Cours regensburg

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Contents

1. Introduction

Let F be a field, F_{sep}/F a separable closure.
Plan : we construct a functor

$$Var_F \longrightarrow \mathrm{Mot}_F(\Lambda)$$

where Var_F is the category of smooth projective varieties over F and $\mathrm{Mot}_F(\Lambda)$ is the category of Grothendieck Chow motives with coefficients in Λ.

2. Chow groups

Definition 2.1. *The group of algebraic cycles on a variety X, denoted by $Z(X)$, is the free abelian group generated by integral closed subvarieties of X.*

Remark 2.2. $Z(X) = \bigoplus_{i=0}^{\dim(X)} Z_i(X)$, where $Z_i(X)$ is the free group generated by closed integral subvarieties of dimension i.
In particular $Z_0(X)$ is generated by closed points of X.

Definition 2.3. *If Y is a (possibly non-integral) closed subscheme of X, we set*

$$[Y] = \sum_{i=1}^{n} l(\mathcal{O}_{Y,y_i}) \cdot [Y_i]$$

where the Y_i's are the irreducible components of Y and where $l(\cdot)$ is the length for generic points $y_1, ..., y_n$.

Definition 2.4. *An irreducible subvariety of $X \times \mathbb{P}^1$ is "not in a fiber" if there is no $t \in \mathbb{P}^1$ such that $Y \subset X \times \{t\}$.*

Definition 2.5. *Two closed subvarieties Y and Z of X are rationally equivalent if there is a $V \in X \times \mathbb{P}^1$ not in a fiber, such that $V \cap (X \times \{0\}) = Y$ and $V \cap (X \times \{\infty\}) = Z$.*

This is an equivalence relation.

Definition 2.6 (Chow groups).

$$\mathrm{CH}_i(X) = Z_i(X)/ < [Y] - [Z],\ Y \textit{ and } Z \textit{ rationally equivalent} >$$

and $\mathrm{CH}(X) = \bigoplus \mathrm{CH}_i(X)$*.*

Example 2.7. X irreductible $\Rightarrow \mathrm{CH}_{\dim(X)} = \mathbb{Z} \cdot [X]$.

Notation 2.8. If X is equidimensional (i.e. its irreducible components have the same dimension), set $\mathrm{CH}^i(X) = \mathrm{CH}_{\dim(X)-i}(X)$.

2.1. Functorial properties.

2.1.1. *Push forward.*

Definition 2.9 ((proper) push-forward). *Let $f : X \longrightarrow Y$ be a proper morphism. f induces a graded morphism $f_* : \mathrm{CH}_*(X) \longrightarrow \mathrm{CH}_*(Y)$ given by*

$$\begin{array}{cccc} f_* : & Z_*(X) & \longrightarrow & Z_*(Y) \\ & [Z] & \longmapsto & \left\{ \begin{array}{c} [F(Z) : F(f(Z))].[f(Z)] \textit{ if } \dim(f(Z)) = \dim(Z) \\ 0 \textit{ otherwise} \end{array} \right. \end{array}$$

Example 2.10. If X is projective, the structure morphism $f : X \longrightarrow \mathrm{Spec}(F)$ is proper. The pushforward of f is the degree, denoted by

$$\deg : \mathrm{CH}(X) \longrightarrow \mathrm{CH}(\mathrm{Spec}(F)) = \mathbb{Z}[pt] \simeq \mathbb{Z}.$$

It is trivial for $\mathrm{CH}_i(X)$, $i \geq 1$, and brings $[x] \in \mathrm{CH}_0(X)$ to the degree of its residue field.

Property 2.11. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper morphisms, then $(g \circ f)_* = g_* \circ f_*$.*

We thus have a functor

$$\begin{array}{ccc} Var(F), proper & \longrightarrow & \textit{graded abelian groups} \\ X & \longmapsto & \mathrm{CH}(X) \end{array}$$

2.1.2. *Pullbacks.*

Definition 2.12. *Let $f : X \longrightarrow Y$ be a flat morphism, of relative dimension n (fibers are equidimensional, of dim n). Then f induces a pullback via*

$$\begin{array}{cccc} f^* : & \mathrm{CH}_*(Y) & \longrightarrow & \mathrm{CH}_{*+n}(X) \\ & [Z] & \longmapsto & [f^{-1}(Z)] \end{array}$$

Example 2.13. If $j : U \longrightarrow X$ is an open immersion, then $j^*([Y]) = [Y \cap U]$.

