On Coherence and Conditionals

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Workshop on ”*Reasoning and uncertainty: probabilistic, logical, and psychological perspectives*”
Regensburg, August 9-10, 2022

Outline

• We recall the *three levels of knowledge* on events (de Finetti 1980), with a hint on the extension to conditional events;

• We describe different *equivalent schemes* for making conditional probability assessments, in particular conditional bets and bets on conditionals;

• We show the equivalence among conditions of coherence based on *random gains* and *geometrical conditions* based on convex hulls;

• We show that the *conjunction* coincides with many (apparently different) *conditional random quantities*; we examine a criticism by Dorothy Edgington and we illustrate the characterization of p-entailment of Adams;

• We briefly examine some intuitively valid probabilistic assertions on complex sentences which we formalize by iterated conditionals;

• We give a look at some basic logical and probabilistic properties valid for unconditional events, *preserved in our approach to compound conditionals*, which are not satisfied in general in the setting of trivalent logics.

The three levels of knowledge on events

Given any event E, in the paper

B. de Finetti (1980), Probabilitá, Enciclopedia, 1146 - 1187, Einaudi

are illustrated "three levels of knowledge" on E: Level 0, or 1, or 2.

Level 0, *logical point of view:* E can be false, 0, or true, 1.

Level 1, *cognitive point of view:* E can be false, 0, or uncertain, ?, or true, 1.

Level 2, *psychological (subjective) point of view:* in case of certainty E is false, 0, or true, 1. In case of uncertainty, E is considered with a (subjective) probability $0 \leq P(E) \leq 1$.

These levels of knowledge, under study in a joint (working) paper with N. Pfeifer, D. Over, G. Sanfilippo, and myself, are discussed in

D.E. Over and J. Baratgin (2016), The "defective" truth table: its past, present, and future, N. Galbraith, D. Over, and E. Lucas, editors, The

Thinking Mind: The use of thinking in everyday life, pages 15–28, Psychology Press

and

J. Baratgin, G. Politzer, D.E. Over, and T. Takahashi (2018), The psychology of uncertainty and three-valued truth tables, Frontiers in Psychology, 9:1479.

Given a family of n events $\mathcal{F} = \{E_1, \ldots, E_n\}$, we can say that:

- we stay at Level 0 on $\mathcal{F}$ if we stay at Level 0 on each $E \in \mathcal{F}$;

- we stay (partially) at Level 1 on the family $\mathcal{F}$ if we stay at Level 1 on some event $E \in \mathcal{F}$;

- we stay (partially) at Level 2 on $\mathcal{F}$ if we stay at Level 2 on some $E \in \mathcal{F}$.

The analysis of Levels 0, 1, 2 could be extended to a conditional event

$A|H$, by considering the associated partition $\pi = \{AH, \overline{A}H, \overline{H}\}$.

- we stay at Level 0 on $A|H$ when we stay at Level 0 on π, in which case we know that $A|H$ is true, or we know that it is false, or we know that it is void.

- we stay at Level 1 on $A|H$ when we stay at Level 1 on π; that is, we are uncertain, among the three possible cases, $AH, \overline{A}H, \overline{H}$, as to which is the true one.

- we can move at Level 2 in different ways when we represent uncertainty by probabilities.

We observe that $P(A) = P(A|\Omega) = P(A \,|\, A \vee \overline{A})$, that is $P(A)$ represents the uncertainty of A with respect to the two alternatives A, or $\overline{A}$.

Similarly, $P(A|H)$ represents the *conditional uncertainty* of AH with respect the two alternatives AH, or $\overline{A}H$. Indeed,

$$P(A|H) = P(AH|H) = P(AH \,|\, AH \vee \overline{A}H)\,.$$

In particular, if we represent our uncertainty on AH and $\overline{A}H$ by $P(AH)$ and $P(\overline{A}H)$, then $P(A|H)$ is uniquely determined, when $P(AH)+P(\overline{A}H) > 0$, by the formula

$$P(A|H) \; = \; P(AH \,|\, AH \vee \overline{A}H) \; = \; \frac{P(AH)}{P(AH) + P(\overline{A}H)}\,.$$

When $P(AH) = P(\overline{A}H) = 0$ we need a direct evaluation of $P(A|H)$.

A similar analysis applies in order to represent other conditional uncertainties.

Notice that in some papers by Gilio & Sanfilippo, see e.g.

A. Gilio and G. Sanfilippo (2014). Conditional random quantities and compounds of conditionals. Studia Logica, 102(4): 709 - 729,

compound conditionals, such as conjunctions and disjunctions, are defined (not as three-valued objects, but) as suitable conditional random quantities, where some of their values are (coherent) probability values.

For instance, given two conditional events $A|H$ and $B|K$, if we assess $P(A|H) = x, P(B|K) = y$, then the possible values of the conjunction $(A|H) \wedge (B|K)$ are $1, 0, x, y, z,$ where z is our assessed prevision of $(A|H) \wedge (B|K)$.

In our approach compound conditionals are directly defined at Level 2.

A general approach to compound conditionals has been developed in

T. Flaminio, A. Gilio, L. Godo, and G. Sanfilippo, Compound conditionals as random quantities and Boolean algebras, 19th Int. Conf. on Principles of Knowledge Representation and Reasoning, KR 2022, July 31 - August 5, 2022, Haifa, Israel.

Some basic aspects on coherence

Coherence-based probability theory of de Finetti:

Probabilities are (coherent) numerical measures of degrees of belief.

Betting framework. Given any (finite) random quantity X, if for its prevision you assess $\mathbb{P}(X) = \mu$, then you accept a bet by agreeing to pay (resp., to receive) an amount $s\mu$, with s an arbitrary real number, and to receive (resp., to pay) the random amount sX.

Random gain: $G = sX - s\mu$; *Coherence:* $\min G \leq 0 \leq \max G$.

Conditional bets. Given a (finite) random quantity X and an event $H \neq \emptyset$, if you assess $\mathbb{P}(X|H) = \mu$, then you accept a bet on X which becomes effective if H is true. If H is true you pay an amount $s\mu$ by receiving the random amount sX. *If H is false the bet has no effect.*

Random gain. H true: $G_H = sX - s\mu$; H false: there is no bet.

Coherence: $\min G_H \leq 0 \leq \max G_H$.

Bets on conditionals (equivalent). If you assess $\mathbb{P}(X|H) = \mu$, then you agree to pay $s\mu$; if H is true, you receive sX; if H is false, you receive back the paid amount $s\mu$ (bet called off).

In other words (as made in the approaches by Gilio & Sanfilippo, and by F. Lad), *by defining*

$$X|H \;=\; XH + \mu\overline{H}\,, \quad \text{where} \quad \mu \;=\; \mathbb{P}(X|H)\,,$$

the bet on $X|H$ works as follows:

if you assess $\mathbb{P}(X|H) = \mu$, then you pay the amount $s\mu$, by receiving the random amount $s\,X|H = s(XH + \mu\overline{H})$.

Random gain: $G = s\,X|H - s\mu = sH(X-\mu)$; $\quad$ H true: $G_H = sX - s\mu$.

Coherence Principle:

(i) we discard all the cases where we receive back the paid amount $s\mu$, whatever it be (that is, we discard the case where H is false);

(ii) the assessment is coherent if in the remaining cases (where the bet has effect) it does not happen that you obtain a sure loss (no Dutch book).

By condition (i), for checking coherence we must refer to G_H.

- Assume, for instance, that the set of possible values of X, when H is true, is $\{x_1, \ldots, x_n\}$; then

$$X|H = XH + \mu\overline{H} \in \{x_1, \ldots, x_n, \mu\}\,;$$

the coherence of μ amounts to: $\min G_H \ \leq\ 0\ \leq\ \max G_H$,
or, equivalently

$$\min\{x_1, \ldots, x_n\} \leq \mu \leq max\,\{x_1, \ldots, x_n\}.$$

Indeed,

$$G_H \in \{g_1, \ldots, g_n\}\,,\ \text{ where: }\ g_h = sx_h - s\mu,\ \ h = 1, \ldots, n,$$

and the condition

$$\min_h (sx_h - s\mu) \leq 0 \leq \max_h (sx_h - s\mu), \quad \forall h, \forall s,$$

is satisfied if and only if : $\min\{x_1, \ldots, x_n\} \leq \mu \leq max\{x_1, \ldots, x_n\}$.

