

Reasoning and uncertainty
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Re-visiting Giles's Game

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Plan of the talks

Part 1

- ▶ Who was Robin Giles? What is Giles's Game about?
- ▶ Dispersive experiments and dialogue rules for connectives
- ▶ Adequateness of Giles's game for Łukasiewicz logic

Part 2

- ▶ Beyond Łukasiewicz logic:
From games to truth tables

Conclusion and outlook

Who was Robin Giles?

A quote from Hmolpedia (via MediaWIKI):

*Robin Giles (c.1935–) is an Irish mathematical physicist noted for 1964 book **Mathematical Foundations of Thermodynamics** ... associated with the **Department of Natural Philosophy at the University of Glasgow** ...*

later Dept. of Mathematics, Queen's University, Kingston, Canada

Of central interest here:

A non-classical logics for physics, *Studia Logica*, 1974

based on an earlier extended version that appeared in *Selected Papers on Łukasiewicz Sentential Calculi*, Warsaw, 1977.

Giles later advertised his game as providing 'tangible meaning' to fuzzy logic / fuzzy set theory:

A formal system for fuzzy reasoning, *Fuzzy Set and Systems*, 1979

The concept of grade of membership, *Fuzzy Set and Systems*, 1988

Giles about reasoning within theories of physics

Key words for Giles's analysis of reasoning:

- ▶ all assertions have to be tested with respect to concrete (instances of) elementary experiments
- ▶ each atomic assertion p is connected to a particular elementary experiment E_p that may either fail or succeed
- ▶ experiments may show dispersion: different instances of the same experiment may yield different results
- ▶ to provide tangible meaning for arbitrary sentences one imagines a dialogue between me and you, both willing to pay 1\$/€ to the opponent for each false atomic assertion, i.e., one where the corresponding instance of the experiment fails
- ▶ since experiments are dispersive, assertions are risky
- ▶ a tenet collects all assertions of a player (me or you) represented by a multiset of occurrences of formulas over some interpreted language

Important observations:

- ▶ I can quantify the **expected loss** for **my** tenet $\{q_1, \dots, q_n\}$ (= **multiset**) of **atomic assertions** by assigning a **subjective failure probability** $\langle q_i \rangle$ to the experiment E_{q_i} .
- ▶ While these **probabilities** may have some **objective grounds** they are still **subjective** in the sense that I don't care which values **you** associate with the same experiments.
- ▶ **Events** are (unrepeatable) **instances** of (repeatable) elementary experiments. In other words: experiments are **event types**, such that the same probabilities are assigned to events of the same type.
- ▶ An **elementary** (or: **atomic, final**) state of the game is denoted by $[p_1, \dots, p_n \parallel q_1, \dots, q_m]$, where $\{p_1, \dots, p_n\}$ is **your** tenet and $\{q_1, \dots, q_m\}$ is **my** tenet of assertions.
My corresponding **risk**, i.e., **my** expected loss of money is

$$\sum_{1 \leq i \leq m} \langle q_i \rangle \text{€} - \sum_{1 \leq j \leq n} \langle p_j \rangle \text{€}$$

What about logically complex statements?

NB: So far, no logical connectives are involved!

For the reduction of logically complex assertions to atomic states Giles refers to the **logical rules** introduced by **Paul Lorenzen** in his dialogue game for constructive reasoning.

Giles states the rules in the following (old fashioned) way:

- ▶ *He who asserts $A \supset B$ agrees to assert B if his opponent will assert A .*
- ▶ *He who asserts $A \vee B$ undertakes to assert either A or B at his own choice.*
- ▶ *He who asserts $A \wedge B$ undertakes to assert either A or B at his opponent's choice.*

Defining $\neg A = A \supset \perp$ leads to

- ▶ *He who asserts $\neg A$ agrees to pay 1€ to his opponent if he will assert A .*

Observations about the dialogue part of Giles's game

- (1) Each assertion can be attacked at most once: we want to respect *repetita juvant*. Since repetitions are risky, the players may choose not to attack an assertion.
- (2) In contrast to Lorenzen:
 - ▶ no regulations on the succession of moves between me and you
 - ▶ no restrictions on which compound statements can be attacked
- (3) Giles defends the \wedge -rule by reference to the principle of *limited liability*: each assertion carries a maximal risk of 1€. There is no rule for strict conjunction ($\&$) – we'll add one.
- (4) While Lorenzen seeks to characterize (intuitionistic) validity by reducing to *ipse dixisti states*, Giles reduces to a given many valued interpretation of atomic statements.
In this respect Giles's game is more like a Hintikka style evaluation game than a Lorenzen style dialogue game.

Adequateness of Giles's game for Ł

Theorem (coarse version):

I have a strategy for avoiding expected loss for precisely those initial statements that are valid in Łukasiewicz logic.

Theorem (refined version):

Suppose we play the game starting with my assertion of F with respect to given assignment $\langle \cdot \rangle$ of risk values to atomic assertions.