Property 2.14. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper morphisms, then $(g \circ f)^* = f^* \circ g^*$.*

2.2. Further properties.

2.2.1. *External product.*

Definition 2.15 (External product). *If X, Y are varieties over F, the external product is given by*

$$\begin{array}{rccc} \times: & \mathrm{CH}(X)\otimes\mathrm{CH}(Y) & \longrightarrow & \mathrm{CH}(X\times Y) \\ & [Z]\otimes[T] & \longmapsto & [Z\times T] \end{array}$$

2.2.2. *Localization sequence.*

X an F-variety, Y a closed subvariety and $U = X \setminus Y$. Denote by $Y \xrightarrow{i} X$ and $U \xrightarrow{j} X$ the immersions. We have an exact sequence

$$\mathrm{CH}_k(Y) \xrightarrow{i_*} \mathrm{CH}_k(X) \xrightarrow{j^*} \mathrm{CH}_k(U) \longrightarrow 0$$

2.2.3. *Homotopy invariance and Projective bundle theorem.*

Theorem 2.16. *Let $f : X \longrightarrow Y$ be an affine bundle of rank n (i.e. a flat morphism s.t. for all $y \in Y$, the fiber X_y is isomorphic to $\mathbb{A}^n_{F(y)}$). Then the pull-back*

$$f^* : \mathrm{CH}_*(Y) \longrightarrow \mathrm{CH}_{*+n}(X)$$

is an isomorphism.

(Proof relies on localisation sequence)

Theorem 2.17. *Let $f : X \longrightarrow Y$ be a projective bundle of rank n (for all $y \in Y$, the fiber X_y is isomorphic to $\mathbb{P}^n_{F(y)}$). Then for any k there is an isomorphism*

$$\mathrm{CH}_k(X) \simeq \bigoplus_{i=0}^{n} \mathrm{CH}_{k-i}(Y)$$

2.3. Internal product.

Now varieties are *smooth projective.*

Let $f : X \longrightarrow Y$ be a morphism of varieties in F. If X and Y are smooth, there is a pullback

$$f^* : \mathrm{CH}^*(Y) \longrightarrow \mathrm{CH}^*(X)$$

Let $\Delta_X : X \longrightarrow X \times X$ be the diagonal embedding. The internal product is defined as

$$\begin{array}{rccc} \cdot: & \mathrm{CH}(X)\times\mathrm{CH}(X) & \longrightarrow & \mathrm{CH}(X) \\ & (\alpha,\beta) & \longmapsto & \Delta_X^*(\alpha\times\beta) \end{array}$$

Remark 2.18. With this $\mathrm{CH}^*(X)$ is a graded ring with identity, and we have $[Y]\cdot[Z] = [Y \cap Z]$.

3. Grothendieck-Chow motives

We want to construct a category $\mathrm{Mot}_F(\Lambda)$ of Chow motives with coefficients in a commutative ring Λ. First, we denote

$$\mathrm{CH}(X;\Lambda) := \mathrm{CH}(X) \otimes_{\mathbb{Z}} \Lambda.$$

The ring Λ is the ring of coefficients. All previous properties are satisfied in $\mathrm{CH}(X;\Lambda)$.

3.0.1. *Morphisms.*

Definition 3.1. *A correspondence of degree k between two F-varieties X (irreducible) and Y (with coefficients in Λ) is an element of $\mathrm{CH}_{\dim(X)+k}(X \times Y; \Lambda)$. These are denoted by $\mathrm{Corr}_k(X, Y; \Lambda)$.*

This is extended to the non-reduced case by taking the direct sum over irreducible components.

Notation 3.2. We will write $X \rightsquigarrow Y$ for a correspondence between X and Y. Set $\mathrm{Corr}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Corr}_i(X, Y; \Lambda)$.

Example 3.3. A morphism of varieties $f : X \longrightarrow Y$ gives rise to a correspondence of degree 0 $[\Gamma_f] \in \mathrm{CH}_{\dim(X)}(X \times Y; \Lambda)$, through its graph.

Define the composition of correspondences

$$\begin{array}{rccc} \circ : & \mathrm{Corr}_i(X, Y; \Lambda) \times \mathrm{Corr}_j(Y, Z; \Lambda) & \longrightarrow & \mathrm{Corr}_{i+j}(X, Z; \Lambda) \\ & (\alpha, \beta) & \longmapsto & \beta \circ \alpha \end{array}$$

by the formula

$$\beta \circ \alpha = (\pi_3)_*(\pi_1^*(\alpha) \cdot \pi_2^*(\beta))$$

$$\begin{tikzcd} & \mathrm{CH}(X \times Y \times Z; \Lambda) \arrow[dl, "\pi_1"'] \arrow[d, "\pi_3"] \arrow[dr, "\pi_2"] & \\ \mathrm{CH}(X \times Y; \Lambda) & \mathrm{CH}(X \times Z; \Lambda) & \mathrm{CH}(Y \times Z; \Lambda) \end{tikzcd}$$

Example 3.4. This composition behaves well with composition of morphisms of varieties: if $X \xrightarrow{f} Y \xrightarrow{g} Z$, then

$$[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{g \circ f}].$$

Note that $[\Delta_X] := [\Gamma_{id_X}]$.