G_H is the restriction to H of the random gain

$$G = sX|H - s\mu = sH(X - \mu) \in \{g_1, \ldots, g_n, 0\}.$$

Conditional probability assessments.
Concerning in particular a conditional probability assessment $P(A|H) = x$, by de Finetti theory we can use a scheme $S1$, where we can consider conditional bets, or bets on conditionals (a further equivalent scheme $S2$ will be considered later).

$S1$. Let $P(A|H) = x$ be your (numerical measure for the) degree of belief on A, assessed when H is uncertain, by supposing H true (and nothing else more).

Conditional bet: "if H is true, I bet on A".

(*B. de Finetti (1936), "La logique de la probabilité", in, Actes du Congrès International de Philosophie Scientifique, Vol. IV, Paris: Hermann et C.ie, 1-8*)

After knowing that H is true, you pay (resp., receive) x by receiving (resp., pay) A. If H is false, the bet is cancelled.

Bet on the conditional "if H, then A".

(*B. de Finetti (1937), "La prévision : ses lois logiques, ses sources subjectives", Annales de l'Institut Henri Poincaré, Tome 7 (1), 1-68.*)

If you assess $P(A|H) = x$, you pay (resp., you receive) x by receiving (resp., by paying) 1, or 0, or x, according to whether AH is true, or $\overline{A}H$ is true, or $\overline{H}$ is true, respectively. The case $\overline{H}$ is discarded (bet called off) because, when $\overline{H}$ is true, you receive back the paid amount x.

Conditional bets and bets on conditionals are equivalent!

Prevision assessments on random vectors

We now consider prevision assessments on two conditional random quantities and we show that the conditions of coherence based on *random gains* are equivalent to suitable *geometrical conditions* based on convex hulls.

Given two events $H \neq \emptyset, K \neq \emptyset$ and two random quantities X, Y, let $\mathcal{P} = (\mu, \eta)$ be a prevision assessment on $\mathcal{F} = \{X|H, Y|K\}$, with $\mu = \mathbb{P}(X|H), \eta = \mathbb{P}(Y|K)$. Moreover, denote by $\{(x_1, y_1), \ldots, (x_m, y_m)\}$ the set of possible values of the random vector $(X|H, Y|K)$ when $H \vee K$ is true.

Notice that, if $\overline{H}K \neq \emptyset$, and/or $H\overline{K} \neq \emptyset$, then there are possible values like (μ, y_j), and/or (x_i, η), for some indices i, j.

In a bet associated with the pair $(\mathcal{F}, \mathcal{P})$ the random gain is

$$G = s_1 H(X - \mu) + s_2 K(Y - \eta) = s_1(X|H - \mu) + s_2(Y|K - \eta) =$$

$$= (s_1 X|H + s_2 Y|K) - (s_1\mu + s_2\eta), \;\; s_1, s_2 \text{ arbitrary real numbers,}$$

which is the difference between what you receive, $s_1X|H + s_2Y|K$, and what you pay, $s_1\mu + s_2\eta$.

Coherence: when $H \vee K$ is false you receive back the paid amount $s_1\mu + s_2\eta$; then, for checking coherence we must only consider the values of the restriction of G to $H \vee K$, $G_{H\vee K}$, that is, we must discard the case $\overline{H}\,\overline{K}$.

- Possible values of $G_{H\vee K}$: we set $Q_h = (x_h, y_h)$, $h = 1, \ldots, m$; then, by considering the linear function $f(x,y) = s_1x + s_2y$, the value g_h of $G_{H\vee K}$ associated with Q_h is

$$g_h = (s_1x_h + s_2y_h) - (s_1\mu + s_2\eta) = f(Q_h) - f(\mathcal{P})\,, \;\; h = 1, \ldots, m.$$

Then, the condition of coherence $\min G_{H\vee K} \;\leq\; 0 \;\leq\; \max G_{H\vee K}$, that is

$$\min_h g_h \;\leq\; 0 \;\leq\; \max_h g_h\,,\; \forall\, s_1, s_2,$$

becomes

$$\min_h f(Q_h) \;\leq\; f(\mathcal{P}) \;\leq\; \max_h f(Q_h)\,,\; \forall\, s_1, s_2,$$

which is satisfied if and only if $\mathcal{P}$ belongs to the convex hull $\mathcal{I}$ of $Q_1, \ldots, Q_m$, that is

$$\mathcal{P} = \sum_{h=1}^{m} \lambda_h Q_h \,, \quad \sum_{h=1}^{m} \lambda_h = 1 \,, \quad \lambda_h \geq 0 \,,\; h = 1, \ldots, m.$$

Indeed:

• if $\mathcal{P} = \sum_{h=1}^{m} \lambda_h Q_h$ then, by observing that

$$\sum_{h=1}^{m} \lambda_h f(Q_h) = f(\sum_{h=1}^{m} \lambda_h Q_h) = f(\mathcal{P}) \,,\; \forall\, s_1, s_2 \,,$$

it holds that

$$\min_h f(Q_h) \;\leq\; f(\mathcal{P}) \;\leq\; \max_h f(Q_h) \,,\; \forall\, s_1, s_2 \,,$$

that is: $\min G_{H \vee K} \;\leq\; 0 \;\leq\; \max G_{H \vee K}$.

• if $\mathcal{P} \notin \mathcal{I}$ there exists a line, with equation $ax + by = c$, which separates $\mathcal{P}$ from the convex hull $\mathcal{I}$. Then, by choosing $s_1 = a$, $s_2 = b$, and by considering the linear function $f(x, y) = s_1 x + s_2 y$, *exactly one of the following alternatives holds:*

$$(i)\;\; f(\mathcal{P}) \;<\; c \;<\; \min_h f(Q_h)\,, \qquad (ii)\;\; f(\mathcal{P}) \;>\; c \;>\; \max_h f(Q_h)\,.$$

As a consequence, when $s_1 = a$ and $s_2 = b$, the condition

$$\min_h f(Q_h) \;\leq\; f(\mathcal{P}) \;\leq\; \max_h f(Q_h)$$

does not hold and the coherence condition

$$\min G_{H \vee K} \;\leq\; 0 \;\leq\; \max G_{H \vee K}\,,\; \forall\, s_1,\, s_2\,,$$

is not satisfied. We illustrate this aspect by an example.

Example. Let be given the probability assessment $\mathcal{P} = (0.3, 0.8)$ on the family $\mathcal{F} = \{B|AHK\,,\;\; AHBK|(HK \vee \overline{A}H \vee \overline{B}K)\}$, where $P(B|AHK) = 0.3$, $P(AHBK|(HK \vee \overline{A}H \vee \overline{B}K)) = 0.8$.

