Isotropic Motives

Alexander Vishik

Lecture 3

The isotropic motivic category $DM(k/k;\mathbb{F}_p)$ is obtained from $DM(k;\mathbb{F}_p)$ by annihilating the motives of anisotropic varieties. It appears that, if k is flexible, it is sufficient to only annihilate varieties of a very special kind.

Let P, Q be smooth projective varieties over k, such that there is a map $f : P \to Q$. Then there are maps $P \xrightarrow{(id,f)} P \times Q \xrightarrow{pr_1} P$ with the identity composition. Thus, $M(P)$ is a direct summand of $M(P \times Q) = M(P) \otimes M(Q)$. Thus, if we annihilate $M(Q)$, we will automatically do $M(P)$.

Proposition 3.1 *If k is flexible, then any p-anisotropic variety P/k can be embeded into a p-anisotropic hypersurface of degree p.*

Proof: Embed P into a projective space. Passing to a Veronese embedding, we may assume that $P \hookrightarrow \mathbb{P}^N$ is defined by quadratic relations. Then it is also defined by relations of degree p (multiply the quadratic relations by all possible monomials of degree $p-2$). Let $|D| \cong \mathbb{P}^m$ be the linear system of all relations of degree p (= all hypersurfaces of degree p passing through P). Consider

$$Y = \{(x, H) | x \in H\} \subset (\mathbb{P}^N \backslash P) \times \mathbb{P}^m.$$

Again, the projection $Y \stackrel{\pi}{\to} (\mathbb{P}^N \backslash P)$ is a $\mathbb{P}^{m-1}$-bundle and so, by the Projective bundle theorem, $\mathrm{Ch}^*(Y) = \oplus_{i=0}^{m-1} \rho^i \cdot \mathrm{Ch}^*(\mathbb{P}^N \backslash P)$. Let $\widetilde{D_\eta}$ be the generic fiber of $Y \stackrel{pr_2}{\to} \mathbb{P}^m$. It is equal to $D_\eta \backslash P$, where D_η is the "generic representative" of the linear system $|D|$ = the "generic" hypersurface of degree p passing through P. It is defined over the function field $F = k(\mathbb{P}^m)$. As before, we have the maps:

$$\mathrm{Ch}^*((\mathbb{P}^N \backslash P)) \stackrel{\pi^*}{\to} \mathrm{Ch}^*(Y) \twoheadrightarrow \mathrm{Ch}(\widetilde{D}_\eta),$$

where the second map is surjective as are all pull-backs for open embeddings. Since ρ^i, for $i > 0$, is supported in positive co-dimension in $\mathbb{P}^m$, the restriction of these classes to the generic fiber is zero. Thus, the composition $\mathrm{Ch}^*((\mathbb{P}^N\backslash P)) \twoheadrightarrow \mathrm{Ch}^*(D_\eta\backslash P)$ is surjective. In particular, the map $\mathrm{Ch}_1((\mathbb{P}^N\backslash P)) \twoheadrightarrow \mathrm{Ch}_0(D_\eta\backslash P)$ sending a 1-cycle u to the zero-cycle $\pi^*(u)|_{D_\eta\backslash P}$ is surjective. But the degree (modulo p) of the zero-cycle $\pi^*(u)|_{D_\eta\backslash P}$ is the same as the degree of $u_F \cdot D_\eta$, since P is anisotropic. And the latter degree is zero (modulo p), since D_η is a hypersurface of degree p. Thus, $D_\eta\backslash P$ is p-anisotropic and hence, so is D_η (as P itself is anisotropic too). So, over a finitely-generated purely transcendental extension F of k, our anisotropic variety P can be embedded into a p-anisotropic hypersurface of degree p. Since k is flexible: $k = k_0(t_1, t_2, \ldots)$, there is an isomorphism F/k_0 with k/k_0 identifying P_F/F with P/k. Hence, P can be embedded into an anisotropic hypersurface of degree p already over k. $\square$

Corollary 3.2 *Over a flexible field, it is sufficient to annihilate only p-anisotropic hypersurfaces of degree p.*

For $p = 2$ can do even better.