Remark 3.5. If $\alpha : X \rightsquigarrow Y$ and $\beta : Y \rightsquigarrow Z$ are correspondences of degrees k and l, then $\beta \circ \alpha$ is of degree $k + l$.

We now define the category $\mathrm{C}_F(;\Lambda)$ of (graded) correspondence with coefficients in Λ as follows :

- Objects : $X[i]$, where X is a (smooth) F-variety, $i \in \mathbb{Z}$;
- Morphisms : $\mathrm{Hom}(X[i], Y[j]) := \mathrm{Corr}_{i-j}(X, Y; \Lambda)$

3.0.2. *The category of motives.*

We want to make computations, to deal with decompositions. The category of correspondences is preadditive.

The category $\mathrm{Mot}_F(\Lambda)$ of (Grothendieck Chow) motives with coefficients in Λ is the pseudo-abelian enveloppe of the additive completion of $\mathrm{C}_F(\Lambda)$. Concretely, $\mathrm{Mot}_F(\Lambda)$ is described as follows :

- Objects: finite direct sums of $(X, \pi)[i]$ where X is a smooth projective variety over F, π is a projector in $End_{\mathrm{C}_F(\Lambda)}(X)$ and $i \in \mathbb{Z}$.

- Morphisms:

  $$\mathrm{Hom}_{\mathrm{Mot}_F(\Lambda)}((X,\pi_1)[i],(Y,\pi_2)[j]) := \pi_2 \circ \mathrm{Hom}_{\mathrm{C}_F(\Lambda)}(X[i],Y[j]) \circ \pi_1$$

  (matrices of such for arbitrary direct sums)

We thus get a functor

$$\begin{array}{ccc} \text{Smooth-proj varieties } /F & \longrightarrow & \mathrm{Mot}_F(\Lambda) \\ X & \longmapsto & M(X) := (X,\Delta_X)[0] \\ X \xrightarrow{f} Y & \longmapsto & [\Gamma_f] \end{array}$$

Example 3.6.

- Symmetric monoidal through

  $$(X,\pi)[i] \otimes (Y,\pi')[j] := (X \times Y, \pi \times \pi')[i+j].$$

- Denote $\Lambda[i] := (Spec(F), \Delta_{Spec(F)})[i]$, for all $i \in \mathbb{Z}$. These are the Tate motives.

- Let X be an irreducible variety over F having a rational point x. The cycle

  $$[x] \times [X] \in \mathrm{Corr}_0(X,X;\Lambda)$$

  is a projector and we have mutual isomorphisms

  $$(X,[x]\times[X]) \xrightarrow{[x]\times[pt]} \Lambda[\dim(X)] \xrightarrow{[pt]\times[X]} (X,[x]\times[X]).$$

  In particular, for instance, $\Lambda[i] = (\mathbb{P}^i, [x] \times \mathbb{P}^i))$.

- This gives a geometric interpretation of the direct sum :

  $$(X,\pi)[i] \oplus (Y,\pi')[i] = (X \sqcup Y, \pi + \pi')[i]$$

  and for different shifts $i < j$,

  $$(X,\pi)[i] \oplus (Y,\pi')[j] = (X,\pi)[i] \oplus [(Y,\pi')[i] \otimes \Lambda[j-i]] = (X,\pi)[i] \oplus (Y \times \mathbb{P}^{j-i}, ?)[i]$$

  and we're back to same shift.

3.1. First examples. Note that for any field extension E/F, we have an additive restriction functor

$$\begin{array}{cccc} res_{E/F}: & \mathrm{Mot}_F(\Lambda) & \longrightarrow & \mathrm{Mot}_E(\Lambda) \\ & M = (X,\pi)[i] & \longmapsto & M_E = (X_E,\pi_E)[i] \end{array}$$

Definition 3.7. *A motive is* split *it is isomorphic to a direct sum of Tate motives. A motive M is geometrically split if $M_{F_{sep}}$ is split, for some separable closure F_{sep}/F.*

Theorem 3.8 (Motives of projective bundles). *Let $f: X \longrightarrow Y$ be a projective bundle of rank n. Then*

$$M(X) \simeq M(Y)[0] \oplus M(Y)[1] \oplus ... \oplus M(Y)[n].$$

Proof. For any smooth projective variety Z, the morphism

$$f_Z : X \times Z \longrightarrow Y \times Z$$

is a projective bundle of rank n. By Theorem 2.17 we have isomorphism

$$\mathrm{CH}_k(X \times Z) \simeq \bigoplus_{i=0}^{n} \mathrm{CH}_{k-i}(Y \times Z).$$

This isomorphism is compatible with morphisms in $\mathrm{Mot}_F(\Lambda)$, we thus have for any motive N natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Mot}_F(\Lambda)}(M(X), N) \simeq \mathrm{Hom}_{\mathrm{Mot}_F(\Lambda)}\left(\bigoplus_{i=0}^{n} M(Y)[i], N\right)$$

and $M(X) \simeq M(Y)[0] \oplus M(Y)[1] \oplus ... \oplus M(Y)[n]$ by Yoneda lemma. □

Example 3.9. $M(\mathbb{P}^n) = \Lambda[0] \oplus \Lambda[1] \oplus ... \oplus \Lambda[n]$.

