The Structure of Special Fibers through Valuations

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Outline

Normal models of curves

- (K, v_K) discrete valued field, $\mathcal{O}_K$ valuation ring with residue ring $k = \mathcal{O}_K/\mathfrak{m}$, $\mathrm{Spec}(\mathcal{O}_K) = \{\eta, s\}$, $\eta = [(0)]$, $s = [\mathfrak{m}]$;
- X algebraic curve (absolutely irreducible, smooth, projective) defined over K.

Definition

A *normal model* of X over $\mathcal{O}_K$ is a normal, flat and proper two-dimensional $\mathcal{O}_K$-scheme $\mathcal{X}$, equipped with an isomorphism $\mathcal{X}_g \cong X$ between its generic fiber and the curve X. As a set: $\mathcal{X} = \mathcal{X}_g \cup \mathcal{X}_s$.

- $\mathcal{X}_s \to \mathrm{Spec}(k)$ *special fiber* (the *reduction of* X *modulo* $\mathfrak{m}$)
  - projective curve, but not necessarily irreducible or smooth

Why normal models?

$\mathcal{X}_s = \Gamma_1 \cup \Gamma_2 \cup \ldots \Gamma_n,\ \Gamma_i = \overline{\{\xi_i\}}$ the *irreducible components*.

Every irreducible component of $\mathcal{X}_s$ induces a discrete valuation on $K(X)$ that extends v_K:

- $\xi \in \mathcal{X}$ point of codimension 1 $\Rightarrow \mathcal{O}_{\mathcal{X},\xi}$ local ring of dimension 1.
- $\mathcal{X}$ normal $\overset{Serre's\ criterion}{\Longrightarrow}$ $\mathcal{X}$ regular in codimension 1;
- $\mathcal{O}_{\mathcal{X},\xi}$ local and a PID $\Rightarrow \mathcal{O}_{\mathcal{X},\xi}$ discrete valuation ring.

Proposition

If (K, v_K) is a Henselian discrete valued field, then every normal model $\mathcal{X}$ of X (isomorphism class) can be identified with a finite set $V(\mathcal{X})$ of valuations on the function field $K(X)$ that extend v_K, whose residue fields have transcendence degree 1 over k.

The projective line and the valuations of its models

(K, v_K) discrete valued field, $X = \mathbb{P}^1_K$, $K(X) = K(t)$

$V(\mathcal{X}) \subset V(K[t]) = \{v : K[t] \to \mathbb{R}_\infty \mid v(t) \geq 0, v|_K = v_K\}$

(pseudovaluations)

Definition

The Gauss valuation:

$$v_0 : K[t] \to \mathbb{R} \cup \{\infty\}$$

$$f = \sum_{i=0}^{n} a_i t^i \mapsto \min_{i=0,n}(v_K(a_i))$$

Order on $V(K[t])$:

$v \leq w \stackrel{def}{\Longleftrightarrow} v(f) \leq w(f), \ \forall f \in K[t]$

v_0 is the least element in the poset $(V(K[t]), \leq)$

Who is $V(K[t])$?

augmentations: modify valuation v to $w = [v; w(\varphi) = r]$, $\varphi \in K[t]$, $r \in \mathbb{R} \cup \{\infty\}$, $r > v(\varphi)$.

$$w : \sum_{i=0}^{n} a_i \varphi^i \mapsto \min_{i=0,n} \left(v(a_i \varphi^i) \right) \Rightarrow w \geq v$$

this can be done repeatedly:

inductive (pseudo)valuation:

$$v = [v_0, v_1(\varphi_1) = \lambda_1, \dots, v_n(\varphi_n) = \lambda_n] = [v_0, \varphi_1 \mapsto \lambda_1, \dots, \varphi_n \mapsto \lambda_n]$$

limit (pseudo)valuation: $v = \lim_{n \to \infty} v_n$, $v_i = [v_{i-1}; v_i(\varphi_i) = \lambda_i]$

Theorem ([ML36])

Any pseudovaluation in $V(K[t])$ is either inductive, or limit.

Berkovich?