Pfister quadrics

Recall that $K_1^M(k) \cong k^\times$. Denote as $\{a\} \in K_1^M(k)$ the element corresponding to $a \in k^\times$ and as $\{a_1, \ldots, a_n\} \in K_n^M(k)$ the product $\{a_1\} \cdot \ldots \cdot \{a_n\}$. Such elements are called *pure symbols*.

For $a \in k^\times$ denote as $\langle\langle a \rangle\rangle$ the 2-dimensional form $\langle 1, -a \rangle = x^2 - ay^2$. It is nothing else, but the norm in the quadratic extension $k(\sqrt{a})/k$.

Let $\alpha = \{a_1, \ldots, a_n\} \in K_n^M(k)/2$ be a pure symbol. The 2^n-dimensional form $\langle\langle a_1, \ldots, a_n \rangle\rangle := \langle\langle a_1 \rangle\rangle \otimes \ldots \otimes \langle\langle a_n \rangle\rangle$ depends only on α and is called the *n-fold Pfister form* q_α. It defines (smooth projective) Pfister quadric Q_α. Moreover,

$$\text{Pfister quadrics} \quad \stackrel{1-to-1}{\longleftrightarrow} \quad \text{pure symbols in } K_*^M(k)/2.$$

Various qualities of the symbol α can be read through the properties of the Pfister quadric Q_α.

Proposition 3.3 *For any extension L/k, TFAE:*

(1) $Q_\alpha|_L$ *has a rational point;*

(2) $Q_\alpha|_L$ *has a zero-cycle of odd degree;*

(3) $Q_\alpha|_L$ *is split* ($q_\alpha = \oplus\langle 1, -1\rangle$)*;*

(4) $\alpha_L = 0 \in K_n^M(L)/2$.

In particular, the equivalence ((2) $\Leftrightarrow$ (4)) shows that Q_α is a *norm-variety* for α.

The following result is known (the proof uses the algebraic theory of quadratic forms):

Proposition 3.4 *Any anisotropic quadric can be embedded into an anisotropic Pfister quadric over some finitely-generated purely transcendental extension.*

Hence, if k is flexible, any anisotropic quadric can be embedded into an anisotropic Pfister quadric already over k.

Corollary 3.5 *Over a flexible field, to construct* $DM(k/k; \mathbb{F}_2)$*, it is sufficient to annihilate only (motives of) anisotropic Pfister quadrics (= norm-varieties for non-zero pure symbols from* $K_*^M(k)/2$*).*

How to compute Homs in the isotropic category? It appears that $DM(k/k; \mathbb{F}_p)$ can be embedded as a full $\otimes - \Delta$-ed subcategory of $DM(k; \mathbb{F}_p)$.

Čech simplicial schemes

Recall that a *smooth simplicial scheme* is a functor: $\Delta^\bullet \to Sm/k$, where $Ob(\Delta^\bullet) = \{[n] | n \in \mathbb{N} \cup \{0\}\}$, where we will view $[n]$ as the collection $\{0, 1, \ldots, n\}$ and $\mathrm{Hom}_{\Delta^\bullet}([n], [m])$ = non-decreasing maps $[n] \to [m]$. Thus, such a simplicial scheme consists of *graded components* $X_n = X([n])$ which are smooth varieties, plus various simplicial maps between them.

Let now Q be some smooth variety. The *Čech simplicial scheme* $\check{C}ech(Q)$ is defined as follows: $\check{C}ech(Q)_n = Q^{n+1}$ and simplicial maps are given by a combination of partial projections and partial diagonals. More precisely, for $[n] \xrightarrow{\varphi} [m]$, the map $Q^{n+1} \xleftarrow{\check{C}ech(Q)_\varphi} Q^{m+1}$ is given by:

$$\left(\check{C}ech(Q)_\varphi(y_0, \ldots, y_m)\right)_i = y_{\varphi(i)}.$$

$$Q \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \begin{matrix} Q \\ \times \\ Q \end{matrix} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \begin{matrix} Q \\ \times \\ Q \\ \times \\ Q \end{matrix} \qquad \cdots$$

In Topology this construction is known as EQ. It produces a contractible space. The same happens in algebraic geometry, if Q has a k-rational point q. The contracting homotopy $\check{C}ech(Q)_n \stackrel{\lambda}{\to} \check{C}ech(Q)_{n+1}$ is given by: $\lambda(x_0, \ldots, x_n) = (q, x_0, \ldots, x_n)$. In particular, over $\overline{k}$, $\check{C}ech(Q)$ is contractible.