DEFINITION 3.10. *A smooth variety X is cellular if there is a filtration given by closed subvarieties*

$$X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset ... \supset X^{(n)} \supset X^{(n+1)} = \emptyset$$

with for any $i = 0, ..., n$ an affine bundle $X^{(i)} \setminus X^{(i+1)} \longrightarrow Y_i$.

THEOREM 3.11 (Karpenko). *Let X be a cellular variety with filtration*

$$X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset ... \supset X^{(n)} \supset X^{(n+1)} = \emptyset$$

and affine bundles $X^{(i)} \setminus X^{(i+1)} \longrightarrow Y_i$, of relative dimension d_i.

Then $M(X) \simeq \bigoplus_{i=0}^{n} M(Y_i)[d_i]$.

Proof. Localization sequence + homotopy invariance. □

Example 3.12 (Motive of $\mathbb{P}^n$). Projective spaces are cellular through the filtration

$$X = \mathbb{P}^n_F \supset \mathbb{P}^{n-1}_F \supset \mathbb{P}^{n-2}_F \supset ... \supset \emptyset$$

and trivial affine bundles $\mathbb{P}^i_F \setminus \mathbb{P}^{i-1}_F \simeq \mathbb{A}^i_F$. We thus get again

$$M(\mathbb{P}^n_F) = \bigoplus_{k=0}^{n} \Lambda[k].$$

The same can be constructed for Grassmann varieties, with the usual Young tableaux stuff. In the next lecture we will see that such cellular structure hold for all flag varieties.

3.1.1. *Motives of split quadrics.* Let q be a split quadratic form (i.e. its Witt index $i_0(q)$ is maximal) (for instance, F_{sep}). Let Q be its projective quadric. Then

- if $\dim(q)$ is odd, then
$$M(Q) \simeq \Lambda[0] \oplus \Lambda[1] \oplus ... \oplus \Lambda[\dim(Q)]$$
- if $\dim(q)$ is even, then
$$M(Q) \simeq \Lambda[0] \oplus \Lambda[1] \oplus ... \oplus \Lambda[\frac{\dim(Q)}{2}]^{\oplus 2} \oplus ... \oplus \Lambda[\dim(Q)]$$

In particular, the functor bringing a (smooth projective) variety to its motive is not conservative.

4. Motivic equivalence of semisimple algebraic groups

Recall that F_{sep}/F is a separable closure.

Definition 4.1. *Let G be a split semisimple group. A flag variety is a G-variety isomorphic to to a quotient G/P by a parabolic subgroup.*
A twisted flag variety is a G-variety X such that $X_{F_{sep}}$ is a flag $G_{F_{sep}}$-variety.

These varieties are smooth, projective.

Example 4.2. Our main examples:

- Flag varieties.
  - $G = PGL_n, SL_n$ (Type A_n): varieties of flags of subspaces (in particular projective spaces).
  - $G = SO(q), Spin(q)$ (B_n, D_n): varieties of flags of totally isotropic subspaces for q (in particular projective quadrics).
- Twisted flag varieties.
  - $G = PGL(A), SL(A)$ (A central simple algebra): flags of ideals in A (in particular Severi-Brauer varieties).
  - B_n, D_n: varieties of flags of isotropic ideals in algebras with involutions (in particular projective quadrics).

For simplicity, we now assume that all semisimple groups are of inner type, i.e. that $Gal(F_{sep}/F)$ acts trivially on their Dynkin diagrams.

Theorem 4.3 (Borel-Tits classification). *Let G be a semisimple group. There is a bijection*

$$\begin{array}{ccc} \{\textit{Subsets of } \Delta(G)\} & \xrightarrow{\sim} & \{\textit{Isom. classes of twisted flag } G\textit{-varieties}\} \\ \Theta & \longmapsto & X_{\Theta,G} \end{array}$$

Definition 4.4. *Let G and G' be two semisimple groups of the same type. We say that G and G' are motivic equivalent with coeff in Λ if there is an isomorphism of diagrams*

$$\varphi : \Delta(G) \xrightarrow{\sim} \Delta(G')$$

such that for any $\Theta \subset \Delta(G)$, $M(X_{\Theta,G}) \simeq M(X_{\varphi(\Theta),G'})$ in $\mathrm{Mot}_F(\Lambda)$.

