of) the classical arithmetic Hilbert-Samuel theorem. In order to use this estimate, one has to be able to compare the L^∞ metrics on $\mathfrak{Y}$ and the subquotient metrics coming from the L^∞ metrics on $\mathfrak{X}$. Such a comparison is given essentially by the following result:

Theorem ([1], Th. B)**.** *Let X be a reduced complex analytic space, $\overline{L}$ an invertible hermitian $\mathcal{O}_X$-module, $\overline{E}$ a locally free hermitian $\mathcal{O}_X$-module of finite type, $i : Y \hookrightarrow X$ a reduced closed analytic subspace, and $\overline{V}$ a locally free hermitian $\mathcal{O}_Y$-module of finite type. Suppose given a coherent $\mathcal{O}_X$-submodule F of E and an epimorphism p of coherent $\mathcal{O}_X$-modules from F onto i_*V. Then:*

(1) *For any $\epsilon > 0$ and any non-empty compact B of Y, there exists a constant $c > 0$ and a non-empty compact A of X, such that for any $n \geq 0$, for any $s \in \Gamma(Y, V\otimes L^{\otimes n})$ and any $\widetilde{s} \in \Gamma(X, F\otimes L^{\otimes n})$ satisfying $p(\widetilde{s}) = i_*s$, one has*

$$\|\widetilde{s}\|_{L^\infty(A)} \geq ce^{-n\epsilon}\|s\|_{L^\infty(B)}.$$

(2) *If moreover X is 1-convex and $\overline{L}$ has strictly positive curvature, there exists an integer n_0 and, for any $\epsilon > 0$ and any non-empty compact A of X, a constant $c' > 0$ and a non-empty compact B of Y, such that for any $n \geq n_0$ and any $s \in \Gamma(Y, V\otimes L^{\otimes n})$ there exists $\widetilde{s} \in \Gamma(X, F\otimes L^{\otimes n})$ satisfying $p(\widetilde{s}) = i_*s$ and*

$$\|\widetilde{s}\|_{L^\infty(A)} \leq c'e^{n\epsilon}\|s\|_{L^\infty(B)}.$$

The proof of this theorem uses a technique introduced by Bost, after ideas of Grauert and others. One considers the unit disc bundle of the total space of the dual of L, and one lifts all the involved sheaves to this disc bundle. The spaces of analytic sections of these sheaves are then equipped with a canonical Fréchet topology, and the first part of the theorem just expresses the continuity of the lifting of p to the disc bundle, while the second is obtained by applying to it Banach's open mapping theorem.

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A quantitative sharpening of the arithmetic Bogomolov inequality

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We reported on [N]. Let K be a number field with ring of integers $\mathcal{O}_K$ and $X/\mathrm{Spec}\,(\mathcal{O}_K)$ an arithmetic surface, i.e. a regular, integral, purely two-dimensional scheme, proper and flat over $\mathrm{Spec}\,(\mathcal{O}_K)$ and with smooth and geometrically connected generic fibre. Attached to a hermitian coherent sheaf on X are the usual characteristic classes with values in the arithmetic Chow-groups $\widehat{CH}^i(X)$ (cf. [GS1], 2.5), and in particular the discriminant of $\overline{E}$

$$\Delta(\overline{E}) := (1-r)\hat{c}_1(\overline{E})^2 + 2r\hat{c}_2(\overline{E}) \in \widehat{CH}^2(X)$$

where $r := \mathrm{rk}(E)$. The arithmetic degree map

$$\widehat{\deg} : \widehat{CH}^2(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is an isomorphism [GS2] and we will use the same symbol to to denote an element in $\widehat{CH}^2(X)_{\mathbb{R}}$ and its arithmetic degree in $\mathbb{R}$, see [GS2], 1.1 for the definition of arithmetic Chow-groups with real coefficients $\widehat{CH}^*(X)_{\mathbb{R}}$. Following [Mo2] we define the positive cone of X to be

$$\hat{C}_{++}(X) := \{x \in \widehat{CH}^1(X)_{\mathbb{R}} \,|\, x^2 > 0 \text{ and } \deg_K(x) > 0\}\ .$$

Given a torsion-free hermitian coherent sheaf $\overline{E}$ of rank $r \geq 1$ on X and a subsheaf $E' \subseteq E$ we endow E' with the metric induced from $\overline{E}$ and consider the difference of slopes

$$\xi_{\overline{E}',\overline{E}} := \frac{\hat{c}_1(\overline{E}')}{\mathrm{rk}(E')} - \frac{\hat{c}_1(\overline{E})}{r} \in \widehat{CH}^1(X)_{\mathbb{R}}.$$

Recall that a subsheaf $E' \subseteq E$ is *saturated* if the quotient E/E' is torsion-free. Our main result is the following.