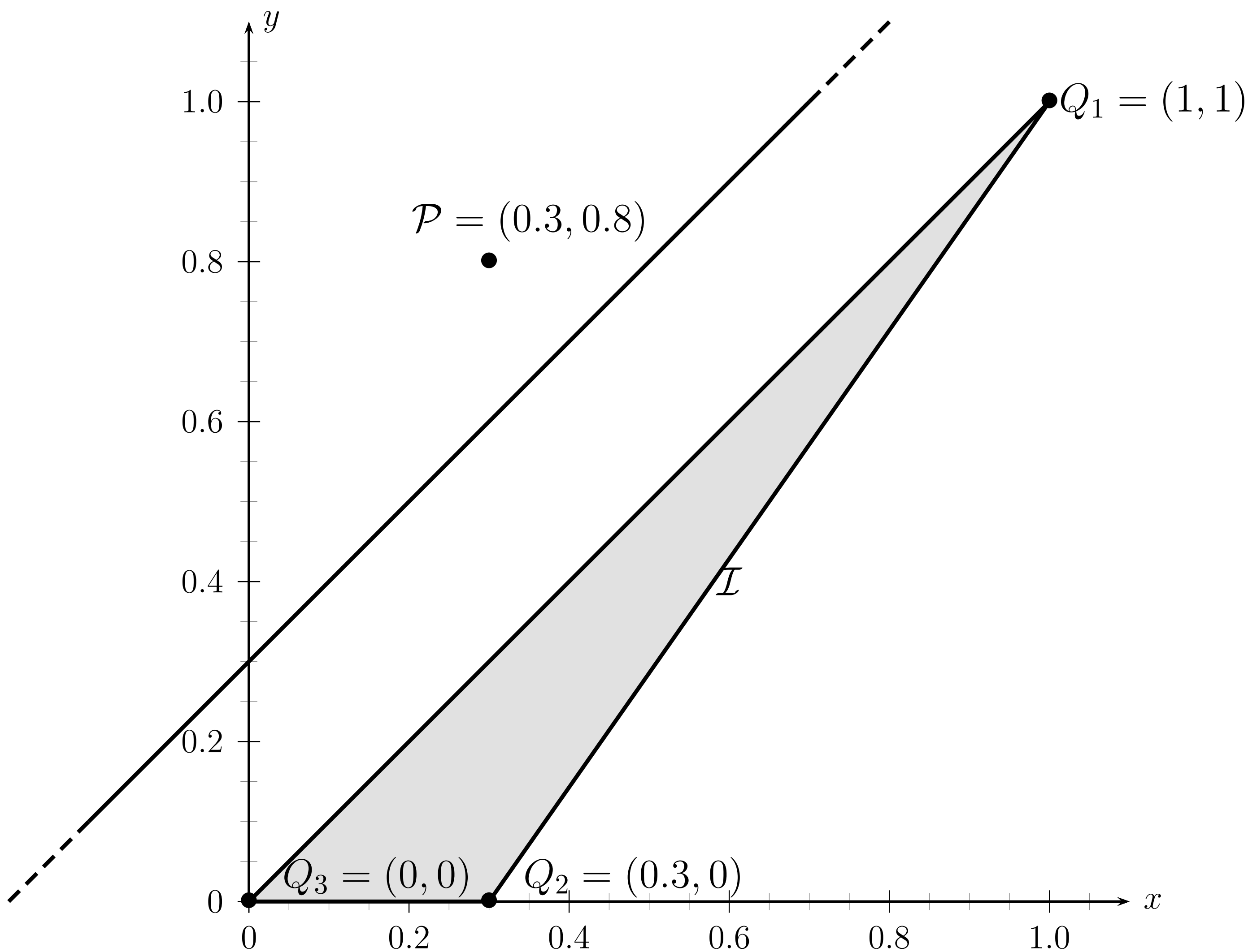

Figure 1: Convex hull $\mathcal{I}$ of the points Q_1, Q_2, Q_3, a triangle.

As shown in Figure 1, the possible values of the random vector

$(B|AHK, AHBK|(HK \vee \overline{A}H \vee \overline{B}K))$ are:

$$Q_1 = (1,1)\,, \;\; Q_2 = (0.3,0)\,, \;\; Q_3 = (0,0)\,, \;\; Q_0 = \mathcal{P} = (0.3,0.8)\,,$$

and $\mathcal{P}$ does not belong to the convex hull $\mathcal{I}$ of Q_1, Q_2, Q_3.

For instance, the line with equation $x - y = -0.3$ separates $\mathcal{P}$ from $\mathcal{I}$. In this case

$$s_1 = 1\,,\; s_2 = -1\,,\; f(x,y) = x - y\,,\; f(\mathcal{P}) = 0.3 - 0.8 = -0.5\,.$$

Then $\mathcal{P}$ is not coherent because

$$g_1 = f(Q_1) - f(\mathcal{P}) = 0 + 0.5 = 0.5 > 0,$$

$$g_2 = f(Q_2) - f(\mathcal{P}) = 0.5 + 0.5 = 1 > 0,$$

$$g_3 = f(Q_3) - f(\mathcal{P}) = 0 + 0.5 = 0.5 > 0.$$

Notice that in the trivalent logic of de Finetti it holds that

$$B|AHK = (B|K)|_{df}(A|H)\,,\ AHBK|(HK\vee\overline{A}H\vee\overline{B}K) = (A|H)\wedge_{df}(B|K)\,.$$

Moreover: $B|AHK \ \geq \ AHBK|(HK \vee \overline{A}H \vee \overline{B}K)$.

Thus, every assessment $\mathcal{P} = (x, y)$ on $\mathcal{F}$ is coherent if and only if $x \geq y$.

Remark. The definition of $X|H$ and $Y|K$ as the random quantities

$$X|H \ = \ XH + \mu\overline{H}\,, \qquad Y|K \ = \ YK + \eta\overline{K}\,,$$

where $\mu = \mathbb{P}(X|H)$ and $\eta = \mathbb{P}(Y|K)$, allows to introduce the points Q_h's, by then obtaining some advantages; for instance:

- we can develop a geometrical approach to the checking of coherence;

- we can represent the possible values g_h's of the random gain $G_{H\vee K}$ as the difference $f(Q_h)-f(\mathcal{P})$, where f is the linear function: $f(x, y) = s_1x+s_2y$;

- we can illustrate the *equivalence* between the conditions:

$$(i) \quad \min G_{H \vee K} \leq 0 \leq \max G_{H \vee K}\,, \qquad\qquad (ii) \quad \mathcal{P} \in \mathcal{I}\,;$$

- In particular, the *numerical counterpart* for the value "void" of $A|H$ is $P(A|H)$; indeed, in the case of conditional events, the components of the points Q_h's are the possible values of the indicators.

The points Q_h's were introduced in

A. Gilio, Criterio di penalizzazione e condizioni di coerenza nella valutazione soggettiva della probabilitá, Bollettino U.M.I., 645–660, 1990,

where the geometrical approach of de Finetti was extended to the case of *conditional events*, by suitably modifying the definition of coherence with the *penalty criterion*.

A further scheme for conditional probability assessments

We describe a further scheme S_2, equivalent to S_1 (conditional bets and bets on conditionals).

$(S2)$. The evaluation of $P(A|H)$ amounts to deciding, under the usual condition of coherence, the value y that you agree to pay in order to receive the random quantity $AH + y\overline{H}$.

In other words, you must choose y such that $y = \mathbb{P}(AH + y\overline{H})$.

Given any events A and H, with $H \neq \emptyset$, let us consider the assessment $\mathcal{P} = (x, y)$ on $\mathcal{F} = \{A|H, AH + y\overline{H}\}$, where $x = P(A|H)$ and $y = \mathbb{P}(AH + y\overline{H})$.

Does coherence require that some relationship be satisfied by x and y?
YES: $x = y$.

First of all, coherence requires that: $0 \leq y \leq 1$.

Indeed, by defining $Y = AH + y\overline{H}$, in a bet associated with the assessment $\mathbb{P}(Y) = y$ you pay, for instance, y by receiving 1, or 0, or y, according to whether AH is true, or $\overline{A}H$ is true, or $\overline{H}$ is true, respectively.

In the case $\overline{H}$ you receive back the paid amount y; hence this case is discarded (bet called off). Then, by coherence: $y \in [0,1]$.

Moreover, by considering the prevision assessment (x, y) on $\{A|H, Y\}$, (under logical independence of A and H) the constituents are $C_1 = AH$, $C_2 = \overline{A}H$, $C_0 = \overline{H}$, and the associated possible values of the random vector $(A|H, Y)$ are:

$$Q_1 = (1,1)\,,\;\; Q_2 = (0,0)\,,\;\; Q_0 = (x,y) = \mathcal{P}\,.$$

In a bet relative to the assessment $\mathcal{P} = (x, y)$ the random gain is

$$G = (s_1 A|H + s_2 Y) - (s_1 x + s_2 y), \quad (s_1, s_2 \text{ arbitrary real numbers}),$$

with the received amount $s_1 A|H + s_2 Y$ equal to the paid amount $s_1 x + s_2 y$ when C_0 is true.

Therefore C_0 must be discarded, that is we must consider the restricted random gain G_H; hence, we must check the condition $\mathcal{P} \in \mathcal{I}$, where the convex hull $\mathcal{I}$ is the segment Q_1Q_2. Then,

$$(x, y) \text{ coherent} \iff 0 \leq x = y \leq 1 .$$

Therefore, in order to assess $P(A|H)$, $S1$ and $S2$ are equivalent.

I recall that, within $S1$, you can consider a *bet on the conditional "if H, then I bet on A"*, or a *conditional bet on "if H, then A"*.

Within $S2$, $P(A|H)$ is the amount y that you agree to pay (resp., receive) by receiving (resp., paying) the random amount $AH + y\overline{H}$.

Coherence: the case $\overline{H}$ is discarded, because when H is false you receive back the paid amount y.

