On graded modules associated with line bundles on tropical curves and integral affine manifolds

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P: a d-dimensional lattice polytope in $\mathbb{R}^d$
X_P: the toric variety of P (over $\mathbb{C}$)
$\mathcal{L}_P$: the (ample) line bundle of P

Theorem 0.1 (..., Danilov, ...)

$$\chi(X_P; \mathcal{L}_P) = \dim_{\mathbb{C}} H^0(X_P; \mathcal{L}_P) = \sharp(P \cap \mathbb{Z}^d).$$

$$\chi(X_P; \mathcal{L}_P^{\vee}) = (-1)^d \dim_{\mathbb{C}} H^d(X_P; \mathcal{L}_P^{\vee}) = (-1)^d \sharp(\mathrm{int}(P) \cap \mathbb{Z}^d).$$

If X_P is smooth, then

$$\sharp(P \cap \mathbb{Z}^d) = \int_{X_P} \mathrm{ch}(\mathcal{L}_P)\, \mathrm{td}(X_P). \tag{1}$$

In particular, $\chi(X_P; \mathcal{O}_{X_P}) = 1 = \chi_{\mathrm{top}}(P)$.

We will consider a certain analog of the above theorem for compact integral affine manifolds and tropical curves.

P: a d-dimensional lattice polytope in $(\mathbb{R}^d)^\vee$

$$f_P(x_1, \ldots, x_d) := \log \left(\sum_{m \in P \cap (\mathbb{Z}^d)^\vee} \exp(\langle m, x \rangle) \right)$$

(f_P is a Laurent poly'l fcn over log semiring.)
$df_P \colon \mathbb{R}^d \to (\mathbb{R}^d)^\vee; (x_1, \ldots, x_d) \mapsto (\frac{\partial f_P}{\partial x_1}, \ldots, \frac{\partial f_P}{\partial x_d});$
df_P can be extended to a continuous map μ_P (called the tropical moment map) on the tropical toric variety X_P^{trop} of P.
$\pi \colon (\mathbb{R}^d)^\vee \to (\mathbb{R}^d)^\vee/(\mathbb{Z}^d)^\vee$; the canonical projection.

Theorem 0.2 (..., Oda, Kajiwara)

μ_P is a homeomorphism from X_P^{trop} onto P and $\pi \circ \mu_P \colon X_P \to (\mathbb{R}^d)^\vee/(\mathbb{Z}^d)^\vee$ gives

$$\sharp(0 \cap \mathrm{graph}(\pi \circ \mu_P)) = \sharp(P \cap (\mathbb{Z}^d)^\vee)(= \chi(X_P; \mathcal{L}_P)). \quad (2)$$

Example 0.3 ($X_{\Delta_2}^{\text{trop}} \simeq \mathbb{T}P^2$)

$\Delta_2 := \text{conv}((0,0),(1,0),(0,1))$, $f_{\Delta_2}(x_1,x_2) = \log(1+e^{x_1}+e^{x_2})$

$$df_{\Delta_2}(x_1,x_2) = \left(\frac{e^{x_1}}{1+e^{x_1}+e^{x_2}}, \frac{e^{x_2}}{1+e^{x_1}+e^{x_2}}\right)$$

f_P also defines an element of the Picard group $\text{Pic}(X_P^{\text{trop}})$;

Example 0.4 ($\mathbb{T}P^1 := \mathbb{R} \cup \{\pm\infty\}$)

$f_{[0,n],-\infty}(x) = \log(1+\sum_{i=0}^n e^{ix})$; a fcn on $\mathbb{R} \cup \{-\infty\}$.
$f_{[0,n],\infty}(x) = \log(\sum_{i=0}^n e^{-ix}+1)$; a fcn on $\mathbb{R} \cup \{\infty\}$.
$f_{[0,n],\infty} - f_{[0,n],-\infty} = nx$ defines a
1-cocycle $D_{[0,n]}$ of $\text{Pic}(\mathbb{T}P^1) \simeq \mathbb{Z}$.
$\sharp(0 \cap \text{graph}(\pi \circ \mu_P)) = \deg(D_{[0,n]}) + \chi_{\text{top}}(\mathbb{T}P^1) = n+1$.

Remark 0.5

We can consider $\sharp(0 \cap \text{graph}(\pi \circ \mu_P))$ as the Euler characteristic of "Lagrangian Floer cohomology".

