Lecture 3: Non-collapsing for mean curvature flow and the Lawson and Pinkall-Sterling conjectures

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Outline of the talk:

- A quick survey of mean curvature flow
- Non-collapsing for mean curvature flow (and some applications)
- The Lawson conjecture (proof after Brendle)
- The Pinkall-Sterling conjecture for CMC tori (joint work with Haizhong Li)
- Generalisations and extensions

The mean curvature flow takes an initial embedding (or immersion) $X_0 : M^n \to \mathbb{R}^{n+1}$ (we assume M is connected and compact) and moves it in the direction of the mean curvature vector. Precisely, this means we produce a smooth map $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ such that $X(.,t)$ is an immersion for each t, and $X(.,0) = X_0$, such that

$$\frac{\partial X}{\partial t} = -H\nu.$$

Basic results:

- A solution exists for a short time and is unique;
- The solution continues to exist as long as the curvature remains bounded;
- Disjoint solutions remain disjoint;
- Embedded solutions remain embedded;
- $T < \infty$;
- (Huisken, 1984, Gage-Hamilton 1986) If X_0 is locally convex (embedded if $n = 1$), then $X_t \to p \in \mathbb{R}^{n+1}$ and

$$\tilde{X}_t = \frac{X_t - p}{\sqrt{2n(T-t)}} \to \tilde{X}_T \text{ in } C^\infty, \quad \tilde{X}_T(M) = S_1^n(0).$$

- (Grayson, 1987) If $n = 1$ and X_0 is an embedding, then the same conclusion holds without assuming convexity.

The analogue of Grayson's theorem is not true for $n \geq 2$: Singularities can appear without the whole hypersurface shrinking to a point. It is also not true without the assumption of embeddedness: A closed curve with a small 'loop' will develop a singularity where the loop shrinks away, with the curvature approaching infinity while the length of the curve does not approach zero.

There are many analogies between the mean curvature flow of hypersurfaces and the Ricci flow of Riemannian metrics, both in the behaviour of the flow and its analysis. In particular Huisken and Sinestrari proved an analogue for mean curvature flow of the Hamilton-Perel'man results for three dimensional Ricci flow:

If $n \geq 3$ and X_0 is 2-convex (i.e. the sum of the smallest two principal curvatures is positive at each point) then any point of high curvature has a neighbourhood which is close (modulo scaling) to either a sphere or a long part of a shrinking cylinder or a 'capped cylinder,' and it is possible to define mean curvature flow with surgery to deduce that the original hypersurface is a sphere or a connected sum of copies of $S^{n-1} \times S^1$.

The $n = 2$ analogue of the last result would be: A compact surface in $\mathbb{R}^3$ with positive mean curvature can be deformed using mean curvature flow with surgery through a finite number of surgeries to pieces which are spheres or tori. This is false without the assumption of embeddedness, and so cannot be proved by the methods Huisken and Sinestrari use. A recent paper of Brendle and Huisken proves this result with the added assumption of embeddedness, using the machinery I will describe.

How can we make use of embeddedness?

I described a method which works for $n = 1$ in my last lecture: The estimate on the isoperimetric profile (first done by Hamilton around 1995).

Another way to do this appeared in work of Huisken, independently but a little later: He considered embedded closed curves γ_t in the plane moving by curve shortening flow, and showed that a certain bound on the 'chord-arc profile' does not get any worse: Precisely, denote by $L(t)$ the total length of the curve γ_0, and by $d(x,y,t)$ and $\ell(x,y,t)$ the chord length and the arc length along γ_t between two points x and y. Then a chord-arc bound of the form

$$d(x,y,t) \geq \frac{L(t)}{c} \sin\left(\frac{\pi \ell(x,y,t)}{L(t)}\right)$$

with $c > \pi$ holds for all positive t provided it holds (for all x and y) at $t = 0$.

Question:
How can something like this work for higher dimensional mean curvature flow?

Using embeddedness: The non-collapsing condition

Brian White (using methods of geometric measure theory), and Weimin Sheng and Xujia Wang (using PDE methods): Regularity results for mean curvature flow of mean-convex hypersurfaces.