By a similar reasoning, a conditional prevision assessment $\mathbb{P}(X|H) = \mu$ can be made by the scheme $S1$, with a conditional bet, or a bet on a conditional, or equivalently by the scheme $S2$.

$S1$. You assess $\mathbb{P}(X|H) = \mu$, by supposing H true. Then, if H is true, you pay μ and you receive X. If H is false, the bet has no effect.

Equivalently, you pay μ by receiving X if H is true, or by receiving back μ if H is false (in this case the bet is called off).

Denoting by $\mathcal{S}$, for instance $\mathcal{S} = \{x_1, \ldots, x_n\}$, the set of possible values of X when H is true, coherence requires that:

$$\min \mathcal{S} = \min\{x_1, \ldots, x_n\} \;\leq\; \mu \;\leq\; \max\{x_1, \ldots, x_n\} = \max \mathcal{S}\,.$$

$S2$. You assess the prevision of $Y = XH + y\overline{H}$, under the condition that $y = \mathbb{P}(Y)$.

In a bet associated with the assessment $\mathbb{P}(Y) = y$, you pay y, by receiving $X \in \{x_1, \ldots, x_n\}$ when H is true, or by receiving back the paid amount y when H is false.

By coherence, the case $\overline{H}$ must be discarded, so that:

$$\min\{x_1, \ldots, x_n\} \;\le\; y \;\le\; \max\{x_1, \ldots, x_n\}\,.$$

In order to verify that $y = \mu$, let us consider the prevision assessment $\mathcal{P} = (\mu, y)$ on the family $\mathcal{F} = \{X|H, Y\}$. We observe that, when H is true, it holds that $(X|H, Y) \in \{(x_1, x_1), \ldots, (x_n, x_n)\}$; when H is false, $(X|H, Y) = (\mu, y)$.

In a bet associated with $\mathcal{P}$, the random gain is

$$G = (s_1 X|H + s_2 Y) - (s_1\mu + s_2 y), \quad (s_1, s_2 \text{ arbitrary real numbers}),$$

with the bet called off when H is false, because in this case you receive back the paid amount $s_1\mu + s_2 y$.

When H is true the bet has effect and the possible values of the random vector $(X|H,Y)$ are

$$Q_1 = (x_1, x_1)\,, \quad \dots \;, \quad Q_n = (x_n, x_n)\,.$$

Then, the condition $\mathcal{P} \in \mathcal{I}$ is satisfied (that is, $\mathcal{P}$ is coherent) if and only if

$$\min\{x_1, \dots, x_n\} \;\leq\; \mu = y \;\leq\; \max\{x_1, \dots, x_n\}\,.$$

Compound prevision theorem: $\mathbb{P}(XH) = \mathbb{P}(X|H)P(H)$

Given an event $\emptyset \neq H \neq \Omega$ and a (finite) random quantity X, denote by $\{x_1, \ldots, x_n\}$ the set of possible values of X when H is true.

The assessment (u, v, z) on $\{H,\, X|H,\, XH\}$ is coherent if and only if :
$(i)\; 0 \leq u \leq 1, \quad (ii)\; \min\{x_1, \ldots, x_n\} = x' \leq v \leq x'' = \max\{x_1, \ldots, x_n\}$,
and $\quad (iii)\; z = uv$. Indeed:

- the condition (i) amounts to coherence of the assessment $P(H) = u$;

- the condition (ii) amounts to coherence of the assessment $\mathbb{P}(X|H) = v$;

- the possible values of the random vector $(H, X|H)$ are: $(1, x_1)$, $\ldots$, $(1, x_n)$, $(0, v)$,, with a convex hull given by the triangle $\mathcal{T}$ with vertices $(0, v)$, $(1, x')$, $(1, x'')$;

- each point (u, v) satisfying conditions (i) and (ii) belongs to $\mathcal{T}$;

- then, coherence of (u, v) amounts to conditions (i) and (ii);

- concerning the extension $z = \mathbb{P}(XH)$, the possible values of the random vector $(H, X|H, XH)$ are:

$$Q_1 = (1, x_1, x_1)\,,\ \ldots\,,\ Q_n = (1, x_n, x_n)\,,\ Q_{n+1} = (0, v, 0)\,;$$

- the condition $\mathcal{P} \in \mathcal{I}$ (convex hull of $Q_1, \ldots, Q_{n+1}$) means that

$$\mathcal{P} = \lambda_1 Q_1 + \cdots + \lambda_n Q_n + \lambda_{n+1} Q_{n+1}\,,\ \lambda_1 + \cdots + \lambda_{n+1} = 1\,,\ \lambda_h \geq 0\,,\ \forall\, h\,,$$

which amounts to solvability of the system

$$\begin{cases} u = \lambda_1 + \cdots + \lambda_n, \\ v = \lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1} v = z + (1-u)v\,, \\ z = \lambda_1 x_1 + \cdots + \lambda_n x_n\,. \end{cases}$$

Therefore: $z = xy$, that is $\mathbb{P}(XH) = \mathbb{P}(X|H)P(H)$ (*compound prevision theorem*).

In particular: $P(AH) = P(A|H)P(H)$ (*compound probability theorem*).

Conjunctions and conditional random quantities

Given two conditionals *(if H then A)* and *(if K then B)*, how can we interpret their conjunction *(if H then A)* $\&$ *(if K then B)*?

In the large part of research on trivalent logics the conjunction is defined as a suitable three-valued object, i.e. still a conditional.

In the approach by Gilio $\&$ Sanfilippo, conditionals are interpreted as conditional events and the conjunction $(A|H) \wedge (B|K)$ is defined in the setting of coherence *(not as a three-valed object, but)* as the following *conditional random quantity*

$$(A|H) \wedge (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}BK \text{ is true,} \\ y, & \text{if } AH\overline{K} \text{ is true,} \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true,} \end{cases}$$

where, *by definition*: $z = \mathbb{P}[(A|H) \wedge (B|K)]$.

Thus, in a bet associated with the assessment $z = \mathbb{P}[(A|H) \wedge (B|K)]$, the quantity z is the amount to be paid in order to receive the random amount $(A|H) \wedge (B|K)$.

Related notions have been given in:

S. Kaufmann (2009), Conditionals right and left: Probabilities for the whole family, Jour- nal of Philosophical Logic, 38:1-53.

V. McGee (1989), Conditional probabilities and compounds of conditionals, Philosophical Review, 98(4):485-541.

We show below that the conjunction $(A|H) \wedge (B|K)$ coincides with many (*apparently different*) conditional random quantities.

We recall that, given a coherent assessment $P(A|H) = x, P(B|K) = y$, the indicators of $A|H$ and $B|K$ are

$$A|H = AH + x\overline{H}\,, \quad B|K = BK + y\overline{K}\,.$$

Now, let us consider for instance the random quantities:

$$Z_1 = AHBK + x\overline{H}BK + yAH\overline{K}\,, \quad Z_2 = \max\{A|H + B|K - 1, 0\}\,,$$
$$Z_3 = (A|H)\cdot(B|K)\,, \quad Z_4 = \min\{A|H, B|K\}\,.$$

We observe that, when $H \vee K$ is true, it holds that

$$Z_1 = Z_2 = Z_3 = Z_4 = AHBK + x\overline{H}BK + yAH\overline{K} \in \{1, 0, x, y\}\,;$$

When $H \vee K$ is false, it holds that

$$Z_1 = 0\,, \; Z_2 = \max\{x + y - 1, 0\}\,, \; Z_3 = xy\,, \; Z_4 = \min\{x, y\}\,,$$

where

$$0 \le \max\{x + y - 1, 0\} \le xy \le \min\{x, y\}\,,$$

so that: $Z_1 \leq Z_2 \leq Z_3 \leq Z_4$. However

$$Z_1|(H \vee K) = \cdots = Z_4|(H \vee K) = (A|H) \wedge (B|K)\,.$$

This result follows by considering the following question:

Does it make any difference between $(A|H) \wedge (B|K)$ and any random quantity Y defined as

$$Y = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}BK \text{ is true,} \\ y, & \text{if } AH\overline{K} \text{ is true,} \\ \mu, & \text{if } \overline{H}\,\overline{K} \text{ is true,} \end{cases} = AHBK + x\overline{H}BK + yAH\overline{K} + \mu\overline{H}\,\overline{K}\,,$$

where $\mu = \mathbb{P}(Y) = \mathbb{P}(AHBK + x\overline{H}BK + yAH\overline{K} + \mu\overline{H}\,\overline{K})$ by definition?