Denote as $\mathcal{X}_Q$ the motive $M(\check{C}ech(Q))$ of our Čech simplicial scheme. We have a natural distinguished Δ in $DM(k;\mathbb{F}_p)$:

$$\mathcal{X}_Q \to T \to \widetilde{\mathcal{X}}_Q \to \mathcal{X}_Q[1].$$

The object $\widetilde{\mathcal{X}}_Q$ measures, how far Q is from having a 0-cycle of degree 1 (a "weak form" of a rational point).

If P is any smooth scheme, or smooth simplicial scheme, such that there exists a map $P \stackrel{f}{\to} Q$, then $P \times \check{C}ech(Q)$ contracts to P. The contracting homotopy $P \times \check{C}ech(Q)_n \stackrel{\lambda}{\to} P \times \check{C}ech(Q)_{n+1}$ is given by

$$\lambda(p, x_0, \ldots, x_n) = (p, f(p), x_0, \ldots, x_n).$$

In particular, $M(Q) \otimes \mathcal{X}_Q = M(Q)$, or, in other words, $M(Q) \otimes \widetilde{\mathcal{X}}_Q = 0$, and also $\mathcal{X}_Q \otimes \mathcal{X}_Q \stackrel{=}{\to} \mathcal{X}_Q$, which implies that $\widetilde{\mathcal{X}}_Q \stackrel{=}{\to} \widetilde{\mathcal{X}}_Q \otimes \widetilde{\mathcal{X}}_Q$ and $\mathcal{X}_Q \otimes \widetilde{\mathcal{X}}_Q = 0$. Thus, $\mathcal{X}_Q$ and $\widetilde{\mathcal{X}}_Q$ are mutually orthogonal $\otimes$-projectors in $DM(k)$, which define a semi-orthogonal decomposition of this Δ-ed category (it can be shown that $\mathrm{Hom}_{DM(k)}(A \otimes \mathcal{X}_Q, B \otimes \widetilde{\mathcal{X}}_Q) = 0$).

The category $DM(k) \otimes \widetilde{\mathcal{X}}_Q$ is equivalent to the localisation of $DM(k)$ by the localising subcategory generated by $M(Q)$. In this construction, Q may consist of any number of connected components. In particular, the *isotropic motivic category* $DM(k/k;\mathbb{F}_p)$ is equivalent to the subcategory $DM(k;\mathbb{F}_p) \otimes \widetilde{\mathcal{X}}_{\mathbf{Q}}$ of $DM(k;\mathbb{F}_p)$, where $\mathbf{Q}$ is the disjoint union of all (isomorphism classes of) p-anisotropic varieties over k and the functor $DM(k;\mathbb{F}_p) \to DM(k/k;\mathbb{F}_p)$ is given by $\otimes \widetilde{\mathcal{X}}_{\mathbf{Q}}$.

When k is flexible and $p = 2$, the projector $\widetilde{\mathcal{X}}_{\mathbf{Q}}$ is the colimit of projectors $\widetilde{\mathcal{X}}_{Q_\alpha}$, where α runs through all non-zero pure symbols in $K^M_n(k)/2$. The colimit is taken over the category:

- *Ob*: non-zero pure symbols $\alpha \in K_*^M(k)/2$;
- *Mor*: $\exists!$ morphism $\alpha \to \beta$, if α divides β.

For a flexible field, this category is directed: if $\alpha_1, \ldots, \alpha_s$ is a finite collection of non-zero pure symbols, then there exists a non-zero $\alpha \in K_*^M(k)/2$ divisible by all $\alpha_1, \ldots, \alpha_s$.