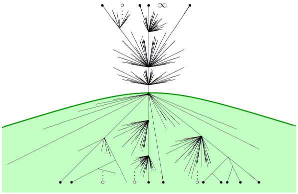
$(K, |\cdot|_K)$ Banach field $\approx (K, v_K)$ complete valued field

$\mathbb{D}^1_{\mathrm{Berk},K}$ = the Berkovich unit disk over K

points in $\mathbb{D}^1_{\mathrm{Berk},K}$ = bounded multiplicative seminorms $\approx$ pseudovaluations in $V(K[t])$

$\mathbb{P}^1_{\mathrm{Berk},K}$, the Berkovich projective line

$\approx V(K[t]) \cup V(K[1/t])$



[BR10]

The Classification of Pseudovaluations/Points on $V(K[t])$:

For every $v \in V(K[t])$ we have the following types of pseudovaluations (corresponding to the types of points in $\mathbb{D}^1_{Berk}$):

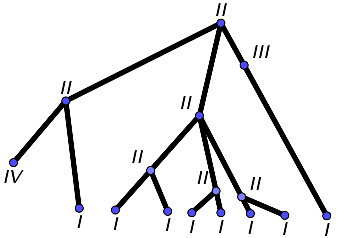
$E_v := \dim_{\mathbb{Q}} \left((\Gamma_v/\Gamma_K) \otimes \mathbb{Q} \right)$

$F_v := \mathrm{trdeg}_k \left(\kappa(v) \right)$

$E_v + F_v \leq \mathrm{trdeg}_K(K(t))$ (*Abhyankar*)

$\mathrm{Kern}(v) = \{ f \in K[t] |\ v(f) = \infty \}$

- *I*, if $\mathrm{Kern}(v) \neq \{0\}$;
- *II*, if $\mathrm{Kern}(v) = \{0\},\ E_v = 0,\ F_v = 1$;
- *III*, if $\mathrm{Kern}(v) = \{0\},\ E_v = 1,\ F_v = 0$;
- *IV*, if $\mathrm{Kern}(v) = \{0\},\ E_v = 0,\ F_v = 0$;



Diskoids

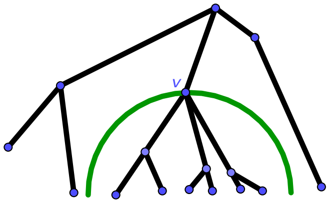
For every $v \in V(K[t])$ we will be interested in the set

$D_v := \{w \in V(K[t]) | \; w \geq v\}$ and call it the diskoid with boundary v.

Theorem

Let (K, v_K) be a Henselian valued field. If $v = [v_0, v_1(\varphi_1) = \lambda_1, \dots, v_n(\varphi_n) = \lambda_n]$, then:

$$D_v = D[\varphi_n, \lambda_n] = \{v \in V(K[t]) | \; v(\varphi_n) \geq \lambda_n\}$$



Theorem

Let (K, v_K) be a Henselian discrete valued field. For any two diskoids $D_1, D_2 \subseteq V(K[t])$ we have either $D_1 \cap D_2 = \emptyset$, $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

Maximal elements and infima

Maximal elements

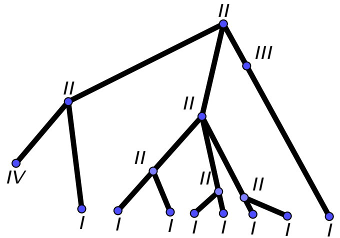
$v \in V(K[t])$ is a maximal element if and only if v is of type I or IV.

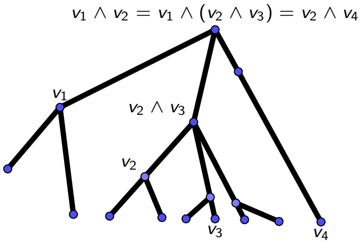
The infimum

$\forall v, w \in V(K[t])$, $\exists! v \wedge w$ so that:

- $v \wedge w \leq v, w$;
- if $u \leq v, w$, then $u \leq v \wedge w$.

$v \wedge w \in V_{II,III}(K[t])$ and if $v \not\leq\not\geq w$, then $v \wedge w \in V_{II}(K[t])$.





Adjacency

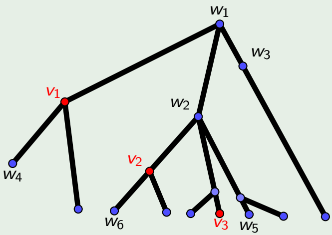
(K, v_K) Henselian discrete valued field, $\mathcal{O}_K$ excellent, $\mathcal{X}$ model of $\mathbb{P}^1_K$.