Definition 1.1 (integral affine manifold)

A n-dimensional integral affine manifold is a pair of n-dimensional real manifold B_0 and an altas $\{(U_i, \psi_i : U_i \to \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R})\}$ such that $\psi_j \circ \psi_i^{-1} \in \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ locally.

Every integral affine manifold is also a tropical manifold.

Example 1.2

Tropical tori $\mathbb{R}^n/\Lambda$, Klein's bottle, and so on.

The integral affine str. of B defines the lattice $T^*_{p,\mathbb{Z}} B_0$ of $T^*_p B_0$ and a canonical torus fibration $\check{X}(B_0) = T^* B_0 / T^*_{\mathbb{Z}} B_0$ and symplectic form ω_{std} on $\check{X}(B_0)$.

B: a tropical mfd, $\mathcal{A}_B^{0,0}$: the sheaf of $(0,0)$-superforms on B
$\mathcal{O}_B^\times$: the sheaf of affine tropical fcns on B.
$\mathcal{A}_B^{0,0}$ is acyclic and there exists the following commutaive diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{R}_B & = & \mathbb{R}_B & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_B^\times & \longrightarrow & \mathcal{A}_B^{0,0} & \longrightarrow & \mathcal{A}_B^{0,0}/\mathcal{O}_B^\times & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{F}^1_{\mathbb{Z},B} & \longrightarrow & \mathcal{Z}^1_B & \longrightarrow & \mathcal{Z}^1_B/\mathcal{F}^1_{\mathbb{Z},B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

$\mathrm{Pic}(B) := H^1(B; \mathcal{O}_B^\times)$ is representable by $H^0(B; \mathcal{A}_B^{0,0}/\mathcal{O}_B^\times)$.

B_0: an integral affine manifold,
$\mathrm{Lag}(E)$: the sheaf of (locally) Lagrangian sections of a fibration $\pi : E \to B_0$. (Lagrangian section = section whose image is a Lagrangian submanifold)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{R}_{B_0} & = & \mathbb{R}_{B_0} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{B_0}^{\times} & \longrightarrow & \mathcal{C}^{\infty}(B_0) & \longrightarrow & \mathcal{C}^{\infty}(B_0)/\mathcal{O}_{B_0}^{\times} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow d & & \| & & \\
0 & \longrightarrow & T_{\mathbb{Z}}^{*}B_0 & \longrightarrow & \mathrm{Lag}(T^{*}B_0) & \longrightarrow & \mathrm{Lag}(\check{X}(B_0)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

$H^1(B_0; \mathcal{O}_{B_0}^{\times})$ is representable by Lagrange sections of $\check{\pi}_{B_0} : \check{X}(B_0) \to B_0$, so we can consider Floer cohomology!

Example 1.3 (Kontsevich–Soibelman'01, Abouzaid'21)

If a compact integral affine manifold B_0 has a Pin structure, every $s \in H^0(B_0; \mathrm{Lag}(\check{X}(B_0))) = H^0(B_0; \mathcal{C}^\infty(B_0)/\mathcal{O}_{B_0}^\times)$ defines a canonical object L_s of $\mathrm{Fuk}(\check{X}(B_0))$.
If $\mathrm{Im}(s)$ intersects to the zero section transversally, then every local function $f_i \colon U_i \to \mathbb{R}$ of $s = \{(U_i, f_i)\}_{i \in I}$ is nondegererate and $p \in 0 \cap \mathrm{Im}(s)$ iff $d(f_i + m_{i,p})(p) = 0$ for some affine function $m_{i,p}$ with integer valued slope locally.

$$\mathrm{CF}^\bullet(L_0, L_s) = \bigoplus_{p \in 0 \cap \mathrm{Im}\, s} \Lambda_{\mathrm{nov}}[-\mathrm{Morse}(-(f_i + m_{i,p}), p)].$$

$\mathrm{CF}^\bullet(L_0, L_s)$ is defined by the only data on B_0 and should be

$$\chi(\mathrm{CF}^\bullet(L_0, L_s)) = \sum_{p \in 0 \cap \mathrm{Im}\, s} (-1)^{s_p} = \int_{\mathcal{X}_{B_0}} \mathrm{ch}(\Theta(L_s))\, \mathrm{td}(\mathcal{X}_{B_0}).$$

If every f_i is concave, then $\chi(\mathrm{CF}^\bullet(L_0, L_s)) = \sharp(0 \cap \mathrm{Im}\, s)$.

$$\chi(\mathsf{CF}^\bullet(L_0, L_s)) = \int_{\mathcal{X}_{B_0}} \mathrm{ch}(\Theta(L_s))\,\mathrm{td}(\mathcal{X}_{B_0}).$$

Question 1.4

Can we replace $\mathcal{X}_{B_0}$ by B_0 and generalize it for more general tropical spaces?

