Refined tropical invariants in positive genus

Uriel Sinichkin (TAU)
Work in progress, joint with Eugenii Shustin

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Outline

Tropical curve counting

Refined invariants

Refined elliptic broccoli invariants

Tropical curve counting

Tropical curve counting

Theorem [Mikhalkin, '03].

For $3d + g - 1$ points $\overline{\boldsymbol{p}}$ in general position we have

$$N_{g,d} = \sum_{\overline{\boldsymbol{p}} \subseteq \Gamma} w(\Gamma)$$

where $w(\Gamma) = \prod_{V \in \Gamma^0} \mu(V)$ and $\mu(V)$ is the Mikhalkin multiplicity of V = the lattice area of the cell corresponding to V in the dual subdivision.

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In particular, this sum is

1. independent of the points (as long as they are generic),
2. and the weight of Γ is a product of terms that are computed locally.

Refined invariants

Block-Gottsche refined invariants

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Definition.

Block-Gottsche weight of a tropical curve Γ is

$$BG_y(\Gamma) = \prod_{V \in \Gamma^0} [\mu(V)]^- \in \mathbb{Z}[y^{\pm 1/2}],$$

where

$$[a]_y^- := \frac{y^{a/2} - y^{-a/2}}{y^{1/2} - y^{-1/2}}$$

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Theorem [Block, Gottsche '14].

Given $3d + g - 1$ points $\overline{\boldsymbol{p}}$ in general position,

$$BG_y(d, g) := \sum_{\Gamma \supset \overline{\boldsymbol{p}},\, g(\Gamma) = g} BG_y(\Gamma)$$

does not depend on the choice of points.

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- This works for (some) other toric surfaces, not only $\mathbb{P}^2$.
- Gottsche-Shende conjecture.
- Mikhalkin interpertaition as quantum index.
- Generating series of log-GW invariants due to Bousseau.

Refined rational broccoli invariant

Definition.

Let $n_e, n_v \in \mathbb{N}$ with $n_e + 2n_v = 3d - 1$ and let $p_1, \dots, p_{n_e+n_v} \in \mathbb{R}^2$. For a rational tropical curve Γ, of degree d, that passes through $p_1, \dots, p_{n_e+n_v}$ and with $p_1, \dots, p_{n_v}$ on its vertices,

$$RB_y(\Gamma) := \prod_{V \in \Gamma^0 \cap \overline{\boldsymbol{p}}} [\mu(V)]_y^+ \cdot \prod_{V \in \Gamma^0 \setminus \overline{\boldsymbol{p}}} [\mu(V)]_y^-$$

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$[\mu(V_1)]_y^+ [\mu(V_2)]_y^-$

Refined rational broccoli invariant

Theorem [Gottsche, Schroeter '16].

1. *The refined rational broccoli invariant*

$$RB_y(d, 0, (n_e, n_v), \overline{\boldsymbol{p}}) := \sum_{\Gamma} RB_y(\Gamma)$$

is independent of the choice of points.

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2. *For $y=1$ we get*

$$RB_1(d,0,(n_e,n_v)) = \langle \tau_0(2)^{n_e}\tau_1(n_v)^{n_v}\rangle_{\Delta}^{0}$$

is the descendant GW invariant.

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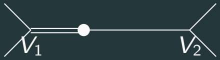
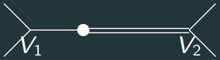
3. *The value* $RB_{-1}(d, 0, (n_e, n_v))$ *equals to the Welschinger invariant with* n_v *pairs of conjugate points.*

Refined elliptic broccoli invariants

Definition of refined elliptic broccoli invariants

To get an invariance in genus 1 we need to consider *collinear cycles*





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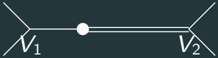


and to assign the pair of curves with the 2 different directions of the collinear cycle the weight [Schroeter, Shustin '16]

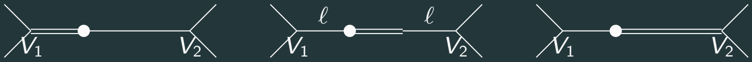
$$\Psi_y^{(2)}(w, \mu(V_1), \mu(V_2)) \cdot \prod_{V \in \Gamma^0 \setminus \overline{\boldsymbol{p}}} [\mu(V)]_y^- \cdot \prod_{\substack{V \in \Gamma^0 \cap \overline{\boldsymbol{p}} \\ V \notin \{V_1, V_2\}}} [\mu(V)]_y^+ .$$

Questions

- Is there a local description of this weight where each individual curve gets its own weight?
- What is the meaning of the values of the refined elliptic broccoli weight for $y = 1, -1$?

Local description

Additional allowed fragment:



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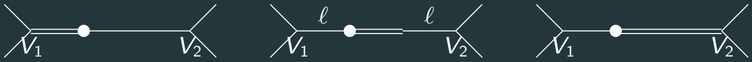

Theorem [Shustin, S. '22+].