Adjacency with respect to a model

$v, w \in V(K[t])$ are $\mathcal{X}$-adjacent ($v \sim_{\mathcal{X}} w$) if

$V(\mathcal{X}) \cap ([v \wedge w, v) \cup (v \wedge w, w]) = \emptyset$.

Example



Let $V(\mathcal{X}) = \{v_1, v_2, v_3\}$. Then:

$w_1 \not\sim_{\mathcal{X}} w_4$, $w_6 \not\sim_{\mathcal{X}} w_5$,

$w_5 \sim_{\mathcal{X}} w_3$, $w_5 \sim_{\mathcal{X}} w_2$.

Adjacency and intersections of special fibers

(K, v_K) Henselian discrete valued field, $\mathcal{O}_K$ excellent;

Theorem

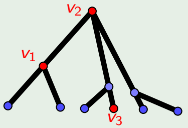
$\forall v_1, v_2 \in V(\mathcal{X})$:

- $|\Gamma_{v_1} \cap \Gamma_{v_2}| \leq 1$;
- $\Gamma_{v_1} \cap \Gamma_{v_2} \neq \emptyset \iff v_1 \sim_{\mathcal{X}} v_2$;
- *if* $v_1 \leq v_2$ *or* $v_2 \leq v_1$ *and* $v_1 \sim_{\mathcal{X}} v_2$, *then* $\Gamma_{v_1} \cap \Gamma_{v_2}$ *belongs to no other component.*

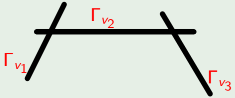
Example

A model with $V(\mathcal{X}) = \{v_1, v_2, v_3\}$

whose valuations are like this:



will have the special fiber:



Adjacency and reduction

(K, v_K) Henselian discrete valued field, $\mathcal{O}_K$ excelent, $\mathcal{X}$ normal model of $\mathbb{P}^1_K \approx \mathrm{Spec}(K[t]) \cup \{\infty\}$.

The type I pseudovaluation induced by a polynomial

$x = [(\varphi)] \in X^0$ with $\varphi \in \mathcal{O}_K[t]$ monic and irreducible

$$\rightsquigarrow v_x = v_{\varphi,\infty} : f \mapsto v_{K^{alg}}(f(\theta))$$

where θ is a root of φ and $v_{K^{alg}}$ is the unique extension of v_K to K^{alg}.

Theorem

Let $x \in X$ be a closed point and denote by $\widetilde{x}$ the reduction of x in $\mathcal{X}$.

Then for every $v \in V(\mathcal{X})$ we have:

$$\widetilde{x} \in \Gamma_v \iff v_x \sim_{\mathcal{X}} v$$

Reduction: an example

Let $(K, v_L) = (\mathbb{Q}_2, \nu_2)$ be the field of 2-adic numbers. We consider the following valuations, along with their diskoids:

$v_1 = [v_0; (t+1) \mapsto 1],\ D_{v_1} = D[t+1, 1]$

$v_2 = [v_0; (t+1) \mapsto 1; (t^2+t+1) \mapsto 2],\ D_{v_2} = D[t^2+t+1, 2]$

$v_3 = [v_0; (t+1) \mapsto 1; (t^2+1) \mapsto 1],\ D_{v_3} = D[t^2+1, 1]$

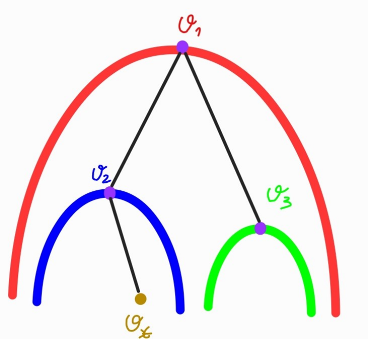
v_2 and v_3 are augmentations of v_1, so $v_2, v_3 \in D_{v_1}$.

$v_3(t^2+t+1) < 2$, so $v_3 \notin D_{v_2}$

$v_2(t^2+1) < 1$, so $v_2 \notin D_{v_3}$

Let $x = [(t^2+t+1)]$. Then $v_x = [v_0; (t^2+t+1) \mapsto \infty] \in D_{v_2}$.

