Isotropic Motives

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Lecture 1

The difference in size and complexity between Topology and Algebraic Geometry can be visualized by the number of "points" in these worlds.

Topology	Algebraic Geometry
one point: $\bullet$	many points: $\mathrm{Spec}(E)$, where E/k-fin. gen. field ext.

Motives provide a "linearization" of varieties/topological spaces.

In Topology, homology and cohomology of a topological space X are defined using the singular chain complex $C_{\bullet}(X)$ of it. It is the *motive* of X. And the category of topological motives is the derived category of abelian groups $D(Ab)$ (a $\otimes$-Δ-ed category).

In Algebraic Geometry, the construction of a motive is more involved.

(classical) Chow motives

Chow groups

X - algebraic variety over k. The Chow group $\mathrm{CH}_r(X)$ of r-dimensional cycles modulo rational equivalence is the quotient-group of $\oplus_V \mathbb{Z} \cdot [V]$ (where V runs over all irreducible subvarieties of $\dim = r$ in X) modulo the subgroup generated by elements $\mathrm{div}(f)$, where f is some rational function on some $(r+1)$-dim. irreducible subvariety U of X and $\mathrm{div}(f)$ is the linear combination of classes of r-dimensional subvarieties of U describing zeroes/poles of f.

Can also use the co-dimensional notations: if X is equidimensional, then $\mathrm{CH}^r(X) = \mathrm{CH}_{\dim(X)-r}(X)$.

Example 1.1 *1) $X = \mathrm{Spec}(E)$, E/k-field ext. Then $\mathrm{CH}^0(X) = \mathbb{Z}$ and $\mathrm{CH}^i(X) = 0$, for any $i \neq 0$.*

2) X - an elliptic curve over k. Then $\mathrm{CH}^0(X) = \mathbb{Z}$, $\mathrm{CH}^1(X) = \mathbb{Z} \oplus X(k)$ - where $X(k)$ is the group of k-points of X, and $\mathrm{CH}^i(X) = 0$, for all $i \neq 0, 1$.

Chow groups - an analogue of singular homology/cohomology in topology.

Chow groups form an *oriented cohomology theory* on Sm/k. That means that for any map $f : X \to Y$ of smooth varieties there is the pull-back map $f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$, for any proper map as above there is the push-forward map $f_* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y)$, which satisfy various properties. In particular, on $\mathrm{CH}^*(X)$ there is a ring structure given by the intersection product $\alpha \cdot \beta = \Delta^*(\alpha \times \beta)$, where $\Delta : X \to X \times X$ is the diagonal embedding.

<u>Category of correspondences</u>

$Ob(Cor(k)) = \{[X] \,|X - \text{smooth proj.}/k\}$

$\mathrm{Hom}_{Cor(k)}([X], [Y]) = \mathrm{CH}_{\dim(X)}(X \times Y)$, if X is equidimensional , - these are *correspondences* from X to Y. The composition $\circ$ is defined as follows:

$\varphi \in \mathrm{CH}_{\dim(X)}(X \times Y)$, $\psi \in \mathrm{CH}_{\dim(Y)}(Y \times Z)$, then

$$\psi \circ \varphi = (\pi_{X,Z})_*((\pi_{X,Y})^*\varphi \cdot (\pi_{Y,Z})^*\psi),$$

where $\pi_{X,Y}$, $\pi_{Y,Z}$ and $\pi_{X,Z}$ are projections from $X \times Y \times Z$ to $X \times Y$, $Y \times Z$ and $X \times Z$, respectively. It is easy to see that this product is associative and also that if we compose this way graphs of (algebro-geometric) maps, it will coinside with the composition of maps themselves. Thus, we get the functor

$$Cor : Sm.Proj./k \to Cor(k)$$

given by: $X \mapsto [X]$ and $(f : X \to Y) \mapsto [\Gamma_F] \in \mathrm{CH}_{\dim(X)}(X \times Y)$ - the *graph* of f.

The advantage of $Cor(k)$ in comparison to $Sm.Proj./k$ - this new category is <u>additive</u>. It also has the natural $\otimes$-structure given by: $[X] \otimes [Y] = [X \times Y]$.