Example 4.5. For $G = PGL_1(A)$, $G' = PGL_1(B)$, $\varphi = id$, this means that for any sequence of integers $i_1 < ... < i_k$, the motives of the flag varieties of ideals $M(SB(i_1, .., i_k; A)$ and $M(SB(i_1, ..., i_k; B)$ are isomorphic.

5. Tits p-indexes

Let G be a semisimple group and p be a prime. We color the Dynkin diagram of G as follows:

- A vertex $i \in \Delta(G)$ is colored if the variety $X_{\{i\},G}$ has a zero-cycle of degree coprime to p.

The data of this colored Dynkin diagram is the Tits p-index of G, denoted by $Tits_p(G)$ (recall that we assumed that G is of inner type).

THEOREM 5.1. *G and G' are motivic equivalent mod p ($\Lambda = \mathbb{F}_p$) iff for any E/F, the Tits p-indexes of G_E and G'_E are isomorphic.*

6. Motives of (twisted) flag varieties

In this section, the ring of coefficients is still arbitrary, but soon its choice will raise some subtleties.

THEOREM 6.1 (Köck, 1991). *The motive of a flag variety $X_{\Theta,G}$ is split and determined by Θ and the root system of G.*
In particular, twisted flag varieties are geometrically split.

Proof. They have a cellular filtration, by Bruhat decomposition. □

Example 6.2. Projective space and split quadrics, as saw during last lecture.

Now that F is arbitrary, the choose of the ring of coefficients becomes important.

- Coefficients in $\mathbb{Q}$ are useless in this context: we get the same classification as when $F = F_{sep}$;

- Coefficients in $\mathbb{Z}$: they are the finest, but we will see they don't behave quite well. We keep these for the moment.

6.0.1. *Motives of quadrics.*

Let q be a nonsingular quadratic form. Denote by $i_W(q)$ the dimension of a maximal totally isotropic subspace. By Witt classification, we can write

$$q \simeq i_W(q) \cdot \mathbb{H} \perp q_{an}$$

where $\mathbb{H}$ is the hyperbolic plane and q_{an} is anisotropic.

THEOREM 6.3 (Rost). *Let q be a quadratic form over F. Then we have*

$$M(Q) \simeq \mathbb{Z}[0] \oplus ... \oplus \mathbb{Z}[i_W(q)-1] \oplus M(Q_{an}) \oplus \mathbb{Z}[\dim(Q)-i_W(q)+1] \oplus ... \oplus \mathbb{Z}[\dim(X)]$$

Proof. Obtained by a right cellular structure. □

After first breakthroughs by Rost, the throughout study of motives of quadrics was carried on by Vishik. Among many others, he provided the following.

THEOREM 6.4 (Vishik). *Let q, q' be two quadratic forms over F. The motives $M(Q)$ and $M(Q')$ are isomorphic if and only if for any field extension E/F, $i_W(q_E) = i_W(q'_E)$.*

Motives of quadrics are thus determined by the "splitting behaviour" of the underlying quadratic forms. We will extend this result in next lectures in several directions.

This condition of "having the same Witt index over any extension" is interesting by itself in the theory of quadratic forms, discarding motives. For forms of odd dimensions (Izhboldin) or on particular fields such as local fields or global fields (Hoffmann) it is equivalent to the forms being isomorphic.

6.0.2. *Severi-Brauer varieties.*

Let A be a central simple algebra over F and $SB(A)$ its Severi Brauer variety. In general, $A \simeq M_n(D)$ (A and D are Brauer-equivalent) for some division algebra D. We have

$$M(SB(A)) \simeq \bigoplus_{k=0}^{n-1} M(SB(D))[k\sqrt{\dim(D)}]$$

More generally, the motive of any twisted flag variety for $PGL(A)$ is isormophic to a direct sum of twists of motives of twisted flag varieties for $PGL(D)$.

Theorem 6.5 (Karpenko). *Let A and B two central simple algebras of the same dimension. The motives of $SB(A)$ and $SB(B)$ are isomorphic (with $\mathbb{Z}$ coefficients) iff $A \simeq B$ or $A \simeq B^{op}$.*

This result does not have the same flavor as Vishik's classification for quadrics: the natural way to define "isotropy" for central simple algebras (the Schur index) does not characterise these motives. Even worse : the motive of a Severi-Brauer variety is then not even a birational invariant.

7. Rost Nilpotence principle

The next property is fundamental.

Definition 7.1. *Let X be a smooth projective variety over F. We say that X satisfies Rost Nilpotence Principle (with coeff in Λ) if the following holds.*
Let $f \in End_{\mathrm{Mot}_F(\Lambda)}(M(X))$ be an endomorphism of $M(X)$, such that $f_E = 0$, for some field extension E/F. Then, f is nilpotent.