The answer is NO. Indeed

$$\mu = z = \mathbb{P}[(A|H) \wedge (B|K)] \quad \text{and} \quad Y = (A|H) \wedge (B|K)\,.$$

Indeed, the possible values of the random vector $((A|H) \wedge (B|K), Y)$ are the points Q_h's associated with the prevision assessment $\mathcal{P} = (z, \mu)$ on the family $\mathcal{F} = \{(A|H) \wedge (B|K)\,,\, Y\}$, that is:

$$Q_1 = (1,1)\,,\; Q_2 = (0,0)\,,\; Q_3 = (x,x)\,,\; Q_4 = (y,y)\,,\; Q_0 = (z,\mu) = \mathcal{P}\,.$$

As we can see, when $H \vee K$ is true, it holds that

$$(A|H) \wedge (B|K) \;=\; Y \;=\; AHBK + x\overline{H}BK + yAH\overline{K} \;\in\; \{1, 0, x, y\}$$

thus, denoting by $\mathcal{I}$ the convex hull of Q_1, Q_2, Q_3, Q_4, it holds that

$$\mathcal{P} \;\in\; \mathcal{I} \quad\Longleftrightarrow\quad z \;=\; \mu\,.$$

Therefore, for each given assessor who evaluates $\mathbb{P}[(A|H) \wedge (B|K)] = z$, the quantity μ such that

$$\mu = \mathbb{P}(Y) = P(AHBK + x\overline{H}BK + yAH\overline{K} + \mu\overline{H}\overline{K})$$

is *uniquely determined* and *coincides* with z.

Thus, for each $i = 1, 2, 3, 4$, it holds that

$$(A|H) \wedge (B|K) \;=\; Z_i \,|\, (H \vee K) \;=\; (AHBK + x\overline{H}BK + yAH\overline{K}) \,|\, (H \vee K)\,.$$

Actually, given any random quantity

$$Z = AHBK + x\overline{H}BK + yAH\overline{K} + t\,\overline{H}\,\overline{K}\,, \quad \textit{(t arbitrary real number)}$$

it holds that: $(A|H) \wedge (B|K) \;=\; Z \,|\, (H \vee K)$.

In (Gilio & Sanfilippo, 2014), by generalizing the formula

$$(A|H) \wedge (B|H) = AB|H = \min\{A|H, B|H\}|H = \min\{A|H, B|H\}|(H \vee H)\,,$$

the conjunction was defined as

$$(A|H) \wedge (B|K) \;=\; \min\{A|H, B|K\}|(H \vee K)\,.$$

A remark on coherence

The condition $\mathcal{P} \in \mathcal{I}$ *is necessary, but in general not sufficient,* for coherence.

For instance, by considering the assessment $\mathcal{P} = (\eta, z)$, where

$$\eta \;=\; \mathbb{P}[\max\{A|H + B|K - 1, 0\}|(H \vee K)]\,, \quad z \;=\; \mathbb{P}[(A|H) \wedge (B|K)]\,,$$

it has been shown before that

$$\eta \;=\; z \qquad \text{and} \qquad \max\{A|H + B|K - 1, 0\}|(H \vee K) \;=\; (A|H) \wedge (B|K)\,.$$

Indeed, the possible values of the random vector

$$(\,\max\{A|H + B|K - 1, 0\}|(H \vee K)\,,\; (A|H) \wedge (B|K)\,)$$

are:

$$Q_1 = (1,1)\,,\; Q_2 = (0,0)\,,\; Q_3 = (y,y)\,,\; Q_4 = (x,x)\,,\; Q_0 = (\eta, z) = \mathcal{P}\,.$$

The convex hull $\mathcal{I}$ of Q_1, Q_2, Q_3, Q_4 is the segment with vertices Q_1, Q_2; then

$$\mathcal{P} \in \mathcal{I} \iff 0 \leq \eta = z \leq 1\,.$$

But, this condition is *not sufficient for coherence*. More precisely, *in order to check coherence*, also the probabilities $P(A|H) = x$ and $P(B|K) = y$ must be taken into account.

Under logical independence of A, H, B, K, the assessment $(x,\, y,\, z,\, \eta)$ on the family

$$\{A|H\,,\; B|K\,,\; (A|H) \wedge (B|K)\,,\; \max\{A|H + B|K - 1, 0\}|(H \vee K)\}$$

is coherent if and only if

$$0 \;\leq\; x\,,\, y \;\leq\; 1\,, \quad \max\{x + y - 1, 0\} \;\leq\; z \;=\; \eta \;\leq\; \min\{x, y\}\,.$$

The previous condition of coherence follows by the result below, given in the paper:

A. Gilio, G. Sanfilippo, "Conditional random quantities and compounds of conditionals", Studia Logica 102 (2014) 709-729.

Fréchet-Hoeffding bounds. Given any logically independent events A, H, B, K, with $H \neq \emptyset, K \neq \emptyset$, the probability assessment $P(A|H) = x, P(B|K) = y$ on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0,1]^2$. Moreover, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ of (x, y) on $(A|H) \wedge (B|K)$ is coherent if and only if

$$max\,\{x + y - 1, 0\} \;=\; z' \;\leq\; z \;\leq\; z'' \;=\; min\,\{x, y\}\,.$$

A criticism by D. Edgington

In our approach, when $HK = \emptyset$, it holds that

$$\mathbb{P}[(A|H) \wedge (B|K)] \ = \ P(A|H)P(B|K)\,,$$

criticized by D. Edgington as a 'strange case' of independence.

Moreover, when for instance $P(A|H) = P(B|K) = 0.5$, some authors "intuitively" evaluate that: $P((A|H) \wedge (B|K)) = P(A|H) = 0.5$.

In particular, this evaluation is made for the conjunction $(A|H) \wedge (A|\overline{H})$.

Our comments:
- if you want to speak of independence you should preliminarily define this notion;
- in our framework, $A|H$, $B|K$, and $(A|H) \wedge (B|K)$ are conditional random quantities; then, the equality above should perhaps be associated to a case of *uncorrelation* (and not independence);
- finally, in our theory, with the conditional events $A|H$ and $B|K$ we

associate the conditional constituents

$$(A|H)\wedge(B|K)\,,\ (A|H)\wedge(\overline{B}|K)\,,\ (\overline{A}|H)\wedge(B|K)\,,\ (\overline{A}|H)\wedge(\overline{B}|K)\,,$$

which satisfy the relations

$$A|H \;=\; (A|H)\wedge(B|K)+(A|H)\wedge(\overline{B}|K)\,,$$

and

$$\overline{A}|H \;=\; (\overline{A}|H)\wedge(B|K)+(\overline{A}|H)\wedge(\overline{B}|K)\,.$$

Then:

$$(A|H)\wedge(B|K)\;+\;(A|H)\wedge(\overline{B}|K)\;+\;(\overline{A}|H)\wedge(B|K)\;+\;(\overline{A}|H)\wedge(\overline{B}|K)\;=$$

$$=\;A|H\;+\;\overline{A}|H\;=\;1.$$

As a consequence, when $P(A|H)=P(B|K)=0.5$, by symmetry the conditional constituents have the same prevision, i.e. 0.25, so that

$$\mathbb{P}[(A|H)\wedge(B|K)]=0.25=0.5\times 0.5=P(A|H)P(B|K)\,.$$

This reasoning also applies to the conjunction $(A|H)\wedge(A|\overline{H})$.

Characterization of p-entailment by conjunctions

The property of p-entailment of Adams can be studied with full generality in the setting of coherence (Gilio 2002), without having to rely on *proper distributions*.

Indeed, by exploiting coherence, there is no need of assuming that *conditioning* events have *positive* probability.

We also avoid the *unnecessary convention* that $P(B|A) = 1$ when $P(A) = 0$.

The property of p-entailment for conditional events can be characterized by conjunctions in the same way as for unconditional events.