Sheng and Wang:
- Deliberate attempt to adapt Perelman's ideas to the mean curvature flow.
- Formulated a geometrically natural analogue of 'non-collapsing' for MCF:

A mean-convex solution of MCF is δ-non-collapsed if for every x in M, there is a ball B of radius $\delta/H(x,t)$ in Ω_t, with $X(x,t) \in \bar{B}$.

Sheng and Wang proved: If X_0 is embedded and mean convex ($H > 0$) then there exists $\delta > 0$ such that the solution $X: \ M \times [0,T) \to \mathbb{R}^{n+1}$ with initial data X_0 is δ-non-collapsed.

Their proof uses a compactness argument after a detailed analysis of the possible limits of rescalings of the flow near a singularity.

Idea: Write the non-collapsing condition as positivity of a two-point function so that a maximum principle to can be applied.

Let $X:\ M\times[0,T)\to\mathbb{R}^{n+1}$ be a (compact) embedded solution of MCF, and let Ω_t be the region enclosed by $M_t = X(M,t)$. Let ν be the 'outward-pointing' unit normal vector. I make the following definitions: For $x,y\in M$ with $y\neq x$, and $t\in[0,T)$,

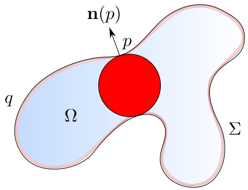
$$k(x,y,t) = \frac{2\left\langle X(x,t)-X(y,t),\nu(x,t)\right\rangle}{\|X(x,t)-X(y,t)\|^2},$$

and

$$\bar{k}(x,t) = \sup\{k(x,y,t):\ y\in M\setminus\{x\}\};$$

I call $\bar{k}$ the *interior ball curvature*.

Claim: $\bar{k}(x,t)$ is the boundary curvature of the largest ball in Ω_t touching at $X(x,t)$.



Claim: $\bar{k}(x,t)$ is the boundary curvature of the largest ball in Ω_t touching at $X(x,t)$.
Proof: Ball B in Ω_t touching at $X(x,t) \Longrightarrow B = B_r(p)$, where $p = X(x,t) - r\nu(x,t))$ for some $r > 0$.

$$\begin{aligned}
B_r(p) \subset \Omega_t &\Longleftrightarrow M_t \subset (B_r(p))^c \\
&\Longleftrightarrow \|X(y,t) - p\|^2 \geq r^2 \text{ for all } y \in M; \\
&\Longleftrightarrow \|(X(y,t) - X(x,t)) + r\nu(x,t)\|^2 \geq r^2 \text{ for all } y \in M; \\
&\Longleftrightarrow \|X(y,t) - X(x,t)\|^2 + 2r\langle X(y,t) - X(x,t), \nu(x,t)\rangle \geq 0 \text{ for all } y \in M; \\
&\Longleftrightarrow k(x,y,t) = \frac{2\langle X(x,t) - X(y,t), \nu(x,t)\rangle}{\|X(x,t) - X(y,t)\|^2} \leq \frac{1}{r} \text{ for all } y \in M \setminus \{x\}; \\
&\Longleftrightarrow \bar{k}(x,t) \leq \frac{1}{r}.
\end{aligned}$$

So a ball of boundary curvature k in Ω_t touches at $X(x,t)$ if and only if $\bar{k}(x,t) \leq k$, and the largest such ball is the one of boundary curvature equal to $\bar{k}(x,t)$. Note that $\bar{k}$ is no smaller than the largest principal curvature of the hypersurface.

The key to the maximum principle proof is the following claim:

(**) The function $\overline{k}$ is a viscosity subsolution of the equation

$$\frac{\partial \overline{k}}{\partial t} = \Delta \overline{k} + |A|^2 \overline{k}.$$

This equation is called the 'linearized MCF' and is very natural: If X_s is a smooth family of solutions to mean curvature flow, then the 'normal variation' $u = \left\langle \frac{\partial X_s}{\partial s}, \nu \right\rangle$ satisfies the equation above. In particular, taking $X_s(x,t) = X(x, s+t)$ we find that the mean curvature H is a solution.