Theorem 1. *Let $\overline{E}$ be a torsion-free hermitian coherent sheaf of rank $r \geq 2$ on the arithmetic surface X, satisfying*

$$\Delta(\overline{E}) < 0 \, .$$

Then there is a non-zero saturated subsheaf $\overline{E}' \subseteq \overline{E}$ such that $\xi_{\overline{E}',\overline{E}} \in \hat{C}_{++}(X)$ and

$$\xi^2_{\overline{E}',\overline{E}} \geq \frac{-\Delta}{r^2(r-1)} \, . \tag{1}$$

Remark 2. *The existence of an $\overline{E}' \subseteq \overline{E}$ with $\xi_{\overline{E}',\overline{E}} \in \hat{C}_{++}(X)$ is the main result of* [Mo2] *and means that $\overline{E}' \subseteq \overline{E}$ is arithmetically destabilising with respect to any polarisation of X, c.f.* loc. cit. *for more details on this. The new contribution here is the inequality (1) which is the exact arithmetic analogue of a known geometric result, c.f. for example* [HL], *Theorem 7.3.4.*

Remark 3. *A special case of Theorem 1 appears in disguised form in the proof of* [So], *Theorem 2: Given a sufficiently positive hermitian line bundle $\overline{L}$ on the arithmetic surface X and some non-torsion element $e \in \mathrm{H}^1(X, L^{-1}) \simeq \mathrm{Ext}^1(L, \mathcal{O}_X)$, C. Soulé establishes a lower bound for*

$$||e||^2 := \sup_{\sigma: K \hookrightarrow \mathbb{C}} ||\sigma(e)||^2_{L^2}$$

by considering the extension determined by e

$$\mathcal{E} : 0 \longrightarrow \overline{\mathcal{O}_X} \longrightarrow \overline{E} \longrightarrow \overline{L} \longrightarrow 0$$

and suitably metrised as to have $\hat{c}_1(\overline{E}) = \overline{L}$ and $2\hat{c}_2(\overline{E}) = \sum_\sigma ||\sigma(e)||^2_{L^2}$, hence $\Delta(\overline{E}) = -\overline{L}^2 + 2\sum_\sigma ||\sigma(e)||^2_{L^2}$ (where we write $\overline{L} = \hat{c}_1(\overline{L})$ following the notation of loc. cit.*).*
If $E_{\overline{\mathbb{Q}}}$ is semi-stable the arithmetic Bogomolov inequality concludes the proof. Otherwise, the main point is to show the existence of of an arithmetic divisor $\overline{D}$ satisfying

$$\begin{aligned} \deg_K(\overline{D}) &\leq \deg_K(\overline{L})/2 \text{ and} \qquad (2)\\ 2(\overline{L} - \overline{D})\overline{D} &\leq [K : \mathbb{Q}] \cdot ||e||^2, \qquad (3) \end{aligned}$$

c.f. (28) *and* (32) *of* loc. cit. *where these inequalities are established by some direct argument. We wish to point out that the existence of some $\overline{D}$ satisfying (2) and (3) is a special case of Theorem 1. In fact, let $\overline{E}' \subseteq \overline{E}$ be as in Theorem 1 and define $\overline{D} := \overline{L} - \hat{c}_1(\overline{E}')$. We then compute*

$$\xi_{\overline{E}',\overline{E}} = \frac{\overline{L}}{2} - \overline{D}$$

and $\xi_{\overline{E}',\overline{E}} \in \hat{C}_{++}(X)$ *implies (2). Furthermore, the inequality (1) in the present case reads*

$$\xi^2_{\overline{E}',\overline{E}} = \frac{\overline{L}^2}{4} + \overline{D}^2 - \overline{L}\,\overline{D} \geq \frac{-\Delta}{4} = \frac{\overline{L}^2}{4} - \frac{1}{2}\sum_{\sigma} ||\sigma(e)||^2_{L^2}\ ,\ i.e.$$

$$2(\overline{L} - \overline{D})\overline{D} \leq \sum_{\sigma} ||\sigma(e)||^2_{L^2},$$

hence the trivial estimate $[K : \mathbb{Q}] \cdot ||e||^2 \geq \sum_\sigma ||\sigma(e)||^2_{L^2}$ *gives (3).*

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