Given a (p-consistent) family of unconditional events $\mathcal{F} = \{E_1, \ldots, E_n\}$ and a further event B, the following assertions are equivalent:

(i) the family $\mathcal{F}$ p-entails B; that is

$$P(E_1) = \cdots = P(E_n) = 1 \implies P(B) = 1\,;$$

(ii) the conjunction $E_1 \cdots E_n$ p-entails B; that is

$$P(E_1 \cdots E_n) = 1 \implies P(B) = 1\,;$$

(iii) $E_1 \cdots E_n \subseteq B$, that is $E_1 \cdots E_n B = E_1 \cdots E_n$; or, in numerical terms: $E_1 \cdots E_n \leq B$.
Notice that the property (iii) can be written as: $P(B|E_1 \cdots E_n) = 1$;
or, equivalently: $B|E_1 \cdots E_n$ constant and equal to 1.

With the notion of conjunction given in our approach, the previous characterization of p-entailment still holds when we consider *conditional events*.

More precisely, given a (p-consistent) family of conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ and a further conditional event $B|A$, the following assertions are equivalent:

(i) the family $\mathcal{F}$ p-entails $B|A$; that is

$$P(E_1|H_1) = \cdots = P(E_n|H_n) = 1 \implies P(B|A) = 1\,;$$

(ii) the conjunction $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ p-entails $B|A$; that is

$$\mathbb{P}[(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)] = 1 \implies P(B|A) = 1\,;$$

(iii) $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \;\leq\; B|A$, or equivalently

$$(E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \wedge (B|A) \;=\; (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)\,.$$

Notice that, by using iterated conditionals, the property (iii) becomes

$$(B|A)\,|\,[(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)] \;=\; 1\,.$$

On iterated conditionals and material conditionals

We recall that, given any events A and $H \neq \emptyset$, for the indicator of $A|H$ it holds that

$$A|H \;=\; AH + P(A|H)\,\overline{H}\,. \tag{1}$$

Then

$$AH \;\leq\; A|H \;\leq\; AH \vee \overline{H}\,, \tag{2}$$

and in particular

$$P(AH) \leq P(A|H) \leq P(AH \vee \overline{H})\,, \tag{3}$$

which shows the relation among the probabilities of the conjunction AH, the conditional event $A|H$, and the material conditional $AH \vee \overline{H}$.
Notice that, in numerical terms, it holds that: $AH \vee \overline{H} = AH + \overline{H}$.

The previous inequalities also hold for our compound conditionals.

We represent the nested conditional "*if 'A when H', then 'B when K'*", by the iterated conditional $(B|K)|(A|H)$ which, based on (1), is defined as

$$(B|K)|(A|H) = (A|H) \wedge (B|K) + \mu\,\overline{A}|H\,,$$

where $\mu = \mathbb{P}[(B|K)|(A|H)] \in [0,1]$.

The associated material conditional, defined as $[(A|H) \wedge (B|K)] \vee (\overline{A}|H)$, by the approach developed in (Flaminio, Gilio, Godo, Sanfilippo, KR 2022) coincides with

$$(A|H)\wedge(B|K)+\overline{A}|H-(A|H)\wedge(B|K)\wedge(\overline{A}|H) \;=\; (A|H)\wedge(B|K)+\overline{A}|H\,,$$

which generalizes the formula $A \vee B = A + B - AB$.

Finally, as $\mu\,\overline{A}|H \leq \overline{A}|H$, we obtain

$$(A|H) \wedge (B|K) \;\leq\; (B|K)|(A|H) \;\leq\; (A|H) \wedge (B|K) + \overline{A}|H\,,$$

and in particular

$$\mathbb{P}[(A|H) \wedge (B|K)] \;\leq\; \mathbb{P}[(B|K)|(A|H)] \;\leq\; \mathbb{P}[(A|H) \wedge (B|K) + \overline{A}|H]\,,$$

which shows that the properties in (2) and (3) still hold when unconditional events are replaced by conditional events.

Another property.

$$A|H = (A|H)\,|\,(A|H) \vee (\overline{A}|H)\,, \quad P(A|H) = P[(A|H)\,|\,(A|H) \vee (\overline{A}|H)]\,.$$

This property can be generalized in the setting of compound conditionals. Indeed, by observing that

$$A = A|\Omega = A\,|\,(A \vee \overline{A})\,, \;\text{ so that }\; P(A) = P(A|\Omega) = P(A\,|\,A \vee \overline{A})\,,$$

it follows that

$$A|H \vee \overline{A}|H = \Omega|H\,, \quad (A|H) \wedge (\Omega|H) = A\Omega|H = A|H\,, \quad \emptyset|H = 0\,.$$

Then, denoting by μ the prevision of $(A|H)\,|\,(A|H) \vee (\overline{A}|H)$, it holds that

$$(A|H)\,|\,(A|H) \vee (\overline{A}|H) = (A|H)\,|\,(\Omega|H) = (A|H) \wedge (\Omega|H) + \mu\,\emptyset|H = A|H\,,$$

so that: $\quad P(A|H) = \mathbb{P}[(A|H)\,|\,(A|H \vee \overline{A}|H)].$

Iterated conditionals allow to generalize the relation $A = A\,|\,(A \vee \overline{A})$, *which in the setting of compound conditionals becomes*

$$A|H = (A|H)\,|\,(A|H \vee \overline{A}|H)\,.$$

(see Assertion (a) in the next slide)

On complex sentences and iterated conditionals

We illustrate some results from the paper:

Gilio A, Sanfilippo G (2021), On compound and iterated conditionals, Argumenta 6(2): 241−266.

Iterated conditionals allow us to give a clear meaning to *intuitively valid* assertions; we examine below some instances.

We interpret the conditional "*if* H *then* A" as $A|H$.

We simply denote $A|H$ by $\mathcal{C}$ and the negation $\overline{A}|H$ by $\overline{\mathcal{C}}$.

We recall that the iterated conditional $(B|K)\,|\,(A|H)$ is defined as

$$(B|K)\,|\,(A|H) = (B|K) \wedge (A|H) + \mu\,\overline{A}|H\,,$$

with

$$(B|K)\,|\,(A|H) \;\in\; \{1, 0, y, x + \mu(1-x), \mu(1-x), z + \mu(1-x), \mu\}\,,$$

where

$$x = P(A|H)\,,\; y = P(B|K)\,,\; z = \mathbb{P}[(B|K)\wedge(A|H)]\,,\; \mu = \mathbb{P}[(B|K)\,|\,(A|H)]\,,$$

and where, by linearity of prevision, $z + \mu(1-x) = \mu$, that is: $z = \mu x$.

Assertion (a).
The probability of $\mathcal{C}$ is (not the probability of its truth, but) the probability of its truth, given that it is true or false.

'$\mathcal{C}$ true' means 'AH true'; '$\mathcal{C}$ true or false' means '$AH \vee \overline{A}H$ true'.

Then, the complex sentence *"if $\mathcal{C}$ is true or false, then $\mathcal{C}$ is true"* is the conditional *"if $AH \vee \overline{A}H$ is true, then AH is true"*, which we represent by the conditional event

$$AH|(AH \vee \overline{A}H) = AH|H = A|H\,.$$

Thus

$$P(\mathcal{C}) = P(AH|(AH \vee \overline{A}H)) = P(A|H) \;\neq\; P(AH) = P(\mathcal{C}\; true)\,,$$

that is: $P(\mathcal{C})$ is the *'probability of its truth, given that it is true or false'.*

Within the formalism of iterated conditionals, as

$$(A|H)\,|\,(A|H)\vee(\overline{A}|H) = (A|H)\,|\,(\Omega|H) = (A|H)\wedge(\Omega|H)+\mu\emptyset|H = A|H\,,$$

it holds that

$$P(\mathcal{C}) = P(A|H) = \mathbb{P}[(A|H)\,|\,(A|H)\vee(\overline{A}|H)] = P(\mathcal{C}\,|\,\mathcal{C}\vee\overline{\mathcal{C}})\,.$$

Assertion (b).
The probability of "$\mathcal{C}$, *given that AH is true*" is 1.

We represent the compound conditional "if AH then $\mathcal{C}$" by the iterated conditional $(A|H)|AH$ and we observe that $AH \subseteq A|H$, so that $(A|H) \wedge AH = AH$.

Then,

$$(A|H)|AH = (A|H)\wedge AH+\mu\overline{AH} = AH+\mu\overline{AH}\,, \quad (\mu = \mathbb{P}[(A|H)|(AH)])\,,$$

which is equal to 1, or μ, according to whether AH is true, or false, respectively.

In a bet on $(A|H)|AH$ we pay μ and we receive 1, if AH is true, or we receive back μ, if AH is false (in this case the bet is called off).