Recall that the statement (**) means the following: If ϕ is a smooth function with $\phi(x_0, t_0) = \overline{k}(x_0, t_0)$ and $\phi(x,t) \geq \overline{k}(x,t)$ for x near x_0 and $t \leq t_0$ near t_0, then

$$\frac{\partial \phi}{\partial t} \leq \Delta \phi + |A|^2 \phi$$

at the point (x_0, t_0). It follows that any inequality $\bar{k} \leq CH$ is preserved, since $\overline{k} - CH$ is a viscosity subsolution of the linearised MCF which is initially non-positive, hence is everywhere non-positive. This implies the Sheng–Wang non-collapsing statement.

Now let us prove the statement (**), that $\overline{k}$ is a subsolution of the linearised MCF. Fix any (x_0,t_0) in $M\times[0,T)$, and let ϕ be a smooth function defined near (x_0,t_0) such that $\phi(x_0,t_0)=\overline{k}(x_0,t_0)$, and $\phi(x,t)\geq\overline{k}(x,t)$ for (x,t) near (x_0,t_0) with $t\leq t_0$. By definition of $\overline{k}$ we also have $\phi(x,t)\geq k(x,y,t)$ for all $(x,y,t)\in M\times M\times[0,t_0]$ with (x,t) close to (x_0,t_0).

We have two possibilities:

Case 1: The supremum $\bar{k}(x_0,t_0)=\sup\{k(x_0,y,t_0):\ y\neq x_0\}$ is not attained. In this case the supremum must be attained for a sequence of points y converging to x_0, and $\bar{k}(x_0,t_0)$ is equal to the largest principal curvature $\kappa_{\max}(x_0,t_0)$. But then we also have $\phi(x,t)\geq\overline{k}(x,t)\geq h_{(x,t)}(e,e)$ for any smooth unit vector field near (x_0,t_0), and we have equality at (x_0,t_0) if we choose $e(x_0,t_0)$ to be the direction of the maximum principal curvature. This implies

$$\left(\frac{\partial}{\partial t}-\Delta\right)\phi\big|_{(x_0,t_0)}\leq\left(\frac{\partial}{\partial t}-\Delta\right)h(e,e)\big|_{(x_0,t_0)}.$$

But under MCF, if we define e by parallel transport from (x_0,t_0) then we have

$$\left(\frac{\partial}{\partial t}-\Delta\right)h(e,e)\big|_{(x_0,t_0)}=-|A|^2\kappa_{\max}=-|A|^2\phi,$$

as required.

Case 2: The supremum is attained — there exists $y_0 \neq x_0$ such that $\overline{k}(x_0,t_0) = k(x_0,y_0,t_0)$. In this case since $\phi(x,t) \geq \overline{k}(x,t) \geq k(x,y,t)$ with equality at (x_0,y_0,t_0) we have

$$\frac{\partial \phi}{\partial t}(x_0,t_0) \leq \frac{\partial k}{\partial t}(x_0,y_0,t_0).$$

To get a good inequality on the second derivatives, it is useful to first look closer at the geometry of the situation: We have by assumption that M_t lies outside the ball of radius $1/\overline{k}$ with centre at $p = X(x) - \frac{1}{\overline{k}}\nu(x)$, and that both $X(x_0,t_0)$ and $X(y_0,t_0)$ lie on the boundary of this ball. Let w be the unit vector from $X(x_0,t_0)$ to $X(y_0,t_0)$. Then $T_{x_0}M_{t_0}$ and $T_{y_0}M_{t_0}$ are related by reflection in the hyperplane orthogonal to w: The reflection

$$R: \ v \mapsto v - 2(w \cdot v)w$$

takes tangent vectors at x_0 to tangent vectors at y_0 and the normal vector $\nu(x_0,t_0)$ to $\nu(y_0,t_0)$. We choose geodesic normal coordinates $\{x^i\}$ for M near x_0 such that the second fundamental form at x_0 is diagonal, and choose normal coordinates $\{y^i\}$ near y_0 such that $\frac{\partial X}{\partial y^i} = R\left(\frac{\partial X}{\partial x^i}\right)$ for each i. Then we have the second derivative inequality

$$\Delta\phi\big|_{(x_0,t_0)} \geq \sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)^2 k\big|_{(x_0,y_0,t_0)} =: Lk\big|_{(x_0,y_0,t_0)}.$$