Our question can be considered as a certain toy model of the following expectation by Auroux, Efimov and Katzarkov.

Conjectural generalization of SYZ (simplified ver. for this talk)

For a given compact "algebraic" tropical manifold B, are there a stratified Lagrangian torus fibration $\check{X}(B)$ and an algebraic variety $\mathcal{X}_B$ over Novikov field Λ_{nov} and HMS?

This dream is proved when B is a trivalent metric graph [Auroux–Efimov–Katzarkov'22] and "essentially proved" for some special polytopes [Futaki–Kajiura] and highly related with toric varieties [Fang–Liu–Treumann–Zaslow], [Kuwagaki].

Cohomological local Morse datum for good smooth Cartier divisors

S: the support of a (finite) rational polyhedral complex in $\mathbb{R}^n$,
$\iota\colon S \to \mathbb{R}^n$: the inclusion map of S, $f\colon \mathbb{R}^n \to \mathbb{R}$; a C^1-fcn.
$f|_S$ can be considered as the restriction of a C^1-fcn f_p on an open neighborhood U_p of $T_pS = \sum_{\sigma\in\Sigma, p\in\sigma} \mathbb{L}(\sigma, p)$ at p.
Assume $df_p(p) \neq 0$ except $p = x$. ($\Longleftrightarrow$: $\mathrm{Crit}(f|_S) = \{x\}$)

Definition 1.5 (Local Morse datum,(..., Schürmann, ...))

$$\mathrm{MF}^\bullet_x(f) := H^\bullet(R\Gamma_{\{f\geqslant f(x)\}}(\iota_*\mathbb{Z}_S)_x) \simeq H^\bullet(R\Gamma_{S\cap\{f\geqslant f(x)\}}(\mathbb{Z}_S)_x)$$
$$\simeq \varinjlim_{x\in U\subset_{\mathrm{open}} S} \tilde{H}^{\bullet-1}(U \cap \{f < f(x)\}; \mathbb{Z}).$$

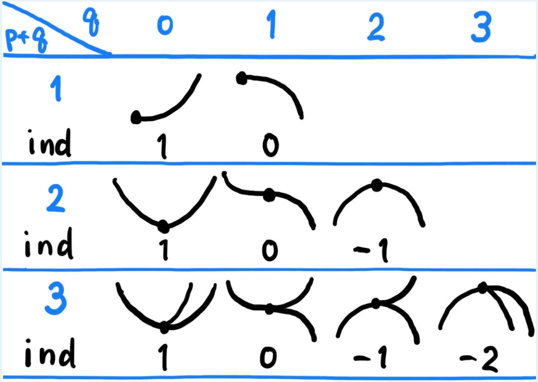
Example 1.6 (Morse function)

$$S = \mathbb{R}^n,\ f\colon \mathbb{R}^n \to \mathbb{R}; (x_1, \ldots, x_n) \mapsto -\sum_{j=1}^i x_j^2 + \sum_{j=i+1}^n x_j^2$$
$$\mathrm{MF}^\bullet_v(f) \simeq \mathbb{Z}[-i], \quad \chi(\mathrm{MF}^\bullet_x(f)) = \mathrm{ind}_{\mathrm{PH}}(\mathrm{grad}(f), x)$$
$$\chi(\mathrm{MF}^\bullet_x(f)) = (-1)^n\chi(\mathrm{MF}^\bullet_x(-f)),$$ special property

Example 1.7 (The local cone of a metric curve at a point)

$S = \Gamma_n := \mathbb{R}_{\geqslant 0}(1, \ldots, 0) \cup \mathbb{R}_{\geqslant 0}(0, \ldots, 1) \cup \mathbb{R}_{\geqslant 0}(-1, \ldots, -1)$
$f : \mathbb{R}^{n-1} \to \mathbb{R}$: a C^1-fcn s.t. $\mathrm{Crit}(f|_{\Gamma_n}) = \{x = (0, \ldots, 0)\}$.

$$\chi(\mathrm{MF}^\bullet_x(f|_{\Gamma_n})) = 1 - \sharp(\pi_0(\{v \in \Gamma_n \mid f(v) < f(x)\})) = 1 - p,$$
$$\chi(\mathrm{MF}^\bullet_x(-f|_{\Gamma_n})) = 1 - \sharp(\pi_0(\{v \in \Gamma_n \mid f(v) > f(x)\})) = 1 - q.$$



To define an analog of Floer complex, we need to add the following condition for $s \in H^0(S; \mathcal{A}_S^{0,0}/\mathcal{O}_S^\times)$.