Then, μ is coherent if and only if $\mu = 1$; thus: $(A|H)|(AH) = 1$.

A more simple equivalent method: we observe that $A|H = AH + x\overline{H}$, where $x = P(A|H)$; in particular $AH|AH = 1$ because $P(AH|AH) = 1$. Then

$$(A|H)|AH = (AH + x\overline{H})|AH = AH|AH + x\,\overline{H}|AH = 1 = \mathbb{P}[(A|H)|AH]\,.$$

Thus, by interpreting "if AH then $\mathcal{C}$" as the iterated conditional $(A|H)|AH$, it coincides with $AH|AH$, i.e. with the constant 1, and its probability is 1.

Assertion (c).
The probability of $\mathcal{C}$, given that A is false and H is true, is 0.

We represent "if $\overline{A}H$ then $\mathcal{C}$" by the iterated conditional $(A|H)|\overline{A}H$.
We set $P(A|H) = x$; moreover $P(AH|\overline{A}H) = P(\overline{H}|\overline{A}H) = 0$, so that

$$AH|\overline{A}H = \overline{H}|\overline{A}H = 0\,.$$

Then

$$(A|H)|\overline{A}H = (AH + x\overline{H})|\overline{A}H = AH|\overline{A}H + x\overline{H}|\overline{A}H = 0 = \mathbb{P}[(A|H)|\overline{A}H]\,.$$

Thus, by interpreting "if $\overline{A}H$ then $\mathcal{C}$" as the iterated conditional $(A|H)|\overline{A}H$, it coincides with the constant 0, and its probability is 0.

Assertion (d).
The probability of $\mathcal{C}$, given that H is false, is $P(A|H)$.

We represent "if $\overline{H}$ then $\mathcal{C}$" by the iterated conditional $(A|H)|\overline{H}$.
We set $P(A|H) = x$; moreover, we observe that $P(AH|\overline{H}) = 0$ and

$P(\overline{H}|\overline{H}) = 1$, so that

$$AH|\overline{H} = 0\,, \quad \overline{H}|\overline{H} = 1\,.$$

Then

$$(A|H)|\overline{H} = (AH + x\overline{H})|\overline{H} = AH|\overline{H} + x\overline{H}|\overline{H} = x\overline{H}|\overline{H} = x = \mathbb{P}[(A|H)|\overline{H}]\,.$$

Thus, by interpreting "if $\overline{H}$ then $\mathcal{C}$" as the iterated conditional $(A|H)|\overline{H}$, it coincides with the constant x, and its probability is $x = P(A|H)$.

Assertion (e).
The probability of $\mathcal{C}$, given that H is true, is $P(A|H)$.

We represent "if H then $\mathcal{C}$" by the iterated conditional $(A|H)|H$.

We set $P(A|H) = x$ and we observe that $\overline{H}|H = 0$.

Then, it holds that

$$(A|H)|H = (AH + x\overline{H})|H = A|H\,, \text{ and } \mathbb{P}[(A|H)|H] = P(A|H)\,.$$

Thus, by interpreting "if H then $\mathcal{C}$" as the iterated conditional $(A|H)|H$, it coincides with $A|H$, and its probability is $P(A|H)$.

Notice that the conditional "if H then $\mathcal{C}$" is equivalent to the conditional "if $\mathcal{C}$ is true or false, then $\mathcal{C}$".

Assertion (f).
The probability of $\mathcal{C}$, given that "if H then A", is 1.

We represent "if $\mathcal{C}$ then $\mathcal{C}$" by the iterated conditional $(A|H)|(A|H)$.
We recall that $(A|H) \wedge (A|H) = A|H$. Then

$$(A|H)|(A|H) = (A|H) \wedge (A|H) + \mu\overline{A}|H = A|H + \mu\overline{A}|H =$$

$$= \begin{cases} 1, & \text{if } AH \text{ is true.} \\ \mu, & \text{if } \overline{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \overline{H} \text{ is true,} \end{cases}$$

where $\mu = \mathbb{P}[(A|H)|(A|H)]$ and $x = P(A|H)$. We observe that

$$\mu = \mathbb{P}[A|H + \mu(1 - A|H)] = P(A|H) + \mu(1 - P(A|H)) = x + \mu(1-x)\,;$$

then, $(A|H)|(A|H) \in \{1, \mu\}$.

In a bet on $(A|H)|(A|H)$ we pay μ by receiving 1 when AH is true, and by receiving back the paid amount μ when AH is false (*bet called off*). Then, by coherence, it must be $\mu = 1$.

Thus, by interpreting "if $\mathcal{C}$ *then* $\mathcal{C}$*" as the iterated conditional* $(A|H)|(A|H)$*, it coincides with the constant 1, and its probability is 1.*

Assertion (g).
The probability of $\mathcal{C}$, given that "if H then $\overline{A}$", is 0.

We represent "if (if H then $\overline{A}$), then $\mathcal{C}$" as the iterated conditional

$(A|H)|(\overline{A}|H)$.
We set $\mathbb{P}[(A|H)|(\overline{A}|H)] = \mu$, $P(A|H) = x$ and we observe that

$$(A|H) \wedge (\overline{A}|H) = 0\,, \quad \overline{\overline{A}|H} = 1 - \overline{A}|H = A|H\,.$$

Then: $(A|H)|(\overline{A}|H) = (A|H) \wedge (\overline{A}|H) + \mu\overline{\overline{A}|H} = \mu A|H \;\in\; \{\mu, 0, \mu x\}$,

so that: $\mu = \mathbb{P}[(A|H)|(\overline{A}|H)] = \mu P(A|H) = \mu x$, that is: $(1-x)\mu = 0$.

If $0 \leq x < 1$, then $\mu = \frac{0}{1-x} = 0$.

If $x = 1$, then $(A|H)|(\overline{A}|H) \in \{0, \mu\}$ and, in a bet on $(A|H)|(\overline{A}|H)$ we pay, for instance, the amount μ by receiving 0, when $\overline{A}H$ is true, or by receiving back the paid amount μ, when $\overline{A}H$ is false (*bet called off*).
Then, by coherence, it must be $\mu = 0$.

Thus, by interpreting "if (if H then $\overline{A}$), then $\mathcal{C}$" as the iterated conditional $(A|H)|(\overline{A}|H)$*, it coincides with 0, and its probability is 0.*

Likewise, by a symmetric reasoning it holds that $(\overline{A}|H)|(A|H) = 0$.

Trivalent logics and compound conditionals

We illustrate below some results from the paper

Gilio A, Sanfilippo G (2022), Subjective probability, trivalent logics and compound conditionals, (submitted).

Conjoined and disjoined conditionals are defined, by the large part of authors, as suitable conditional events.

However, when compound conditionals are defined as conditional events many basic logical and probabilistic properties are lost.

In the paper above we considered the following notions of conjunctions.

- Kleene-Lukasiewicz-Heyting conjunction ($\wedge_K$), or de Finetti conjunction ($\wedge_{df}$):

$$(A|H) \wedge_K (B|K) = AHBK|(AHBK \vee \overline{A}H \vee \overline{B}K)\,;$$

- Lukasiewicz conjunction ($\wedge_L$):

$$(A|H) \wedge_L (B|K) = AHBK|(AHBK \vee \overline{A}\,H \vee \overline{B}\,K \vee \overline{H}\,\overline{K})\,;$$

- Bochvar internal conjunction, or Kleene weak conjunction ($\wedge_B$);

$$(A|H) \wedge_B (B|K) = AHBK|HK = AB|HK\,;$$

- Sobocinski conjunction, or quasi conjunction ($\wedge_S$):

$$(A|H) \wedge_S (B|K) = [(AH \vee \overline{H}) \wedge (BK \vee \overline{K})]|(H \vee K)\,.$$

The disjunctions $\vee_K, \vee_L, \vee_B, \vee_S$, associated with the previous conjunctions, are defined by exploiting De Morgan Laws.

We list below some basic properties, *valid in the case of unconditional events and preserved in our approach*, which are *not satisfied in general* by the pairs in the set

$$\{(\wedge_K, \vee_K)\,,\ (\wedge_L, \vee_L)\,,\ (\wedge_B, \vee_B)\,,\ (\wedge_S, \vee_S)\}\,,$$

when unconditional events, say A and B, are replaced by conditional events, say $A|H$ and $B|K$.