It follows that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi\big|_{(x_0,t_0)} \leq \left(\frac{\partial}{\partial t} - L\right)k\big|_{(x_0,y_0,t_0)}.$$

The time derivative is as follows: (where $d = \|X(y) - X(x)\|$ and $w = \frac{X(y)-X(x)}{\|X(y)-X(x)\|}$):

$$\begin{aligned} \frac{\partial}{\partial t} k &= \frac{\partial}{\partial t}\left(-\frac{2\langle X(y) - X(x), \nu(x)\rangle}{\|X(y) - X(x)\|^2}\right) \\ &= -\frac{2}{d^2}\left(\langle dw, \nabla H(x)\rangle + \langle -H(y)\nu(y) + H(x)\nu(x), \nu(x) + kdw\rangle\right) \end{aligned}$$

Next we compute first derivatives of k:

$$\begin{aligned} \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right) k &= \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)\left(-\frac{2\langle X(y) - X(x), \nu(x)\rangle}{\|X(y) - X(x)\|^2}\right) \\ &= -\frac{2}{d^2}\left(\left\langle dw, h_i^p(x)\frac{\partial X}{\partial x^i}\right\rangle + \left\langle \frac{\partial X}{\partial y^i} - \frac{\partial X}{\partial x^i}, \nu(x) + kdw\right\rangle\right) \end{aligned}$$

Note that at (x_0, y_0, t_0) we have the following first derivative conditions:

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial k}{\partial x^i} = \frac{2(k - \kappa_i)}{d}\left\langle w, \frac{\partial X}{\partial x^i}\right\rangle;$$

while the derivatives of k with respect to y_i vanish.

$$\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)k=-\frac{2}{d^2}\left(\left\langle dw, h_i^p(x)\frac{\partial X}{\partial x^i}\right\rangle+\left\langle\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i}, \nu(x)+k\,dw\right\rangle\right)$$

$$\begin{aligned}
Lk &= \sum_i\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)^2 k\\
&= -\frac{2}{d^2}\Bigg(\left\langle dw, \nabla H(x)\ -|A(x)|^2\nu(x)\right\rangle+\left\langle\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i},\ 2\ h_i^p(x)\frac{\partial X}{\partial x^p}+2\frac{\partial \phi}{\partial x^i}dw\right\rangle\\
&\qquad +\langle -H(y)\nu(y)+H(x)\nu(x), \nu(x)+k\,dw\rangle\ +k\left\|\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i}\right\|^2\Bigg)
\end{aligned}$$

From this we get

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-L\right)k &= |A|^2k+\frac{2}{d^2}\left(2\left\langle\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i}, \kappa_i\frac{\partial X}{\partial x^i}+\frac{\partial \phi}{\partial x^i}dw\right\rangle+k\left\|\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i}\right\|^2\right)\\
&= |A|^2k+\frac{2}{d^2}\left(4(k-\kappa_i)\left\langle w, \frac{\partial X}{\partial x^i}\right\rangle^2-4\frac{\partial \phi}{\partial x^i}\left\langle dw, \frac{\partial X}{\partial x^i}\right\rangle\right)\\
&= |A|^2\phi-\sum_i\frac{2}{k-\kappa_i}\left(\frac{\partial \phi}{\partial x^i}\right)^2\le |A|^2\phi.
\end{aligned}$$

So we have $(\partial_t-\Delta)\phi\le|A|^2\phi$ as required, proving the non-collapsing result for mean curvature flow. Haslhofer and Kleiner recently used this estimate as the basis for a regularity theory for the mean curvature flow of mean-convex hypersurfaces, and Brendle and Huisken used it to perform mean curvature flow with surgery on mean-convex surfaces.