The following are equivalent:

- ▶ F evaluates to $1-x$ in Łukasiewicz logic under the interpretation that assigns $1 - \langle p \rangle$ to each atom p .
- ▶ My best strategy guarantees that the play ends in an elementary state, where my risk is not higher than $x \in$, but you have a strategy enforcing an elementary state, where my risk is not less than $x \in$.

Remarks on the proof of Giles's Theorem

We have to show that **my risk** $\langle \cdot \rangle$ can be extended from elementary to arbitrary states in such a way that

$$\langle \Gamma \parallel A \supset B, \Delta \rangle = \max(\langle \Gamma \parallel \Delta \rangle, \langle \Gamma, A \parallel B, \Delta \rangle) \quad (1)$$

$$\langle \Gamma, A \supset B \parallel \Delta \rangle = \min(\langle \Gamma \parallel \Delta \rangle, \langle \Gamma, B \parallel A, \Delta \rangle) \quad (2)$$

(analogous conditions have to hold for other connectives)

This can be achieved by defining

$$\langle \Gamma \parallel \Delta \rangle^v =_{def} |\Delta| - |\Gamma| + \sum_{G \in \Gamma} v(G) - \sum_{F \in \Delta} v(F).$$

for the valuation v assigning $1 - \langle p \rangle$ to each atom p .

The fact that **no regulations** are needed in Giles's game falls out from the proof. From a game theoretic point of view it is more natural to assume regulations, and **prove** that they don't affect the players' respective 'power'. (See [FM, StudLog09])

Truth functional probabilities ?!

Claim: no confusion between *probabilities* and *degrees of truth* is implied by Giles's model of reasoning:

The following is perfectly true:

Any instance of a binary experiment either succeeds or does not succeed.

However, $p \vee \neg p$ is analyzed by Giles as:

*An instance of experiment E_p succeeds or **another** instance of experiment E_p does not succeed.*

If E_p has success probability 0.8 then the risk associated to the assertion $p \vee \neg p$ is $\min(0.8, 1 - 0.8) = 0.2$ according to the rules providing 'tangible meaning' to disjunction and negation.

NB: The *final (total) risk* associated with complex statements (according to Giles's reduction rules) refers to **independent events**. It can be computed according to the **min-max principle** of game theory and thus **propagates functionally**.

Towards Part 2: Adding strong conjunction

Recall Giles's rule for conjunction:

- ▶ *He who asserts $A \wedge B$ undertakes to assert either A or B at his opponent's choice.*

Why don't we have to assert both conjuncts?

Limited liability principle: one is never forced to risk more than 1€.

However there is an even more direct way to respect that principle:
[Giles-style:]

- ▶ *He who asserts $A \& B$ undertakes to assert either both, A and B , or else pay 1€ to the opponent, at his own choice.*

[Chris-style:]

- ▶ *Asserting $A \& B$ obliges one to assert either A and B , or else to pay 1€.*

Even better: forget about the 'or else'-part of the rule and add the following **general principle of limited liability**:

- ▶ **To any attack on a (logically complex) assertion one has either to reply according to the appropriate rule or else pay to 1€.**

Beyond Łukasiewicz logic

Note: all n -valued Łukasiewicz logics \mathbf{L}_n are characterized if possible risk values are restricted to $\{\frac{i}{n-1} : 0 \leq i < n\}$.

CHL is characterized by removing experiments that always fail.

There are also game characterizations of **Gödel logic G** and of **Product logic P**. However there is a high(?) price to pay:

- ▶ the **refined version** of the adequateness theorem **fails**:
only **validity**, but **not (graded) truth** is captured
- ▶ An **additional flag** has to be introduced.
(Alternatively: two types of states are needed.)
- ▶ The **implication rule** has to be **extended** in a somewhat problematic manner.

NB: these latter game variants are still well worth investigating. They directly correspond to a kind of **uniform hypersequent calculus** for \mathbf{L} , \mathbf{G} , and \mathbf{P} .

Liberating from risk: the case of Abelian logic **A**

NB: The interpretation of intermediary truth values as risk values and the corresponding story about dispersive experiments is completely **independent** from the dialogue game.

From a game theoretic perspective, one does not need to talk about probabilities or risk at all. These numbers are nothing but **inverted payoff values**.

From now on, we will reverse the inversion and assume that players are **maximizing payoff**, instead of minimizing expected payments according to a particular betting scheme.

Three simple modifications turn Giles's game for **L** into one for Abelian logic **A**:

- (1) allow **arbitrary real numbers** as payoff values
- (2) **drop** the **option not to attack** an implication
- (3) **drop** the ('or else pay 1€') **principle of limited liability**

General payoff principles

We are (still) interested in games ending in states where **my** atomic assertions faces **your** atomic assertions.

