Berkovich spaces over $\mathbb{Z}$: étale morphisms

Dorian Berger

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Definition

A map $|.| : \mathcal{A}[T_1, \dots, T_n] \to \mathbb{R}_+$ is a *multiplicative semi-norm on* $\mathcal{A}[T_1, \dots, T_n]$ *whose restriction to* $\mathcal{A}$ *is bounded* if, for all $P, Q \in \mathcal{A}[T_1, \dots, T_n]$ and $a \in \mathcal{A}$:

- $|0| = 0$,
- $|1| = 1$,
- $|P + Q| \leq |P| + |Q|$,
- $|PQ| = |P||Q|$,
- $|a| \leq \|a\|_{\mathcal{A}}$.

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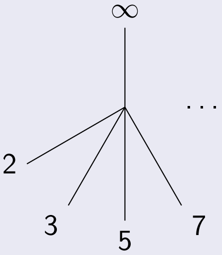
- Spectrum of $\mathbb{Z}$:



Let $x \in \mathbb{A}^n_{\mathcal{A}}$. Then:

- $\ker(|.|_x) \subset \mathcal{A}[T_1, \ldots, T_n]$ is a prime ideal,
- $|.|_x$ induces an absolute value on

$$\mathrm{Frac}\left({\mathcal{A}[T_1, \ldots, T_n]}\big/{\ker(|.|_x)} \right).$$

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The image of $f \in \mathcal{A}[T_1, \ldots, T_n]$ in $\mathcal{H}(x)$ is denoted by $f(x)$.

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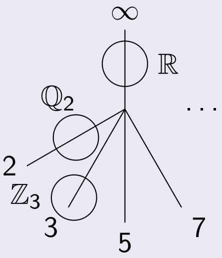

- More generally, we can define $\mathcal{A}$-analytic spaces as spaces locally of the form $\mathrm{Supp}\,(\mathcal{O}_U/\mathcal{I})$ with $U \subset \mathbb{A}^n_{\mathcal{A}}$ open and $\mathcal{I} \subset \mathcal{O}_U$ a coherent ideal sheaf.

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- In general, a closed disc is *not* an $\mathcal{A}$-analytic space.
- $\mathcal{A}$-analytic spaces come with a notion of morphisms.

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Remark

Let $f : X \to S$ be a morphism of $\mathcal{A}$-analytic spaces and $s \in S$.

- $\kappa(s)$ is valued but not complete,
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Note that κ is always henselian (Poineau, 2013).

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- discrete valuation rings,
- trivially valued Dedekind rings,
- etc ...

In those cases, Lemanissier and Poineau proved:

- existence of finite fiber products,
- existence of analytification functor,
- Oka coherence theorem,
- finite mapping theorem,
- Rückert Nullstellensatz,
- etc . . .

Definition

Let $f : X \longrightarrow S$ be a morphism of $\mathcal{A}$-analytic spaces, $x \in X$ and $s = f(x)$. We call f *flat* (resp. *unramified*, resp. *étale*) at x if the induced morphism $f_x^\sharp : \mathcal{O}_s \longrightarrow \mathcal{O}_x$ is flat (resp. unramified, resp. étale).

Recall that:

- $\mathcal{O}_s \longrightarrow \mathcal{O}_x$ is flat if $\_ \otimes_{\mathcal{O}_s} \mathcal{O}_x$ is exact,
- $\mathcal{O}_s \longrightarrow \mathcal{O}_x$ is unramified if $\mathfrak{m}_x = \mathfrak{m}_s \mathcal{O}_x$ and $\kappa(x)$ is a finite separable extension field of $\kappa(s)$,
- $\mathcal{O}_s \longrightarrow \mathcal{O}_x$ is étale if it is flat and unramified.

Example

The natural morphism $\mathcal{M}(\mathbb{Z}[i]) \longrightarrow \mathcal{M}(\mathbb{Z})$ is:

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Let $\mathscr{S}$ and $\mathscr{X}$ be $\mathcal{A}$-schemes locally of finite presentation and $\rho : \mathscr{X}^{\mathrm{an}} \longrightarrow \mathscr{X}$ the natural morphism.

Theorem (B., 2021)

Let $f : \mathscr{X} \longrightarrow \mathscr{S}$ be a morphism and $x \in \mathscr{X}^{\mathrm{an}}$. Then f is unramified (resp. étale) at $\rho(x)$ if and only if f^{an} is unramified (resp. étale) at x.

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Definition

We call f:

- *thick rigid* at x if $\kappa(x)$ is a finite extension field of $\kappa(s)$,
- *purely locally transcendental* at x if $\mathfrak{m}_x = \mathfrak{m}_s \mathcal{O}_x$.

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Remark: unramified morphisms are thick rigid and purely locally transcendental.

Example

The natural projection $\pi : \mathbb{A}^n_S \longrightarrow S$ is:

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If f is thick rigid and purely locally transcendental at x then $\dim_x f = 0$.

Idea of the proof

Up to shrinking X and S, it exists Y such that:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \overset{f}{\searrow} & \downarrow \\ & & S \end{array}$$

with $X \longrightarrow Y$ thick rigid at x and $Y \to S$ purely locally transcendental at y.

Then ${\mathcal{O}_y}\big/_{\mathfrak{m}_s\mathcal{O}_y}$ is a field and ${\mathcal{O}_y}\big/_{\mathfrak{m}_s\mathcal{O}_y} \longrightarrow \mathcal{O}_{Y_s,y}$ is flat.

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As $X \longrightarrow Y$ is thick rigid at x, up to composing with an endomorphism of X, we may assume $\mathcal{O}_x$ is a quotient of a subring of $\mathcal{O}_y[\![T_1, \ldots, T_n]\!]$. We can then conclude using the local criterion for flatness with $(T_1, \ldots, T_n) \subset \mathcal{O}_x$.

Corollary: fiber criterion for flatness

In the following situation:

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array},$$

if $X \longrightarrow S$ and $g_s : X_s \longrightarrow Y_s$ are flat at x then g is flat at x and $Y \longrightarrow S$ is flat at y.

What about the fiber criterion for ramification?

Goal:

$$\begin{cases} \kappa(x) \text{ finite sep. over } \kappa(s) \\ \mathfrak{m}_x = \mathfrak{m}_s \mathcal{O}_x \end{cases} \iff \begin{cases} \mathcal{H}(x) \text{ finite sep. over } \mathcal{H}(s) \\ \mathfrak{m}_{X_s,x} = 0 \end{cases}$$

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However, it is true if f is thick rigid at x, according to the following:

Proposition (B.,2021)

If f is thick rigid at x then $\mathfrak{m}_{X_s,x} = \sqrt{\mathfrak{m}_x \mathcal{O}_{X_s,x}}$.

Corollary: fiber criterion for étaleness

In the following situation:

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array},$$

if $X \longrightarrow S$ is flat at x and $g_s : X_s \longrightarrow Y_s$ is étale at x then g is étale at x (and $Y \longrightarrow S$ is flat at y).