The 'Lawson conjecture' dates back to a paper of Lawson from 1970:

Any embedded minimal torus in S^3 is congruent to the Clifford torus $\mathcal{C} = S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$.

Some background behind the conjecture:

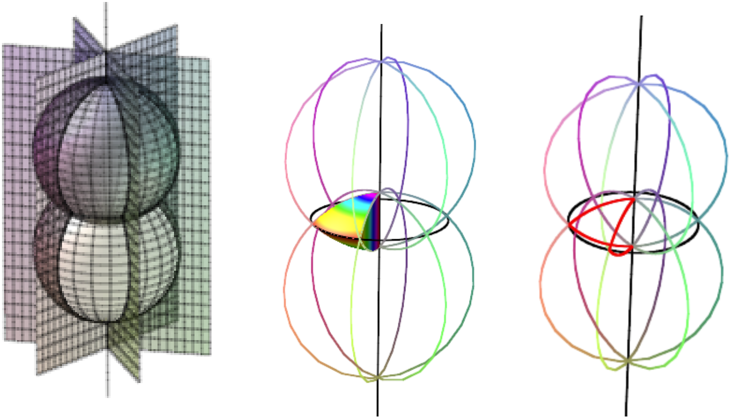
• First, the conjecture only makes sense for tori:

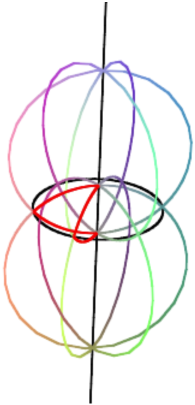
For spheres, Almgren proved using the holomorphicity of the Hopf differential that the only minimal 2-spheres in S^3 are the equators (given by the intersection of a 3-plane in $\mathbb{R}^4$ with S^3).

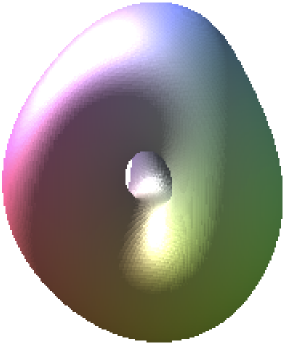
In another 1970 paper, Lawson constructed examples of embedded minimal surfaces in S^3 of any higher genus, and there are now several other constructions known as well, but no reasonable conjecture for an analogous classification or rigidity statement in the higher genus case.

A quick sketch of Lawson's construction:

The idea is to cut the three-sphere $\{(x,y,w,z): \ x^2+y^2+w^2+z^2=1\}$ into congruent pieces by symmetrically-placed great two-spheres. For example, take the orthogonal two-spheres $\{w=z\}$ and $\{w=-z\}$, and the three two-spheres given by $\{y=0\}$, $\{y=\sqrt{3}x\}$ and $\{y=-\sqrt{3}x\}$ which meet at equal angles $\pi/3$, decomposing S^3 into 24 identical convex tetrahedra.







By slicing by k equal-angle two-spheres in the $x-y$ plane and m equal-angle two-spheres in the $w-z$ plane, this construction produces a compact embedded minimal surface of genus $(k-1)(m-1)$.

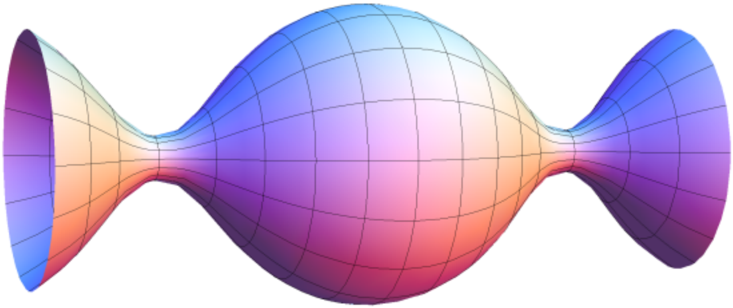
If $k=m=2$ this is exactly the Clifford torus $\mathcal{C}$.