While the **order of assertions** is irrelevant, **repetitions** are not. In other words:

- ▶ **final states** take the form of pairs of **multisets of atomic statements (tenets)**, denoted $[\Gamma \parallel \Delta]$
- ▶ the corresponding **payoff value** ($\in \mathbb{R}$) is denoted by $\langle \Gamma \mid \Delta \rangle$

A payoff function $\langle \cdot \mid \cdot \rangle$ is called

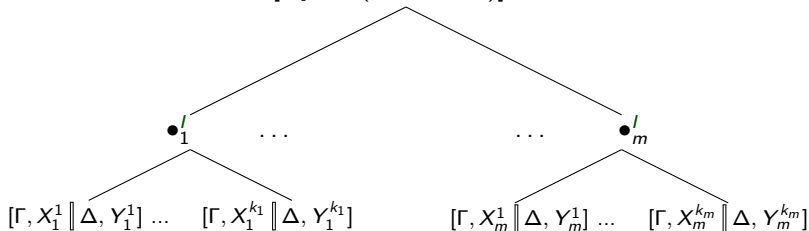
- ▶ **context independent** if $\langle \Gamma' \mid \Delta' \rangle = \langle \Gamma'' \mid \Delta'' \rangle$ implies $\langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle = \langle \Gamma, \Gamma'' \mid \Delta'', \Delta \rangle$
- ▶ **monotone** if $\langle \Gamma' \mid \Delta' \rangle \leq \langle \Gamma'' \mid \Delta'' \rangle$ implies $\langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle \leq \langle \Gamma, \Gamma'' \mid \Delta'', \Delta \rangle$
- ▶ **symmetric** if $\langle \Gamma \mid \Delta \rangle = -\langle \Delta \mid \Gamma \rangle$

We call payoff functions that are context independent, monotone, and symmetric **discriminating**.

General format of (decomposing) dialogue rules

You may attack my assertion of $\diamond(A_1, \dots, A_n)$ in different ways:

$$[\Gamma \parallel \Delta, \diamond(A_1, \dots, A_n)]^{You}$$



where X_i^j, Y_i^j are multisets of the form $\{A_1^{\ell_1}, \dots, A_n^{\ell_n}\} \cup C$ for some multiset C of truth constants.

Rules for my attacks on your assertions of $\diamond(A_1, \dots, A_n)$ are dual!

The presence of a node $[\Gamma \parallel \Delta]$ amounts to granting (no attack).

Main result

Theorem

Let \mathcal{D} be a game with discriminating payoff function $\langle \cdot \mid \cdot \rangle$ and decomposing dialogue rules respecting duality. Then one can extract from $\langle \cdot \mid \cdot \rangle$ and the rules a set truth functions $\mathcal{F}_{\mathcal{D}}$ over \mathbb{R} such that the following two values are equivalent for every formula A :

- ▶ the optimal payoff guaranteed by my best strategy for a \mathcal{D} -play starting in $[\parallel A]$,
- ▶ the truth value of A according to $\mathcal{F}_{\mathcal{D}}$ under the interpretation that assigns $\langle \mid p \rangle$ to p for all atomic formulas p .

In other words: discriminating payoff and dual decomposing rules are sufficient for a game to characterize a many valued logic!

Some comments

- ▶ The proof of the theorem relies on the fact that **context independent** payoff is sufficient to guarantee the existence of an AC-function \circ s.t. $\langle \Gamma, \Gamma' \mid \Delta, \Delta' \rangle = \langle \Gamma \mid \Delta \rangle \circ \langle \Gamma' \mid \Delta' \rangle$.
The decomposing and dual rules induce truth functions via the **min-max principle**.
- ▶ **Restricted truth value sets** (like $[0, 1]$ for \mathbf{L}) are obtained by examining which payoffs for single formulas result from corresponding restrictions of $\langle \mid p \rangle$ for atomic p .
- ▶ A lot of interesting many-valued logics, like **G** and **P** (provably) **do not admit** a characterization in terms of Giles-style dialogue games.
- ▶ Among the games covered are: \mathbf{L}_n for all $n \geq 2$, **L**, **CHL**, **A**; but also **extensions** of such logics by arbitrary truth constants and new connectives.

Summary (of Part 2)

- ▶ We have **isolated** the essential **principles** underlying Giles's characterization of Łukasiewicz logic and synthesized a corresponding general 'toolkit' for **assembling games** that are **adequate** for a certain type of **truth functional logics**.

Topics for further research:

- ▶ Concise **characterization** of the class of logics where truth functions can be extracted from a Giles style game.
NB: this should allow to **prove also negative results**.
- ▶ **Connection to proof theory**: do winning strategies always correspond to analytic proofs in a hypersequent system?
- ▶ **Generalizing further** in different directions, e.g.:
 - ▶ only care about **winning conditions** / **designated values**
 - ▶ allow for **non-decomposing rules**
 - ▶ consider **other types of states**
 - ▶ ...

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