<u>Chow motives</u>

The category $Chow_{eff}(k)$ of *effective Chow motives* is the Pseudo-Abelian (Karobian) envelope of $Cor(k)$. That is,

- $Ob(Chow_{eff}(k))$ are pairs (X,ρ), where X is smooth projective and $\rho \in \mathrm{End}_{Cor(k)}([X])$ is a *projector*: $\rho \circ \rho = \rho$.
- $\mathrm{Hom}_{Chow_{eff}}((X,\rho),(Y,\eta)) = \eta \circ \mathrm{Hom}_{Cor(k)}([X],[Y]) \circ \rho \subset \mathrm{Hom}_{Cor(k)}([X],[Y])$.

Thus, we formally add kernels and cokernels of projectors.
Have the *motivic functor*:

$$M : Sm.Proj/k \to Chow_{eff}(k)$$

mapping a smooth projective variety X to the pair (X, id).

In the Chow motivic category motives of some varieties will split into smaller pieces - we get new objects.

$X = \mathbb{P}^1$; The identity map is given by the class $[\Delta] \in \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ of the diagonal. Due to the rational function $\dfrac{x_0y_1 - x_1y_0}{x_0y_0}$, this class is rationally equivalent to the sum $[\mathbb{P}^1 \times \bullet] + [\bullet \times \mathbb{P}^1]$ of mutually orthogonal projectors. Considering the maps $\bullet \xrightarrow{f} \mathbb{P}^1 \xrightarrow{g} \bullet$, we see that the first of these projectors gives a direct summand in $M(\mathbb{P}^1)$ isomorphic to $M(\bullet) =: T$ - the *trivial Tate-motive*. The complementary summand given by the projector $[\bullet \times \mathbb{P}^1]$ is denoted $T(1)[2]$ - the *Tate-motive*. Thus, $M(\mathbb{P}^1) = T \oplus T(1)[2]$.

The category $Chow(k)$ of *Chow motives* is obtained from $Chow_{eff}(k)$ by formally inverting $T(1)[2]$. At the same time, $Chow_{eff}(k)$ is a full subcategory of $Chow(k)$, so it doesn't matter in which category to compute Homs.

The category of Chow motives is $\otimes$-additive:

$$(X,\rho) \otimes (Y,\eta) := (X \times Y, \rho \times \eta).$$

Define: $T\{k\} := T(k)[2k] := T(1)[2]^{\otimes k}$.

If X is smooth projective, one has the identifications:

$$\begin{aligned} \mathrm{CH}_r(X) &= \mathrm{Hom}_{Chow(k)}(T\{r\}, M(X)) \quad \text{and} \\ \mathrm{CH}^r(X) &= \mathrm{Hom}_{Chow(k)}(M(X), T\{r\}). \end{aligned}$$

In particular, correspondences act on Chow groups via the composition.

If $N \in Chow(k)$, we can define:

$$\mathrm{CH}_r(N) = \mathrm{Hom}_{Chow(k)}(T\{r\}, N)$$

and similar for $\mathrm{CH}^r(N)$.

<u>Fact</u>: The map $\otimes T\{1\} : \mathrm{CH}_r(N) \to \mathrm{CH}_{r+1}(N\{1\})$ is an isomorphism.

Example 1.2 *1) $X = \mathbb{P}^m$. One can see: $\rho_i = [\mathbb{P}^{m-i} \times \mathbb{P}^i]$ are mutually orthogonal projectors on $\mathbb{P}^m$, s.t. $\sum_{i=0}^m \rho_i = id$ and $(\mathbb{P}^m, \rho_i) \cong T\{i\}$. Thus,*

$$M(\mathbb{P}^m) = \bigoplus_{i=0}^{m} T\{i\} \quad \textit{and} \quad \mathrm{CH}_i(\mathbb{P}^m) = \begin{cases} \mathbb{Z}, & \textit{if } 0 \leqslant i \leqslant m \\ 0, & \textit{otherwise} \end{cases}.$$

2) More generally, if $V \to X$ is an $(m+1)$-dimensional vector bundle, then

$$M(\mathbb{P}_X(V)) = \bigoplus_{i=0}^{m} M(X)\{i\}.$$

The Chow motivic category $Chow(k)$ contains motives of smooth projective varieties. This category can be embedded as a full $\otimes$-additive subcategory into the Voevodsky category of motives $DM(k)$. The details can be found in [5]. $DM(k)$ is a $\otimes - \Delta$-ed category. It contains motives of arbitrary varieties and even that of simplicial schemes. Applying the shift functor [1] to *pure Tate-motives* $T(i)[2i]$ (residing in Chow motivic category), we obtain Tate-motives $T(i)[j]$, for arbitrary i and j.