Embeddedness is crucial: Without embeddedness the result is false.

Surfaces of rotation:

• minimal surface equation reduces to an ODE;

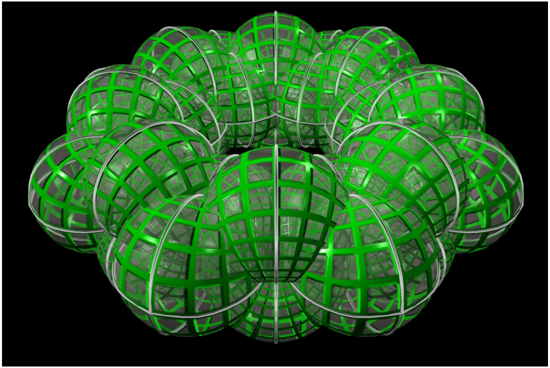
• solutions similar to the constant mean curvature Delaunay surfaces in $\mathbb{R}^3$



• rational 'period integral' produces an immersed minimal torus , so there are infinitely many examples.

• Integrality condition for embeddedness. Otsuki (1970): No embedded examples of this kind other than the Clifford torus.

Minimal and CMC surfaces have an integrable structure, used by Pinkall and Sterling to construct (all) immersed CMC tori in S^3 (or $\mathbb{R}^3$ or $\mathbb{H}^3$) as 'finite gap' solutions which can be constructed algebraically (and in fact written explicitly in terms of elliptic functions, as shown by Bobenko). Very difficult to say anything about embeddedness (approach by Hauswirth, Kilian and Schmidt seems likely to work in this situation).



Picture thanks to Mathias Heil

Simon Brendle's proved the Lawson conjecture, using some clever adaptations of the machinery I have just described. One potential obstacle is that we are now looking at a surface in the 3-sphere instead of in Euclidean space. In fact this causes no problems:

- Exactly as before, the function $\overline{k}(x) = \sup_{y \in M \setminus \{x\}} \left\{ \frac{2\langle X(x)-X(y), \nu(x)\rangle}{\|X(x)-X(y)\|^2} \right\}$ is the boundary curvature of the largest ball which touches the surface at $X(x)$ in the $-\nu(x)$ direction.
- The function $\underline{k}(x) = \inf_{y \in M \setminus \{x\}} \left\{ \frac{2\langle X(x)-X(y), \nu(x)\rangle}{\|X(x)-X(y)\|^2} \right\}$ is minus the boundary curvature of the largest ball which touches the surface at $X(x)$ in the $+\nu(x)$ direction.
- The computation I gave above to prove non-collapsing changes in only small ways.

The time derivative is as follows: (where $d = |X(y) - X(x)|$ and $w = \frac{X(y)-X(x)}{|X(y)-X(x)|}$):

$$\frac{\partial}{\partial t}k = \frac{\partial}{\partial t}\left(-\frac{2\langle X(y)-X(x),\nu(x)\rangle}{|X(y)-X(x)|^2}\right)$$
$$= -\frac{2}{d^2}\left(\langle dw, \nabla H(x) + H(x)X(x)\rangle + \langle -H(y)\nu(y) + H(x)\nu(x), \nu(x) + kdw\rangle\right)$$

Next we compute first derivatives of k:

$$\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)k = \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)\left(-\frac{2\langle X(y)-X(x),\nu(x)\rangle}{|X(y)-X(x)|^2}\right)$$
$$= -\frac{2}{d^2}\left(\left\langle dw, h_i^p(x)\frac{\partial X}{\partial x^i}\right\rangle + \left\langle \frac{\partial X}{\partial y^i} - \frac{\partial X}{\partial x^i}, \nu(x) + kdw\right\rangle\right)$$

Note that at (x_0, y_0, t_0) we have the following first derivative conditions:

