Wild models of curves via nonarchimedean geometry

young researchers' in algebraic number theory 2022

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- Assume the residue field $\widetilde{K}$ is algebraically closed of characteristic p.

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Time depending I'll discuss an application to wild models of curves

Overview

skeleta

(log) differents

application to wild ramification of curves

Reminders on models and their skeleta

- A pure-dimensional normal K°-variety $\mathcal{X}$ is called a **model** of its generic fiber X/K, call its special fiber a **degeneration**.
- Each component of a degeneration induces a discrete valuation on $K(X)$, and so a point in the Berkovich analytification X^{an}, such points are called **divisorial points**. They lie dense in X^{an}.
- Say a model is snc if it is regular and its special fiber is a snc divisor.

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- Say a model is snc if it is regular and its special fiber is a snc divisor.
- The skeleton $\operatorname{sk} \mathcal{X}$ of an snc model $\mathcal{X}$ is the dual intersection complex of the components of $\mathcal{X}_s$, it is a compact integral piecewise affine simplicial space in X^{an} whose vertices are the divisorial points of $\mathcal{X}_s$.

Approaches to canonical skeleta in Berkovich geometry

classical for hyperbolic curves via minimal snc models.
in higher dimension there are at least two approaches:

1. *(Kontsevich-Soibelman '08, Mustaţă-Nicaise '15)*
   essential skeleton via weight functions (as will be explained)
   (Nicaise-Xu '16)
   Equivalently as the skeleton of a good minimal dlt model
2. *(Temkin '16)*
   minimality locus of the Kähler valuation on the sheaf of Kähler differentials

weight functions (after Mustaţă-Nicaise '15)

For simplicity assume X admits a snc model

- Pick any nonzero ω meromorphic pluricanonical form on X
- associate an integral piecewise affine **weight function**

$$\mathrm{wt}_\omega : X^{\mathrm{an}} \to \mathbb{R}.$$

If a divisorial point $x \in X^{\mathrm{an}}$ corresponds to a component E of a snc model $\mathcal{X}$, the weight $\mathrm{wt}_\omega(x)$ measures how ω degenerates at E

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- More precisely, ω seen as a rational form on $\mathcal{X}$ defines a Cartier divisor $\mathrm{div}_{\mathcal{X}}(\omega)$, suppose locally around E we have $\mathrm{div}_{\mathcal{X}}\omega = \nu E$ with $\nu \in \mathbb{Z}$. Then

$$\mathrm{wt}_\omega(x) := \frac{\nu + 1}{\mathrm{mult}_{\mathcal{X}_s}(E)} - 1$$

  (the $+1$ & -1 come from considering logarithmic forms on $\mathcal{X}$)

Example

$E : y^2 = x^3 + 2/\widehat{\mathbb{Q}_2^{\mathrm{ur}}}$ has a snc model $\mathcal{X}$ with special fiber

special fiber type II.png

Let $\omega = dx/2y$ be the invariant differential. Then wt_ω looks like

wt function example.png

where wt_ω increases with constant slope 1 with respect to the induced metric by the $\mathbb{Z}$-affine structure.
For example if E is the green component one computes $\mathrm{div}_{\mathcal{X}}\omega = 4E$ around E and so $\mathrm{wt}_\omega(E) = \frac{4+1}{6} - 1 = -\frac{1}{6}$

- If X has a snc model, $\mathrm{wt}_\omega(x)$ is bounded below, and we can define the **Kontsevich-Soibelman skeleton** as

$$\mathrm{sk}(X,\omega) = \{x \in X^{\mathrm{an}} : \ \mathrm{wt}_\omega(x) \text{ is minimal}\}.$$

  In fact, it is a union of faces of the skeleton of any snc model.
- Define the **essential skeleton** as

$$\mathrm{sk}(X) = \bigcup_{\omega \neq 0} \mathrm{sk}(X,\omega).$$

Example (continued)

In the example before, $\mathrm{sk}(X)$ is a singleton corresponding to the divisorial point corresponding to the green component.

Proposition *(Mustață-Nicaise '15)*

$\mathrm{sk}(X)$ is a connected union of faces of any snc model (assuming the latter exists)

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(logarithmic) differents

Definition

Given a seperable extension of discretely valued fields L/K we define the **different** as

$$\delta_{L/K} := \mathrm{length}\left(\Omega_{L^\circ/K^\circ}\right)_{\mathrm{tor}},$$

similarly the **log different** $\delta^{\log}_{L/K} := \mathrm{length}\left(\Omega^{\log}_{L^\circ/K^\circ}\right)_{\mathrm{tor}}$.

in case of finite seperable extensions, δ (resp. $\delta^{\log}$) measures ramification (resp. wild ramification), namely it is zero iff the extension is unramified (resp. tame). Also,

$$\delta_{L/K} = \delta^{log}_{L/K} + e(L/K) - 1.$$

differents in algebraic geometry

Example (Riemann-Hurwitz formula)

If $f : X \to Y$ is a generically étale morphism of smooth K-curves, adjunction says

$$\omega_X \cong f^*\omega_Y \otimes \omega_{X/Y}.$$

A computation shows

$$\omega_{X/Y} \cong \Omega_{X/Y}/\mathcal{O}_X.$$

So the ramification divisor is

$$\sum_{x \in X \text{ closed}} \delta_{(K(X),v_x)/(K(Y),v_{f(x)})} (x).$$

differents in algebraic geometry

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Remark

Actually, the above argument still works for regular K°-varieties, as well as in higher dimensions, and there is a logarithmic version.

pullback of weight functions

The previous remark can be used to show

Proposition (W.)