In the category $DM_{\widetilde{\alpha}}(k) := DM(k; \mathbb{F}_2) \otimes \widetilde{\mathcal{X}}_{Q_\alpha}$ we still have Tate-motives and Homs between these

$$\begin{aligned}\mathrm{Hom}_{DM_{\widetilde{\alpha}}(k)}(T, T(a)[b]) &= \mathrm{Hom}_{DM(k;\mathbb{F}_2)}(\widetilde{\mathcal{X}}_{Q_\alpha}, \widetilde{\mathcal{X}}_{Q_\alpha}(a)[b]) = \\ &= \mathrm{Hom}_{DM(k;\mathbb{F}_2)}(T, \widetilde{\mathcal{X}}_{Q_\alpha}(a)[b])\end{aligned}$$

are just motivic homology groups of the (reduced) Čech simplicial scheme. Denote:

$$\mathrm{H}^{*,*'}_{\mathcal{M},\widetilde{\alpha}}(k; \mathbb{F}_2) = \oplus_{a,b} \mathrm{Hom}_{DM_{\widetilde{\alpha}}(k)}(T, T(a)[b]).$$

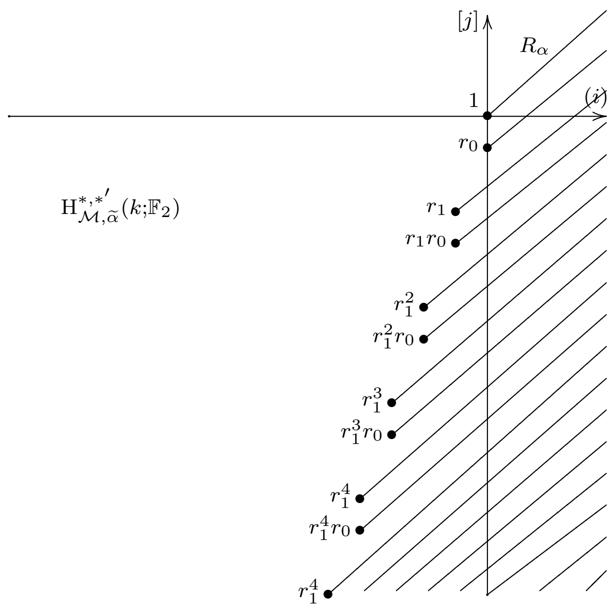
These can be computed using Voevodsky techniques.

For $\alpha = \{a_1, \ldots, a_n\} \in K_n^M(k)/2$, denote $R_\alpha = K_*^M(k)/\operatorname{Ker}(\cdot\alpha)$. The latter ring is isomorphic to the principal ideal $\alpha \cdot K_*^M(k)/2$ as a $K_*^M(k)$-module.

Proposition 3.6 *There are classes r_i, $0 \leqslant i \leqslant n-1$, of degree $\deg(r_i) = (1-2^i)[1-2^{i+1}]$ such that*

$$\mathrm{H}^{*,*'}_{\mathcal{M},\widetilde{\alpha}}(k; \mathbb{F}_2) = R_\alpha[r_i|_{i=0,\ldots,n-1}] / \left(r_i^2 - r_{i+1} \cdot \{-1\}|_{0 \leqslant i \leqslant n-2}\right).$$

Thus, we have $(n-1)$ "external" generators plus a polynomial one, and it is a free module over R_α. Here is the picture for a symbol $\alpha = \{a_1, a_2\}$ of degree 2.



Each diagonal ray represents a copy of R_α (multiplied by the specified generator).

In the colimit (over the category of symbols described above) we obtaine *isotropic motivic cohomology* of a point (with $\mathbb{F}_2$-coefficients). In this colimit, R_α degenerates into $\mathbb{F}_2$ (as all elements of Milnor's K-theory of positive degree eventually get into $\mathrm{Ker}(\cdot\alpha)$), while all the generators become "external".