There exist a refined weight of elliptic tropical curves which satisfies:

1. *The weight of a curve is a product of multiplicities on its local fragments.*
2. *The sum of weights of curves passing through $p_1, \ldots, p_{n_e+n_v}$ is equal to the refined elliptic broccoli invariant $RB_y(d, 1, (n_e, n_v))$.*

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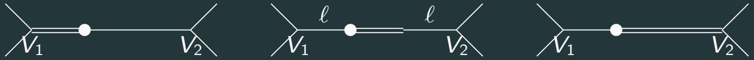

$[\mu(V_2)]^- \varphi^{(1)}(\mu(V_1), w_1, w_2)$ $[\mu(V_1)]^- \varphi^{(1)}(\mu(V_2), w_1, w_2)$

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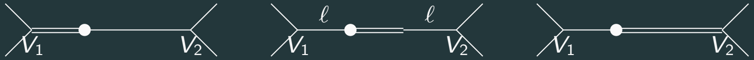

$[\mu(V_2)]^- \varphi^{(1)}(\mu(V_1), w_1, w_2)$

$[\mu(V_1)]^- [\mu(V_2)]^- \varphi^{(0)}(w_1, w_2)$

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2. *The sum of weights of curves passing through* $p_1, \ldots, p_{n_e+n_v}$ *is equal to the refined elliptic broccoli invariant* $RB_y(d, 1, (n_e, n_v))$.

Refined elliptic broccoli invariants at $y = 1$

Theorem [Shustin, S. '22+].

The value $RB_1(d, 1, (n_e, n_v))$ is equal to the number of elliptic curves of degree d that pass through $n_e + n_v$ points and have prescribed tangent directions in n_v of those points.

Example

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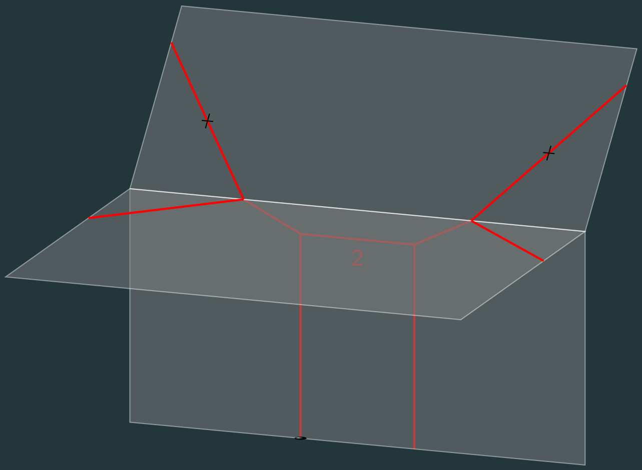
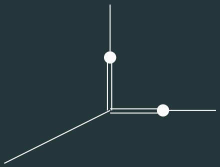


Higher genera

- In higher genera we need to include the fragment



  for which we can not assign a weight that will give an invariant count.
- This fragment does not appear if $n_v = 1$ or if the points are in Mikhalkin position, in those situations we get a refined invariant.

Psi classes

- The calculation of characteristic numbers is related to descendant GW invariants through the work of Graber, Kock, Pandharipande on modified GW invariants.
- We are working on relating those invariants to tropical psi classes as studied by Cavalieri, Gross, Kerber, Markwig, Rau and others.

Questions?

$$
\begin{aligned}
\Psi_z^{(2)}\left(m,\nu_1,\nu_2\right) = {} & \frac{1}{\left(z-z^{-1}\right)^3\left(z+z^{-1}\right)}\times \\
& \times\left[\frac{2\left(z^{\nu_2 m}-z^{-\nu_2 m}\right)\left(z^{\nu_1 m-1}-z^{1-\nu_1 m}\right)}{z-z^{-1}}-\right. \\
& -\frac{2m\left(z^{\nu_2 m}-z^{-\nu_2 m}\right)\left(z^{\nu_1 m-m}-z^{m-\nu_1 m}\right)}{z^m-z^{-m}}+ \\
& +(m-1)\left(z^{\nu_1 m}-z^{-\nu_1 m}\right)\left(z^{\nu_2 m}+z^{-\nu_2 m}\right)- \\
& -\frac{2\left(z^{\nu_2 m}-z^{-\nu_2 m}\right)\left(z^{\nu_1 m-\nu_1}-z^{\nu_1-\nu_1 m}\right)}{z^{\nu_1}-z^{-\nu_1}}- \\
& \left.-\frac{2\left(z^{\nu_1 m}-z^{-\nu_1 m}\right)\left(z^{\nu_2 m-\nu_2}-z^{\nu_2-\nu_2 m}\right)}{z^{\nu_2}-z^{-\nu_2}}\right].
\end{aligned}
$$

where $\mu(V_1)=m\nu_1$ and $\mu(V_2)=m\nu_2$.

$$\varphi_z^{(0)}(k_1, k_2) = \frac{2}{z + z^{-1}} \cdot \frac{[k_1]_z^- [k_2]_z^-}{[k_1 + k_2]_z^-},$$

$$\varphi_z^{(1)}(k_1, k_2, \nu) = [k_1 \nu]_z^- [k_2 \nu]_z^- - \frac{[k_1]_z^- [k_2]_z^-}{[k_1 + k_2]_z^-} [(k_1 + k_2)\nu]_z^-$$