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial k}{\partial x^i} = -\frac{2(k-\kappa_i)}{d}\left\langle w, \frac{\partial X}{\partial x^i}\right\rangle;$$

while the derivatives of k with respect to y_i vanish.

$$\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)k=-\frac{2}{d^2}\left(\left\langle dw,h_i^p(x)\frac{\partial X}{\partial x^i}\right\rangle+\left\langle\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i},\nu(x)+kdw\right\rangle\right)$$

$$\begin{aligned}Lk&=\sum_i\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)^2k\\&=-\frac{2}{d^2}\left(\left\langle dw,\nabla H(x)-|A(x)|^2\nu(x)-H(x)X(x)\right\rangle+\left\langle\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i},2h_i^p(x)\frac{\partial X}{\partial x^p}+2\frac{\partial\phi}{\partial x^i}dw\right\rangle\right.\\&\qquad\left.+\langle -H(y)\nu(y)+H(x)\nu(x)-nX(y)+nX(x),\nu(x)+kdw\rangle+k\left\|\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i}\right\|^2\right)\end{aligned}$$

From this we get

$$\begin{aligned}\left(\frac{\partial}{\partial t}-L\right)k&=|A|^2k+\frac{2}{d^2}\left(2\left\langle\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i},\kappa_i\frac{\partial X}{\partial x^i}+\frac{\partial\phi}{\partial x^i}dw\right\rangle+k\left\|\frac{\partial X}{\partial y^i}-\frac{\partial X}{\partial x^i}\right\|^2+\cdots\right)\\&=|A|^2k+\frac{2}{d^2}\left(4(k-\kappa_i)\left\langle w,\frac{\partial X}{\partial x^i}\right\rangle^2-4\frac{\partial\phi}{\partial x^i}\left\langle dw,\frac{\partial X}{\partial x^i}\right\rangle+\cdots\right)\\&=|A|^2\phi-\sum_i\frac{2}{k-\kappa_i}\left(\frac{\partial\phi}{\partial x^i}\right)^2\leq|A|^2\phi.+2H-n\phi\end{aligned}$$

So we have $(\partial_t-\Delta)\phi\leq|A|^2\phi$ as required, proving the non-collapsing result for mean curvature flow. $(\partial_t-\Delta)\phi\leq|A|^2\phi+2H-n\phi$, so $\bar{k}$ satisfies $\partial_t\bar{k}\leq\Delta\bar{k}+|A|^2\bar{k}+2H-n\bar{k}$ in the viscosity sense. Since $\partial_tH=\Delta H+H|A|^2+nH$, we can preserve $\bar{k}\leq cH$ for $c\geq 1/n$.

Of course, any minimal hypersurface is a (stationary) solution of mean curvature flow, so the computation above holds in particular for minimal embedded hypersurfaces in the sphere. However, in mean curvature flow we could only get a useful conclusion by comparing the interior ball curvature $\overline{k}$ with the mean curvature, and this is zero for a minimal surface.

Brendle's beautiful idea is to compare $\overline{k}$ with the maximum principal curvature κ instead of the mean curvature. One nice reason for trying this is that the holomorphic Hopf differential allows one to prove that a minimal torus has no umbilic points, so κ is nowhere zero and so is a smooth positive function.

A second reason why it is a great idea to compare $\overline{k}$ to κ is because in the Clifford torus we have $\overline{k} = \kappa$ at every point: The lines of curvature in the Clifford torus are circles in $\mathbb{R}^4$, which are in the boundary of the largest touching sphere at every point they pass through.

Brendle in fact shows using the maximum principle that $\overline{k} = \kappa$ everywhere on an embedded minimal torus.