This property was first proved for quadrics by Rost.

Remark 7.2. Some direct consequences of Rost Nilpotence Principle:

- Let X and Y be smooth projective varieties satisfying Rost Nilpotence principle (with coefficients in Λ) and $f : M(X) \longrightarrow M(Y)$ be a morphism in $\mathrm{Mot}_F(\Lambda)$. If there is an extension E/F such that f_E is an isomorphism, then f is an isomorphism.
- if $f \in End(M(X))$ is such that f_E is idempotent for some E/F, then there is an idempotent $p \in End(M(X))$ such that $f_E = p_E$.

Theorem 7.3 (Chernousov, Gille, Merkurjev).
Twisted flag varieties satisfy Rost Nilpotence Principle.

8. Motivic decompositions

Definition 8.1. *A pseudo-abelian category C satisfies the Krull-Schmidt principle (KSP) if objects of C have a "unique" decomposition, as direct sum of indecomposable object (up to permutation and isomorphisms of the summands).*

Remark 8.2. As C is pseudo-abelian, $X \in Ob(C)$ is idemcomposable iff the ring $End(X)$ is connected (its only idempotents/projectors are 0 and 1).

8.0.1. *KSP with $\mathbb{Z}$-coefficients.*

We begin with two positive examples for the Krull-Schmidt property.

THEOREM 8.3 (Vishik). *Motives of quadrics in $\mathrm{Mot}_F(\mathbb{Z})$ satisfy KSP.*

THEOREM 8.4 (Karpenko). *Let D be a division algebra. The the motive of its Severi-Brauer variety $M(SB(D))$ is indecomposable.*

Remark 8.5. The previous decompositions with $A \simeq M_n(D)$

$$M(SB(A)) \simeq \bigoplus_{k=0}^{n-1} M(SB(D))[k\sqrt{\dim(D)}]$$

is thus the "complete" decomposition of $M(SB(A))$.

Yet, KSP fails with $\mathbb{Z}$-coefficients.

Example 8.6 (Chernousov-Merkurjev). Let A and B two division algebras of such that:

(1) $< A >=< B >$ in the Brauer group of F;

(2) B is not isomorphic to A or A^{op}.

Consider the two projections

$$\begin{tikzcd} & SB(A) \times SB(B) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ SB(A) & & SB(B) \end{tikzcd} .$$

By property 1), π_1 is a projective bundle, hence by Theorem 3.8

$$M(SB(A)\times SB(B)) \simeq M(SB(A))\oplus M(SB(A))[1]\oplus ...\oplus M(SB(A))[\dim(SB(B))]$$

The same reasoning with π_2 gives

$$M(SB(A)\times SB(B)) \simeq M(SB(B))\oplus M(SB(B))[1]\oplus ...\oplus M(SB(B))[\dim(SB(A))]$$

Since A and B are division, these are complete non-isomorphic motivic decompositions. Some other examples in the absolutely simple case are provided by Calmès-Petrov-Semenov-Zainoulline.

9. KRULL-SCHMIDT PROPERTY WITH $\mathbb{F}_p$ COEFFICIENTS

We now fix coefficients in $\Lambda = \mathbb{F}_p$.

DEFINITION 9.1. *A Regensburg variety is a variety X such that:*

(1) *X is geometrically split, geometrically irreducible;*
(2) *X satisfies Rost Nilpotence Principle.*

Theorem 9.2. *The full subcategory of* $\mathrm{Mot}_F(\mathbb{F}_p)$ *generated by motives of Regensburg varieties is Krull-Schmidt.*

9.1. Existence.

Lemma 9.3. *Assume that* X *satisfies Rost Nilpotence Principle and let* π, π' *be two projectors in* $End(M(X)$. *If* $\pi_E = \pi'_E$ *for some* E/F, *then* (X,π) *and* (X,π') *are isomorphic.*

Proof. $(X,\pi) \xrightarrow{\pi'\circ\pi} (X,\pi')$ is an isomorphism over E, hence over F by Rost Nilpotence Principle. □

Consequence 9.4. *Set*

$$\overline{End}(M(X)) := Im\left(res_{F_{sep}/F} : End(M(X)) \longrightarrow End(M(X_{F_{sep}})\right).$$

Motivic decompositions of $M(X)$ *are determined by decompositions of the diagonal as a sum of orthogonal projectors in* $\overline{End}(M(X))$.

If X *is Regensburg,* $\overline{End}(M(X))$ *is finite. We can thus write a finite number of projectors* $\pi_1, ..., \pi_k$ *such that the* (X,π_i) *are indecomposable and such that*

$$M(X) \simeq (X,\pi_1) \oplus ... \oplus (X,\pi_k).$$

Motivic decompositions do exist.