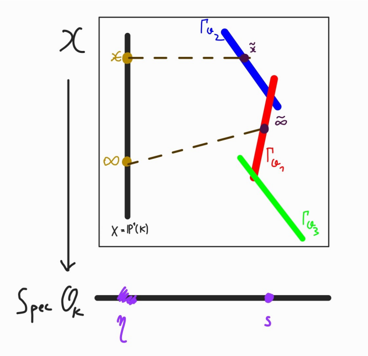
The diskoid/model image



$v_x \in D_{v_2}$

$D_{v_2} \cap D_{v_3} = \emptyset$

$D_{v_2}, D_{v_3} \subset D_{v_1}$



$\widetilde{x} \in \Gamma_{v_2}, \ \widetilde{x} \notin \Gamma_{v_1}, \ \widetilde{x} \notin \Gamma_{v_3}$

$\widetilde{\infty} \in \Gamma_{v_1}, \ \widetilde{\infty} \notin \Gamma_{v_2}, \ \widetilde{\infty} \notin \Gamma_{v_3}$

References

- Matthew Baker and Robert Rumely, *Potential Theory on the Berkovich Projective Line*, American Mathematical Society, 2010.
- Tudor Micu, *Pseudovaluations on polynomial rings, diskoids and normal models of the projective line*, Ph.D. thesis, 2020.
- Saunders Mac Lane, *A Construction for Absolute Values in Polynomial Rings*, Transactions of the American Mathematical Society **40** (1936), no. 3, 363–395.
- Julian Rüth, *Models of Curves and Valuations*, Ph.D. thesis, Ulm University, 2015.

What is a valuation?

R commutative ring,

$v : R \to \mathbb{R} \cup \{\infty\}$ valuation

1. $v(x+y) \geq \min\left(v(x), v(y)\right)$;
2. $v(x \cdot y) = v(x) + v(y)$;
3. $v(1) = 0$;
4. $v(x) = \infty \iff x = 0$

$\to$ can be extended to $\mathrm{Frac}(R)$

Examples

1. $R = \mathbb{Z}$, p prime in $\mathbb{Z}$,

   $\nu_p(n) = \max\{k \in \mathbb{Z} | p^k \text{ divides } n\}$

   p-adic valuation

2. $R = \mathcal{H}(U)$, $U \subset \mathbb{C}$ connected open,

   $P \in U$

   $\mathrm{ord}_P(f) = \min\{n \in \mathbb{N} |\ f^{(n)}(P) \neq 0\}$

   multiplicity in P

pseudovaluation: allows $x \neq 0$ with $v(x) = \infty$

$\mathrm{Kern}(v) := \{x \in R |\ v(x) = \infty\}$ ideal in R

Key polynomials

Definition

Let $v \in V(K[t]) \setminus V_I(K[t])$ and $f, g \in K[t]$. We define the relations:

- $f \sim_v g \overset{def}{\iff} v(f-g) > v(f)$ or $f = g = 0$;
- $g \mid_v f$ if $\exists q \in K[t]$ so that $f \sim_v qg$.

For $v \in V(K[t])$, a polynomial $\varphi \in K[t]$ is:

- v-irreducible if $\forall f, g \in K[t]: \ \varphi \mid_v fg \Rightarrow \varphi \mid_v f$ or $\varphi \mid_v g$;
- v-minimal if $\forall f \in K[t] \setminus \{0\}: \ \varphi \mid_v f \Rightarrow \deg(\varphi) \leq \deg(f)$;
- key polynomial of v if φ is v-minimal and v-irreducible.

In particular, every key polynomial is irreducible over K and its completion $\widehat{K}$.

Augmentations

Definition

Let $v \in V(K[t])$ be a Mac Lane valuation, $\varphi \in K[t]$ a key polynomial over v and $\lambda \in \mathbb{R} \cup \{\infty\}$ with $\lambda > v(\varphi)$. We define the *augmentation corresponding to* (φ, λ) as:

$$w : K[t] \to \mathbb{R} \cup \{\infty\}$$
$$f \mapsto \min_{i=0,m} \left(v(a_i) + i\lambda\right) = \min_{i=0,m} \left(v(a_i\varphi^i)\right),$$

where $f = \sum_{i=0}^{m} a_i\varphi^i$ is the φ-adic expansion of f, that is, we have $a_i \in K[t]$ with $\deg(a_i) < \deg(\varphi)$ for every i.