Definition 1.8 (The microsupport of a sheaf $\mathcal{F}$ on a C^1-mfd X)

$(x; y) \notin SS(\mathcal{F})(\subset T^*X) :\iff \exists$ an open nbhd U of $(x; y)$ s.t. $\forall (f \in C^1(X)$ and $df(x) \in U)$ satisfies $(R\Gamma_{\{f \geqslant f(x)\}}\mathcal{F})_x \simeq 0$.

$SS(\mathcal{F})_x := SS(\mathcal{F}) \cap T_x^*X$

Example 1.9

(i) $X = S$, $SS(\mathbb{Z}_S) = T_X^*X = 0_X$,
(ii) $X = \mathbb{R}^n$, $S = \Gamma_{n+1}$, $\mathrm{Span}(SS(\iota_*\mathbb{Z}_S)_0) = \mathbb{R}^n$.

Definition 1.10 (T.)

$f \in C^\infty(X)$ is admissible at $p(\in S)$ if
$df_p(p) \notin (\mathrm{Span}(SS(\iota_{p,*}\mathbb{Z}_{U_p})_p) + T^*_{p,\mathbb{Z}}S) \setminus T^*_{p,\mathbb{Z}}S, (\iota_p \colon U_p \to T_pS)$.

We can extend the admissibility condition for $s = \{(U_i, f_i)\}_{i \in I}$.

B: a cpt rational polyhedral space s.t. every atlas is in $\mathbb{R}^n$.
$s = \{(f_i, U_i)\}_{i \in I} \in H^0(B; \mathcal{A}_B^{0,0}/\mathcal{O}_B^\times)$
$s_0 \cap s := \{p \in B \mid df_{i,p}(p) \in T^*_{p,\mathbb{Z}}B \text{ for some } f_i\}$
$m_{i,p}$: an affine fcn s.t. $d(f_{i,p} + m_{i,p})(p) = 0$.
Suppose s is admissible and $\sharp(s_0 \cap s) < \infty$.

Definition 1.11 (T.)

$$\mathrm{MF}^\bullet_p(s) := (R\Gamma_{\{f_i + m_{i,p} \geqslant f_i(p) + m_{i,p}(p)\}} \iota_{p,*}(\mathbb{Z}_{B \cap U_i}))_p, \tag{3}$$

$$\mathrm{MF}^\bullet(B, s) := \bigoplus_{p \in s_0 \cap s} \mathrm{MF}^\bullet_p(s). \tag{4}$$

Remark 1.12

If B is an integral affine mfd and s intersect transversely as a Lagrangian submfd, then the graded module $\mathrm{MF}^\bullet(B, s) \otimes_{\mathbb{Z}} \Lambda_{\mathrm{nov}}$ corresponds to that of usual Floer complex $\mathrm{CF}^\bullet(L_0, L_{s^{-1}})$, but rank $\mathrm{MF}^\bullet_p(s)$ may be larger than 1 in contrast to classical Floer cohomology or AEK's.

C: a compact tropical curve with no 1-valent vertex,
$V(C) := \{v \in C \mid \mathrm{val}(v) \neq 2\}$
$C_0 := C \backslash V(C)$: the regular part of C.
$\check{X}(C_0) := T^*C_0 / T^*_{\mathbb{Z}} C_0$: a disjoint union of cylinder.
$i : C_0 \to C$:inclusion map, $s_0' : C_0 \to \check{X}(C_0)$: the zero section
$\check{X}_0(C) := C \sqcup_{i, s_0'} \check{X}(C_0)$: the pushout of i and s_0'. (tentative)

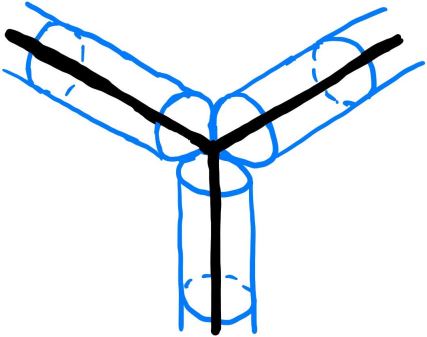


Figure: C and $\check{X}_0(C)$

Every admissible Cartier data $s = \{(U_i, f_i)\}_{i \in I}$ defines a continuous section $s' \colon C \to \check{X}_0(C)$ by $\{df_{i,p}(p)\}_{p \in C}$ and $s_0 \cap s = s_0'(C) \cap s'(C)$.