$Property\ P1$.

$$A|H \subseteq B|K \iff (A|H) \wedge (B|K) = A|H\,,$$

where $\subseteq$ denotes the *Goodman-Nguyen inclusion relation*, defined as

$$A|H \subseteq B|K \iff AH \subseteq BK \text{ and } \overline{B}K \subseteq \overline{A}H\,.$$

Property P1 is *not satisfied* by $\wedge_L, \wedge_B$, and $\wedge_S$, while it is *satisfied* by $\wedge_K$.

$Property\ P2$.

$$A|H = [(A|H) \wedge (B|K)] \vee [(A|H) \wedge (\overline{B}|K)], \tag{4}$$

$$A|H = (A|H) \wedge (K|K), \tag{5}$$

$$(A|H) \wedge [(B|K) \vee (\overline{B}|K)] = [(A|H) \wedge (B|K)] \vee [(A|H) \wedge (\overline{B}|K)]\,. \tag{6}$$

Formula (4) is *not satisfied* by any of the pairs $(\wedge_K, \vee_K)$, $(\wedge_L, \vee_L)$, $(\wedge_B, \vee_B)$, and $(\wedge_S, \vee_S)$;

Formula (5) is *not satisfied* by any of the conjunctions $\wedge_K$, $\wedge_L$, $\wedge_B$, and $\wedge_S$;

Formula (6) is *not satisfied* by the pair $(\wedge_L, \vee_L)$, while it is *satisfied* by the pairs $(\wedge_K, \vee_K)$, $(\wedge_B, \vee_B)$, and $(\wedge_S, \vee_S)$.

$Property\ P3$.

$$(A|H) \vee (B|K) = (A|H) \vee [(\overline{A}|H) \wedge (B|K)] \,.$$

Property P3 is *not satisfied* by the pairs $(\wedge_K, \vee_K)$, $(\wedge_L, \vee_L)$, and $(\wedge_S, \vee_S)$, while it is *satisfied* by $(\wedge_B, \vee_B)$.

Property P4.

$$P[(A|H) \wedge (B|K)] \;\leq\; P(A|H) \;\leq\; P[(A|H) \vee (B|K)] \,. \qquad (7)$$

Property P4 is *not satisfied* by the pairs $(\wedge_B, \vee_B)$ and $(\wedge_S, \vee_S)$, while it is *satisfied* by the pairs $(\wedge_K, \vee_K)$ and $(\wedge_L, \vee_L)$.

Property P5.

$$P[(A|H) \vee (B|K)] = P(A|H) + P(B|K) - P[(A|H) \wedge (B|K)]\,. \qquad (8)$$

Property P5 is *not satisfied* by any of the pairs $(\wedge_K, \vee_K)$, $(\wedge_L, \vee_L)$, $(\wedge_B, \vee_B)$, and $(\wedge_S, \vee_S)$.

Property P6.

$$\max\{P(A|H)+P(B|K)-1, 0\} \leq P[(A|H)\wedge(B|K)] \leq \min\{P(A|H), P(B|K)\}\,.$$

$$\max\{P(A|H), P(B|K)\} \;\leq\; P[(A|H)\vee(B|K)] \;\leq\; \min\{P(A|H)+P(B|K), 1\}\,,$$

Property P6 is *not satisfied* by any of the pairs $(\wedge_K, \vee_K)$, $(\wedge_L, \vee_L)$, $(\wedge_B, \vee_B)$, and $(\wedge_S, \vee_S)$, while it is *satisfied* by the pair $(\wedge_{gs}, \vee_{gs})$, the conjunction and disjunction in the approach by Gilio & Sanfilippo.

We recall that $(A|H) \vee (B|K) = 1 - (\overline{A}|H) \wedge (\overline{B}|K)$, by De Morgan Law, and hence

$$P[(A|H) \vee (B|K)] = 1 - P[(\overline{A}|H) \wedge (\overline{B}|K)] \,.$$

Then, the lower and upper bounds on $P[(A|H) \vee (B|K)]$ can be derived from the lower and upper bounds on $P[(\overline{A}|H) \wedge (\overline{B}|K)]$.

A criticism to an example in Bradley's theory

As a final example, let us consider the compound conditional "$\mathcal{C}$, given $\overline{AH}$".

At page 51 of the paper:

Edgington, D. 2020, "Compounds of Conditionals, Uncertainty, and Indeterminacy", in Elqayam, Douven, Evans et al. 2020, 42-56,

referring to the theory illustrated in

Bradley, R. 2012, "Multidimensional Possible-World Semantics for Conditionals", The Philosophical Review, 121, 4, 539-71,

it is observed that *the probability of "$\mathcal{C}$, given $\overline{AH}$" is 0.*

Question: is it the case that the probability of $\mathcal{C}$, given $\overline{AH}$, is 0 ?

Our answer is NO.

In our approach $P(\mathcal{C},\ given\ \overline{AH})$ is equal to $P(A|H)P(\overline{H}|\overline{AH})$.

We represent "if $\overline{AH}$ then $\mathcal{C}$" by the iterated conditional $(A|H)|\overline{AH}$.
We set $P(A|H) = x$, $\mathbb{P}[(A|H)|\overline{AH}] = \mu$, and we observe that $AH|\overline{AH} = 0$.

Then

$$(A|H)|\overline{AH} = (AH + x\overline{H})|\overline{AH} = AH|\overline{AH} + x\,\overline{H}|\overline{AH} = x\,\overline{H}|\overline{AH}\,, \quad (9)$$

and hence

$$\mu = \mathbb{P}[(A|H)|\overline{AH}] = x\,P(\overline{H}|\overline{AH}) = P(A|H)P(\overline{H}|\overline{AH})\,,$$

which in general is not 0.

Interpretation of Bradley's example by conditional bets and bets on conditionals

We can make two equivalent bets on the compound conditional $(A|H)|\overline{AH}$.

Conditional bet: if $\overline{AH}$ *is true, then we bet on* $A|H$.

- all probabilistic evaluations are made when $\overline{AH}$ is uncertain;

- if $\overline{AH}$ (that is $\overline{A}H \vee \overline{H}$) turns out to be true, then the bet becomes effective;

- in this case we pay the amount μ and we receive the random amount $A|H = AH + x\overline{H}$, that is:

we receive 0 if $\overline{A}H$ is true (with probability $P(\overline{A}H|\overline{AH})$);

we receive x if $\overline{H}$ is true (with probability $P(\overline{H}|\overline{AH})$).

Thus, in order the bet be fair, the amount to be paid μ must coincide with the prevision of the random amount that we receive, that is

$$\mu \;=\; 0 \cdot P(\overline{A}H|\overline{AH}) + x\, P(\overline{H}|\overline{AH}) \;=\; P(A|H)\, P(\overline{H}|\overline{AH})\,.$$

Bet on the conditional $(A|H)|\overline{AH}$.

We set $P(A|H) = x$ and $\mathbb{P}[(A|H)|\overline{AH}] = \mu$; then

- we pay μ and we receive the value assumed by the iterated conditional

$$(A|H)|\overline{AH} \;=\; (A|H) \wedge \overline{AH} + \mu AH\,,$$

where

$$(A|H)\wedge\overline{AH} \;=\; (A|H)\wedge(\overline{A}H\vee\overline{H}) \;=\; \begin{cases} 0, & \text{if } H \text{ is true,} \\ x, & \text{if } H \text{ is false,} \end{cases} \;=\; x\,\overline{H} \;\in\; \{0, x\}\,.$$

Thus

$$(A|H)|\overline{AH} \;=\; x\,\overline{H} + \mu\, AH \;\in\; \{0, x, \mu\}\,.$$

When AH is true we receive back the paid amount μ; then, by coherence, this case is discarded. Therefore $(A|H)|\overline{AH}$ coincides with the conditional random quantity

$$(x\,\overline{H} + \mu\,AH)|\overline{AH} \;=\; x\,\overline{H}|\overline{AH}\,,$$

and its prevision is

$$\mu \;=\; \mathbb{P}(x\,\overline{H}|\overline{AH}) \;=\; xP(\overline{H}|\overline{AH}) \;=\; P(A|H)P(\overline{H}|\overline{AH})\,.$$

The *Conditional bet "if* $\overline{AH}$ *is true, then I bet on* $A|H$*"*

and

the *Bet on the conditional* $(A|H)|\overline{AH}$ are equivalent.