9.2. Unicity.

Theorem 9.5 (Bass). *Let* C *be a pseudo-abelian category and* $X = X_1 \oplus ... \oplus X_n$ *a complete decomposition of an object of* C. *If the* $End(X_i)$ *are local, for any* $i = 1, ..., n$, *then this decomposition is "unique".*

Proof of Unicity. Let $M = (X,\pi)[i]$ be an indecomposable summand of $M(X)$. We have to show that $End(M)$ is local.

Let $f \in End(M)$ be a non-invertible element. Denote by $\bar{f}$ its image in

$$\overline{End}(M) = \bar{\pi} \circ \overline{End}(M(X)) \circ \bar{\pi}.$$

As $\overline{End}(M)$ is a finite ring, an appropriate power of $\bar{f}$ is idempotent. Now $End(M)$ is connected so this power is trivial and $\bar{f} = 0$. In particular by Rost Nilpotence, f is nilpotent. Any non-invertible element in $End(M)$ is then nilpotent: the ring is local. □

9.3. Upper motives.

Let X be a Regensburg variety and let

$$M(X) \simeq \bigoplus_{i=1}^{n} M_i$$

be a complete decomposition of its $\mathbb{F}_p$-motive.

Problem: carry on the qualitative study of the M_i's.

Note that we have

$$\mathrm{CH}^0(X) \simeq \mathrm{CH}^0(M_1) \oplus ... \oplus \mathrm{CH}^0(M_n).$$

$(\mathrm{CH}^i(M) := \mathrm{Hom}(M, \Lambda[i]))$
In particular, there is a unique $i \in \{1, ..., n\}$ such that $\overline{\mathrm{CH}}^0(M_i) \neq 0$.

Definition 9.6. *Assume X is Regensburg. A direct summand M of $M(X)$ is upper, if $\overline{\mathrm{CH}}^0(M) \neq 0$. The "unique" upper indecomposable direct summand of X is the upper motive of X (with coefficients in $\mathbb{F}_p$).*

Remark 9.7. Equivalent definition: M is upper in $M(X)$ if over a separable closure, it contains a direct summand isomorphic to $\mathbb{F}_p[0]$.

Notation 9.8. The upper motive of X will be denoted $U(X)$.

Example 9.9 (Karpenko). Let D be a division algebra of p-primary dimension. Then $M(SB(D))$ is indecomposable, i.e. $U(SB(D)) = M(SB(D))$.

In general, write $A = M_r(D)$. Then in $\mathrm{Mot}_F(\mathbb{F}_p)$, we have

$$M(SB(A)) = \bigoplus_{k=0}^{?} M(SB(D_p))[k \cdot \sqrt{\dim(D_p)}]$$

where D_p is the p-primary component of D. This is the complete decomposition of $M(SB(A))$.

Karpenko's structure theorem generalizes the previous example to all twisted flag varieties.

Theorem 9.10. *If X is a twisted flag variety, then any indecomposable summand of $M(X)$ is isomorphic to a Tate twist of the upper motive of a twisted flag variety.*

(the varieties appearing are projective homogeneous for the same group)

Example 9.11. Let $A \simeq M_r(D)$ be a central simple algebra and D_p the p-primary component of D. Write

$$U(SB(p^l; D_p)) = U_{l,D_p}.$$

Then for any flag $PGL_1(A)$-variety $SB(i_1, ..., i_k; A)$, we have

$$M(SB(i_1, ..., i_k; A)) \simeq \bigoplus_{s=0}^{?} U_{l_s,D_p}[n_s]$$

for some nonnegative integers $n_s \geq 0$ and l_s such that $l_s \leq v_p(\gcd(i_1, ..., i_k))$.

9.4. Classification of upper motives.

Definition 9.12 (Multiplicity of correspondences). *Let*

$$\alpha : X \rightsquigarrow Y$$

be a correspondence of degree 0, with X irreducible. Denote by $\pi_1 : X \times Y \longrightarrow X$ the first projection. The multiplicity $mult(\alpha)$ of α is the integer such that $(\pi_1)_*(\alpha) = mult(\alpha) \cdot [X]$.

Remark 9.13.

- We have
$$mult(\beta \circ \alpha) = mult(\beta) \cdot mult(\alpha),$$
in particular multiplicity of a projector is an idempotent.

- A summand (X, π) is upper in $M(X)$ if and only if $mult(\pi) = 1$.

The next Proposition relates upper motives and the birational geometry of twisted flag varieties.

PROPOSITION 9.14. *Let X and Y be two Regensburg varieties. The upper motives of X and Y are isomophic if and only if there are two multiplicity 1 correspondences $\alpha : X \rightsquigarrow Y$ and $\beta : Y \rightsquigarrow X$.*

9.5. PROOF OF THE CRITERION. We give details for groups of inner type A_n, whose proof is simpler, and with $\varphi = id$.