Let $f : X \to Y$ be a generically étale morphism of smooth K-varieties. Let $x \in X^{\mathrm{an}}$ be divisorial. Then for all nonzero rational pluricanonical forms ω on Y we have

$$\mathrm{wt}_{f^*\omega}(x) - \mathrm{wt}_{\omega}(f(x)) = \frac{1}{e(\mathcal{H}(x)/K)} \delta^{\log}_{\mathcal{H}(x)/\mathcal{H}(f(x))} \quad =: \mathfrak{d}(\mathbf{x})$$

As a corollary, one can extend $\mathfrak{d}$ to a positive integral piecewise affine function on X^{an}.

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Remarks

- It follows essential skeleta pull back along tame Galois covers.
- In the algebraically closed case, Temkin e.a. have shown the existence of a similar different function

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Base change of nonarchimedean curves

Let C be a smooth, geometrically connected K-curve, and L/K Galois. Consider $\pi : C_L \to C$ and $\mathfrak{d} : C_L^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$.

Example: $y^2 = x^3 + 2/(K = \widehat{\mathbb{Q}_2^{\mathrm{ur}}})$ continued

The normalised base change of the minimal snc model to $L = K(\sqrt{2})$ has a unique singularity, which is resolved after one normalised blowup; at the node we have $\mathfrak{d} = \delta_{L/K}^{log} = 2$.

blowup normalised base change.png

Equip C and C_L with the metrics induced by the integral piecewise affine structures, the so-called **stable metric**. *(Baker-Nicaise '15)*

The potential theory of $\mathfrak{d}$ is less clear than in the algebraically closed case due to:

bad news

$\pi^{\mathrm{an}} : C_L^{\mathrm{an}} \to C^{\mathrm{an}}$ is not an isometry in general, and disagrees with the metric introduced by Ducros.

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definition

The **temperate part** of X^{an} is the closure of the tame divisorial points, i.e. those divisorial points x with $p \nmid e(\mathcal{H}(x)/K)$.

good news

π^{an} is an isometry above the temperate part of C^{an}. In general, locally the contraction factor is an integer dividing $[L : K]$.

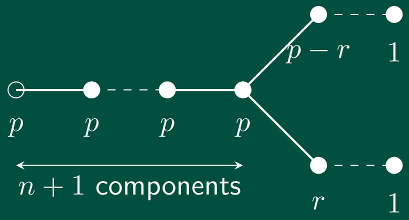
So in concrete cases like degree p extensions, we can work out the slopes of $\mathfrak{d}$ from the slopes of weight functions. Also, $\mathfrak{d} \equiv \delta_{L/K}^{\log}$ on the temperate part.

Curves with potential good ordinary reduction

Suppose moreover that C has bad reduction and C_L has good ordinary reduction and $[L : K] = p$. Let $\mathcal{X}$ be the smooth model of C_L and $x \in \mathcal{X}/\mathrm{Gal}(L/K)$ be singular.

Theorem

- *(Lorenzini '14)* The singularity in x is isolated and its resolution has dual graph



for some $n \in p\mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{\geq 0}$ with $p \nmid r$.

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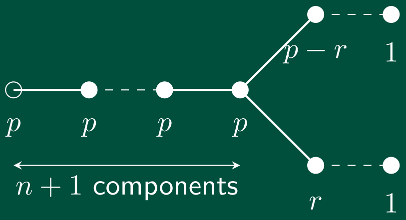
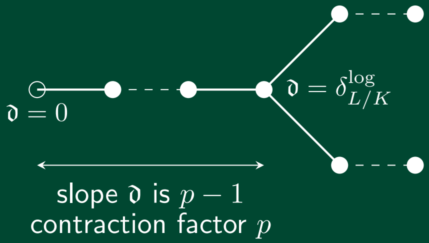


  for some $n \in p\mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{\geq 0}$ with $p \nmid r$.
- *(Obus-Wewers '20, W. '22)* $n = p \cdot (\text{jump in the ramification filtration of } L/K)$

How to determine n using $\mathfrak{d}$

- Base change the previous resolution graph to L:



- $\mathfrak{d} = 0$ on the strict transform (the smooth component), as the residue field extension is unramified
- $\mathfrak{d} \equiv \delta_{L/K}^{\log}$ above the temperate part of C^{an}, which includes the node
- one computes $\mathfrak{d}$ has constant slope $p-1$ on the segment between the strict transform and the node
- Can show directly that stable metric is not isometry, so length segment changes by contraction factor p.