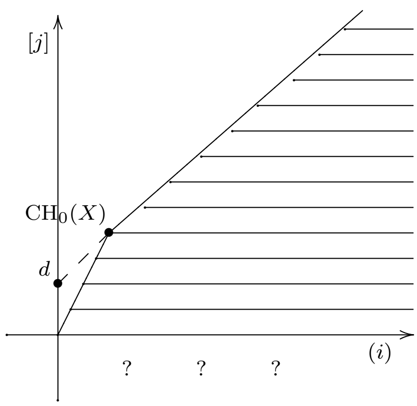
Have the motivic functor:

$$M : Sm/k \to DM(k)$$

extending Chow motivic functor for smooth projective varieties. Can define *motivic cohomology* of $X \in Sm/k$ as

$$\mathrm{H}^{j,i}_{\mathcal{M}}(X, \mathbb{Z}) := \mathrm{Hom}_{DM(k)}(M(X), T(i)[j]).$$

For a smooth variety X of dimension d, the motivic cohomology $\mathrm{H}^{*,*'}_{\mathcal{M}}(X, \mathbb{Z})$



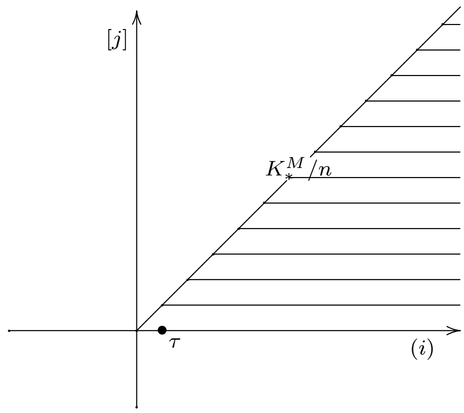
will disappear above the d-diagonal, as well as above the line with slope two. And the groups residing on the latter line are exactly Chow groups of X. For X projective one can see it from: $\mathrm{H}^{2i,i}_{\mathcal{M}}(X,\mathbb{Z}) = \mathrm{Hom}_{DM(k)}(M(X),\mathbb{Z}(i)[2i]) = \mathrm{Hom}_{Chow(k)}(M(X),\mathbb{Z}(i)[2i]) = \mathrm{CH}^i(X)$. In particular, the corner where the two pieces of the boundary meet is the group of 0-cycles on X.

Motivic cohomology is an analogue of singular cohomology in Topology. The only thing, now these groups are numbered by two numbers $(i)[j]$.

There is a version of Voevodsky motivic category with coefficients: $DM(k;R)$, where R is any commutative ring. Respectively,

$$\mathrm{H}^{j,i}_{\mathcal{M}}(X;R) := \mathrm{Hom}_{DM(k;R)}(M(X),T(i)[j]).$$

If $n \in \mathbb{N}$ and k contains a primitive n-th root ζ_n of 1, then $\mathrm{H}^{*,*'}_{\mathcal{M}}(\bullet,\mathbb{Z}/n)$ can be computed explicitly:



By the result of Voevodsky,

$$\mathrm{H}^{*,*'}_{\mathcal{M}}(\mathrm{Spec}(k),\mathbb{Z}/n) = K^M_*(k)/n[\tau],$$

where $\tau \in \mathrm{H}^{0,1}_{\mathcal{M}}(\mathrm{Spec}(k),\mathbb{Z}/n) = \{n\text{−th roots of} \, 1 \, \text{in} \, k\}$ is a choice of ζ_n. Here the Milnor's K-theory

$$K^M_*(k) = \mathrm{T}_{\mathbb{Z}}(k^\times)/\big(a\otimes(1-a),\, a\in k^\times\backslash\{1\}\big)$$

is the tensor algebra of the multiplicative group of k considered as a $\mathbb{Z}$-module, modulo specified quadratic relations (see [3]).

The algebro-geometric motivic category $DM(k)$ is much larger than the topological motivic category $D(Ab)$. In particular, the subcategory of compact objects in $D(Ab)$ consists of complexes with cohomology of finite (total) rank and Homs between such objects are abelian groups of finite rank. In contrast, the compact part $DM_{gm}(k)$ of Voevodsky category consists of objects which can be obtained from Chow motives by finitely many Cone operations. In particular, all Chow motives and Tate-motives are such. But motivic cohomology of varieties (in particular, their Chow groups and Milnor's K-theory of the ground field) can be arbitrarily large.

References

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