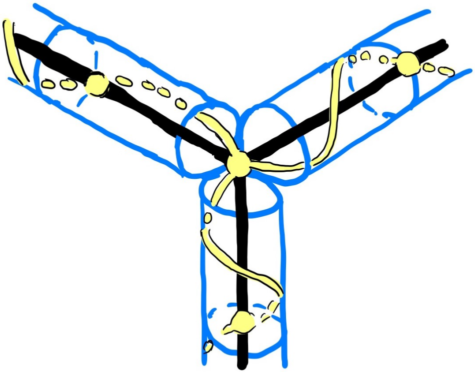


Figure: the zero section s_0' and an admissible section s'

C: a compact tropical curve with no 1-velent vertex,
s: an admissible smooth Cartier divisor s.t. $\sharp(s_0 \cap s) < \infty$.
$D_s \in \mathrm{Pic}(C)$: the divisor class of s.
The continuous section $s' \colon C \to \check{X}_0(C)$ of s defines the rotation number on each edge of C since $s_0'(C) \cap s'(C) \supset V(C)$. (cf. Auroux–Efimov–Katzarkov'22)
$\mathrm{rot}(s)$: the sum of the rotation number of s on each edge.

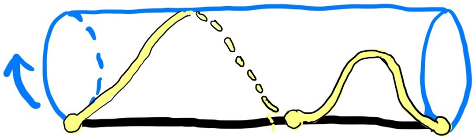


Figure: $\mathrm{rot}(s) = 1$ on an edge

Theorem 2.1 (T., cf. Auroux–Efimov–Katzarkov'22)

$$\chi(\mathrm{MF}^\bullet(C, s)) = \mathrm{rot}(s) + \chi_{\mathrm{top}}(C) = \deg(D_s) + \chi_{\mathrm{top}}(C)$$

Theorem 3.1 (T.)

B_0: a n-dimensional cpt integral affine manifold with a Hessian form.
$s = \{(f_i, U_i)\}$: a smooth Cartier data which intersect to the zero section transversely as a Lagrangian section,
D_s: the divisor class of s. Then,

$$\chi(\mathrm{MF}^\bullet(B_0, s)) = \frac{1}{n!} c_1(D_s)^n.$$

$\mathcal{F}^p_{\mathbb{Z},B_0} := \bigwedge_{i=1}^p T^*_{\mathbb{Z}} B_0$, $H^{p,q}(B_0;\mathbb{Z}) := H^q(B_0; \mathcal{F}^p_{\mathbb{Z},B_0})$

Sketch of proof: (i) By Cheng–Yau's result, B_0 has a finite cover by a tropical torus.
(ii) Use the pushforwards of tropical Borel–Moore homology $H_{p,q}(B_0;\mathbb{Z}) := \mathbb{H}^0(R\mathcal{H}om(\mathcal{F}^p_{\mathbb{Z},B_0}[q], \omega^\bullet_{B_0}))$ [Gross–Shokrieh'19]. ($\omega^\bullet_{B_0} := a^!_{B_0}\mathbb{Z} \simeq \mathcal{F}^n_{\mathbb{Z},B_0}[n]$ where $a_{B_0} : B_0 \to \{\mathrm{pt}\}$.)

Remark 3.2

We can generalize our approrach for more general tropical spaces (e.g. tropical toric varieties) by extending the definition of smooth function on tropical spaces. In this case, we can get $\chi(\mathrm{MF}^\bullet(X_P^{\mathrm{trop}}; -\mu_P)) = \sharp(\mathrm{int}(P \cap \mathbb{Z}^n))$.

Slogan: Mirror HRR for tropical manifolds

B: a compact tropical manifold
$\mathrm{td}(B)$: the Todd class of B (should be) defined by the theory of the Chern–Schwartz–Macpherson cycles of matroids. [Mdedrano–Rincón–Shaw]
$s_{\mathcal{L}}$: a smooth admissible Cartier data of $\mathcal{L}$ s.t. $\sharp(s_0 \cap s_{\mathcal{L}}) < \infty$
Then,

$$\chi(\mathrm{MF}^\bullet(B, s_{\mathcal{L}})) = \int_B \mathrm{ch}(\mathcal{L})\,\mathrm{td}(B). \tag{5}$$