Recall that our argument for mean curvature flow in the sphere proved that $\bar{k}$ satisfies

$$\frac{\partial}{\partial t}\bar{k} \le \Delta\bar{k} + |A|^2\bar{k} + H - n\bar{k} - \sum_{i=1}^{n} \frac{2}{\bar{k} - \kappa_i}\left(\frac{\partial \bar{k}}{\partial x^i}\right)^2.$$

So for a minimal 2-surface we have (since $H = 0$ and $n = 2$)

$$0 \le \Delta\bar{k} - \sum_{i=1}^{2} \frac{2}{\bar{k} - \kappa_i}\left(\frac{\partial \bar{k}}{\partial x^i}\right)^2 + (|A|^2 - 2)\bar{k}.$$

Choose $\kappa_1 = \kappa > 0$, so $\kappa_2 = -\kappa$. Then $\bar{k} \ge \kappa$, so $\bar{k} - \kappa_i \le \bar{k} + \kappa \le 2\bar{k}$. This gives

$$0 \le \Delta\bar{k} - \frac{|D\bar{k}|^2}{\bar{k}} + (|A|^2 - 2)\bar{k} = \bar{k}\left(\Delta \log \bar{k} + (|A|^2 - 2)\right).$$

The Simons' identity for a minimal 2-surface gives

$$0 = \Delta\kappa - \frac{|D\kappa|^2}{\kappa} + (|A|^2 - 2)\kappa = \kappa\left(\Delta \log \kappa + |A|^2 - 2\right).$$

Combining these, we find

$$0 \le \Delta\left(\log \bar{k} - \log \kappa\right),$$

so $\bar{k} = C\kappa$ for some $C \ge 1$. But any surface has points where $\bar{k} = \kappa$, so $C = 1$.

This shows that $\bar{k} = \kappa$ everywhere, so the osculating sphere at each point does not intersect the surface.

It follows that $\nabla_1 h_{11} = 0$. The minimal surface equation then gives $\nabla_1 h_{22} = 0$.

Now consider balls touching on the other side of the surface: The same argument shows that $\underline{k} = -\kappa$, and it follows that $\nabla_2 h_{22} = 0$. The minimal surface equation gives $\nabla_2 h_{11} = 0$. By the Codazzi equation, the second fundamental form is parallel. The curvature of the surface is therefore constant, hence zero by the Gauss-Bonnet theorem: The surface is flat. It now follows easily that the surface is congruent to the Clifford torus, and the Lawson conjecture is proved.

I will now briefly describe my work with Haizhong Li extending Brendle's idea to constant mean curvature (CMC) surfaces:

At first sight it seems unlikely that Brendle's method could work, because there are known examples of embedded CMC tori which are not just product of circles. Examples can be constructed in the class of surfaces of rotation, by solving a certain ordinary differential equation. In fact it was shown by Perdomo that for any nonzero value of H other than $H = \pm\frac{2}{\sqrt{3}}$ there exists an embedded CMC torus with this value of H which is not a product torus.

However, Haizhong and I proved that one can recover something similar to Brendle's argument by considering spheres touching only on one side of the surface (the side that the mean curvature vector points towards). We conclude that $\bar{k} = \kappa$, where κ is the largest principal curvature. It follows that $\nabla_1 h_{11} = \nabla_1 h_{22} = 0$. This is enough for us to conclude that the surface is a surface of rotation, and we end up with a complete classification of embedded CMC tori in S^3. In particular we prove that CMC tori with mean curvature $\pm\frac{2}{\sqrt{3}}$ are product tori.

We later found out that the axial symmetry of embedded CMC tori in S^3 was an explicit conjecture in a paper of Ulrich Pinkall and Ivan Sterling in 1989.

I will finish with a quick description of some recent work with Xuzhong Chen (currently at ECNU in Shanghai). We were interested in extending the argument to cover surfaces satisfying other kinds of curvature equations. This can be done for a reasonable wide variety of examples, of which I mention only one:

Theorem

Let Σ be a compact embedded torus in S^3 satisfying $\kappa_2 + a\kappa_1 = b$, for $0 < a \leq 1$ and $b \geq 0$. Then Σ is rotationally symmetric (with circles of symmetry parallel to the principal direction corresponding to κ_1). If $b = 0$ then Σ is a Clifford torus.

An additional issue to be overcome here is that the equation is now fully nonlinear, and we have to handle the fact that principal directions at the two points x and y might not align favourably.