Theorem 3.7 *If k is flexible, then*

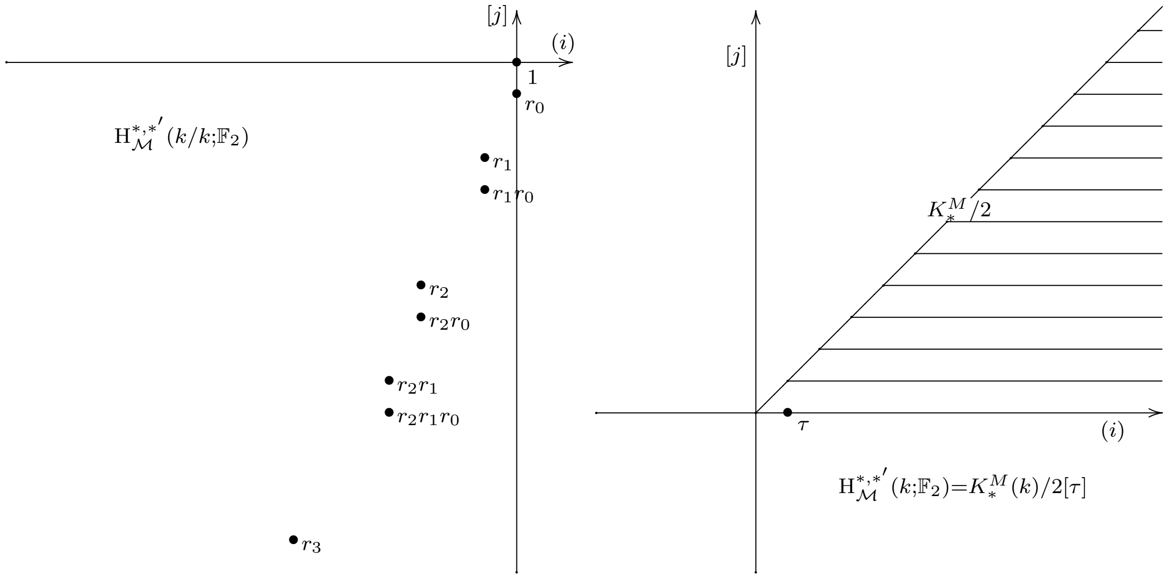
$$\mathrm{H}^{*,*'}_{\mathcal{M}}(k/k;\mathbb{F}_2) = \Lambda_{\mathbb{F}_2}\left(r_i|_{i\geqslant 0}\right).$$

is the external algebra over $\mathbb{F}_2$ with generators r_i.

Observe that the answer doesn't depend on the choice of a field, as long as it is flexible. Also, we see that the Homs in isotropic motivic category between various Tate-motives are finite groups. The same should be true for Homs between arbitrary compact objects of this category. Notice that "local" motivic cohomology of a point is drastically different from the "global" one (computed by Voevodsky). The former reside in the third quadrant, while the latter live in the first one. In particular, the global to local map

$$\mathrm{H}^{*,*'}_{\mathcal{M}}(k;\mathbb{F}_2) \to \mathrm{H}^{*,*'}_{\mathcal{M}}(k/k;\mathbb{F}_2)$$

is zero outside the bidegree (0)[0].



Let me add that this computation of the *isotropic motivic cohomology* of a point permitted Tanania to compute the (2-completed) *isotropic stable homotopy groups of spheres*. It appears that these can be identified with the E_2-term of the classical Adams spectral sequence converging to the topological stable homotopy groups of spheres, thus demonstrating a peculiar interaction between isotropic and topological worlds.

References

[1] W.Fulton, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **2**, Springer-Verlag, Berlin-New York 1984.

[2] F.Tanania, *Isotropic stable motivic homotopy groups of spheres*, Advances Math., **383** (2021), p.107696.

[3] A.Vishik, *Isotropic motives*, J. of the Inst. of Math. of Jussieu, **21** (2022), n.4, 1271-1330. DOI:10.1017/S1474748020000560

[4] V. Voevodsky, *Triangulated category of motives over a field*, in Cycles, transfers and motivic cohomology theories, Annals of Math Studies, pp. 188-238, Princeton: Princeton Univ. Press, 2000.