Disks

$a \in K$, $f \in K[t]$, $r \in \mathbb{R} \cup \{\infty\}$

$D_K[a, r] := \{v \in V(K[t]) |\ v(t - a) \geq r\}$ non-archimedean disk

If $a, b \in \mathcal{O}_K$ and $r, s \in \mathbb{R}_{>0} \cup \{\infty\}$, then:

$$D[a, r] \cap D[b, s] = \emptyset \iff v_K(a - b) < \min(r, s)$$

$$D[a, r] \subset D[b, s] \iff v_K(a - b) \geq \min(r, s) \text{ and } r > s$$

$$D[a, r] = D[b, s] \iff v_K(a - b) \geq \min(r, s) \text{ and } r = s$$

$$D[a, r] \supset D[b, s] \iff v_K(a - b) \geq \min(r, s) \text{ and } r < s$$

Seminorms

R commutative ring.

$|\cdot| : R \to \mathbb{R}_+$ seminorm on R if:

1. $|f-g| \leq |f|+|g|, \ \forall f,g \in R$;
2. $|fg| \leq |f| \cdot |g|, \ \forall f,g \in R$;
3. $|1| = 1$;
4. $|0| = 0$.

$|\cdot|$ will be called

- a norm, if

  $|f| = 0 \iff f = 0, \ \forall f \in R$;
- non-archimedean, if

  $|f-g| \leq \max(|f|,|g|), \ \forall f,g \in R$;
- archimedean, if it is not non-archimedean;
- multiplicative, if

  $|fg| = |f| \cdot |g|, \ \forall f,g \in R$.

The type classification

We have obtained the following classification theorem:

Theorem

(K, v_K) discrete valued field, $v \in V(K[t])$.

Then we have the following possibilities:

v inductive,

$v = [v_0, \dots, v_n(\varphi_n) = \lambda_n]$

and v is of type

$$\begin{cases} I, & \text{if } \lambda_n = \infty \\ II, & \text{if } \lambda_n \in \mathbb{Q} \\ III, & \text{if } \lambda_n \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

v limit,

$v = \lim_{n\to\infty} v_n$

and v is of type:

$$\begin{cases} I, & \text{if } (\deg(\varphi_n))_{n\in\mathbb{N}} \text{ constant for } n \geq m, \\ & \text{and } \exists g \in K[t] : \lim_{n\to\infty} \varphi_n \mid g \in \hat{K}[t] \\ IV, & \text{otherwise} \end{cases}$$

The type classification ($\mathcal{O}_K$ excellent)

We have obtained the following classification theorem:

Theorem

(K, v_K) discrete valued field, $v \in V(K[t])$, $\mathcal{O}_K$ excellent.

Then we have the following possibilities:

v inductive,

$$v = [v_0, \dots, v_n(\varphi_n) = \lambda_n]$$

and v is of type

$$\begin{cases} I, & \text{if } \lambda_n = \infty \\ II, & \text{if } \lambda_n \in \mathbb{Q} \\ III, & \text{if } \lambda_n \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

v limit,

$$v = \lim_{n \to \infty} v_n$$

and v is of type:

$$\begin{cases} I, & \text{if } (\deg(\varphi_n))_{n \in \mathbb{N}} \text{ constant for } n \geq m, \\ & \text{and } \varphi := \lim_{n \to \infty} \varphi_n \in K^h[t] \\ IV, & \text{otherwise} \end{cases}$$

The type classification (K complete)

We have obtained the following classification theorem:

Theorem

(K, v_K) complete discrete valued field, $v \in V(K[t])$.

Then we have the following possibilities:

v <u>inductive</u>,

$$v = [v_0, \dots, v_n(\varphi_n) = \lambda_n]$$

and v is of type $\begin{cases} I, \text{ if } \lambda_n = \infty \\ II, \text{ if } \lambda_n \in \mathbb{Q} \\ III, \text{ if } \lambda_n \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

v <u>limit</u>,

$$v = \lim_{n \to \infty} v_n,$$

$(\deg(\varphi_n))_{n \in \mathbb{N}}$ unbounded

and v is of type IV