Let A and B two central simple algebras, of the same dimension. By definition, $PGL(A)$ and $PGL(B)$ are motivic equivalent mod p if and only if for any sequence of integers $i_1 < ... < i_k$, the motives of $SB(i_1, ..., i_k; A)$ and $SB(i_1, ..., i_k; A)$ are isomorphic in $\mathrm{Mot}_F(\mathbb{F}_p)$.

9.5.1. *Step 1: upper equivalence.*
In the same way as motivic equivalence, we say that two semisimple groups G and G' are *upper equivalent mod p* if for any subset Θ, the *upper motives* of $X_{\Theta,G}$ and $X_{\varphi(\Theta),G'}$ are isomorphic.

Clearly, motivic equivalent implies upper equivalent, by the Krull-Schmidt property.

THEOREM 9.15. *G and G' are upper equivalent (mod p) iff for any field extension E/F, the Tits p-indexes of G_E and G'_E are isomorphic.*

The proof relies on Kersten and Rehmann's generic splitting fields for reductive groups.

9.5.2. *Step 2: upper equivalence implies motivic equivalence.*

Assume that $U_{l,A_p} \simeq U_{l,B_p}$, for any l.

We prove that $\mathrm{PGL}_1(A)$ and $\mathrm{PGL}_1(B)$ are motivic equivalent by induction on $n = v_p(ind(A))$ $(= v_p(ind(B)))$.

$(n = 0)$ nothing to prove, contained in the split case.

$(n > 0)$ Assume that $X = M(SB(i_1, ..., i_k; A) \not\simeq M(SB(i_1, ..., i_k; B) = Y$. Write

$$M(X) \simeq \bigoplus_j U_{l_j,A_p}[n_j]$$

and

$$M(Y) \simeq \bigoplus_s U_{l_s,B_p}[n_s].$$

Notation 9.16.
We denote by $n_{X,l,i}$ the number of $U_{l,A_p}[i]$ in the motivic decomposition of X.

We denote by $n_{Y,l,i}$ the number of $U_{l,B_p}[i]$ in the motivic decomposition of Y.

As $PGL_1(A)$ and $PGL_1(B)$ are upper equivalent, we can set

$$i_0 = \min\{i \in \mathbb{N}, n_{X,l,i} \neq n_{Y,l,i}, \text{ for some } l\}.$$

Now extend the scalars to the compositum field E/F of function fields $F(SB(p^{n-1}; A))$ and $F(SB(p^{n-1}; B))$. We have :

- $v_p(ind(A_E)) = v_p(ind(B_E) = n - 1$, so $PGL_1(A_E)$ and $PGL_1(B_E)$ are motivic equivalent mod p, by induction;

- $(U_{l,A_p})_E = U_{l,A_{E,p}}[0] \oplus ...$ and $(U_{l,B_p})_E = U_{l,B_{E,p}}[0] \oplus ...$, where "..." denotes a direct sum of upper motives shifted by a *positive* integer.

We thus get

$$n_{X_E,l,i_0} = n_{X,l,i_0} + N \text{ and } n_{Y_E,l,i_0} = n_{Y,l,i_0} + N'$$

for some integers N and N', which correspond to the contribution given by Tate twists of upper motives U_{l,A_p} and U_{l,B_p}, shifted by less than i_0. In particular byy definition of i_0, N and N' are equal and $n_{X_E,l,i_0} \neq n_{Y_E,l,i_0}$. Contradiction.

Corollary 9.17 (Inner type A_n). *The following conditions are equivalent:*
1. *$PGL(A)$ and $PGL(B)$ are motivic equivalent modulo p;*
2. *$< A_p >=< B_p >\subset Br(F)$;*
3. *$SB(A_p)$ and $SB(B_p)$ are stably birationnally equivalent.*
4. *For any E/F, $v_p(ind(A_E)) = v_p(ind(B_E))$.*

Corollary 9.18. *If q and q' be two quadratic forms of the same dimension, the following are equivalent:*
1. *$SO(q)$ and $SO(q')$ are motivic equivalent mod 2;*
2. *$M(Q) \simeq M(Q')$;*
3. *for any field extension, $i_W(q_E) = i_W(q'_E)$.*

9.6. General case. The generalized case follows the same strategy but is much more tedious, as one can not rely anymore on the clasification.

Step 1 becomes : G and G' are upper equivalent if and only if their Tits p-indices are isomorphic.

Step 2 relies on the same computation, with E being the composite of the function field $F(X)$ and $F(Y)$. One then use Chernousov-Gille-Merkurjev motivic decompositions to carry on the same algorithm, which allows to reconstruct $M(X)$ and $M(Y)$ from $M(X_E)$ and $M(Y_E)$.