On the Growth Effects of North-South Trade:
The Role of Labor Market Flexibility

Technical Appendix on Transitional Dynamics

Lutz G. Arnold

This technical appendix contains the proof of Theorem 2 in our paper “On the Growth Effects of North-South Trade: The Role of Labor Market Flexibility”. It is demonstrated that for initial values $\xi(0)$ close to $\xi^*$ and $L^N(0)$ close to $L^N*$ there exists a unique trajectory converging to the steady state $(g^*, \xi^*, L^N*)$ and that this convergent path may feature either monotonic convergence or damped oscillations:

**Theorem 2.** The system of linear differential equations obtained by linearizing the system (6), (7), (8) in a neighborhood of its steady state possesses a unique convergent path. Depending on parameter values, this convergent path is monotonic or oscillatory.

The proof is difficult. We proceed in several steps. Let $\tilde{y}(t) \equiv y(t) - y^*$ for $y \in \{g, \xi, L^N\}$. Then:

**Result 1.** The linearized version of the system (6), (7), (8) is:

$$
\begin{pmatrix}
\dot{\tilde{g}} \\
\dot{\tilde{\xi}} \\
\dot{\tilde{L}}^N
\end{pmatrix} =
\begin{pmatrix}
J^*_{gg} & J^*_g \xi & J^*_g L^N \\
\frac{m}{g^* + m} & -(g^* + m) & 0 \\
ma & 0 & -(\beta + m)
\end{pmatrix}
\begin{pmatrix}
\tilde{g} \\
\tilde{\xi} \\
\tilde{L}^N
\end{pmatrix},
$$

(T.1)

where

$$
J^*_{gg} = m + (g^* + m + \rho) \left(1 + \frac{\alpha}{1 - \alpha} g^* \right),
$$

$$
J^*_g \xi = \frac{\alpha}{1 - \alpha} \left(g^* + m + \rho\right)^2,
$$

$$
J^*_g L^N = -\frac{(\beta + m) + (g^* + m + \rho)}{a}.
$$

Proof: Let $J_{yz} \equiv \partial \dot{y}/\partial z$ for $y, z \in \{g, \xi, L^N\}$. Then

$$
J^*_{yy} = m - \left[\rho + m + g^* - \frac{1}{\alpha \xi^*} \left(L^N* - g^* \right) \right] + \left(L^N* - g^* \right) \left(1 + \frac{1 - \alpha}{\alpha \xi^*} \right).
$$

1
The system contains two state variables (\(\tilde{\xi}\) and \(\tilde{L}^N\)) and one jump variable (\(\tilde{g}\)). So in order for a unique convergent path to exist, exactly two eigenvalues of the system must have negative real parts.

To prove this, we let \(\varphi \equiv (\tilde{g}, \tilde{\xi}, \tilde{L}^N)'\) and denote the Jacobian matrix in (T.1) as \(J^*\). Then (T.1) can be rewritten as \(\dot{\varphi} = J^* \varphi\). Suppose there exist solutions to this system of the form \(\varphi(t) = b e^{qt}\), where \(b = (b_g, b_{\xi}, b_{LN})'\). Then \(\dot{\varphi} = qbe^{qt} = q\varphi\). Hence \(J^* \varphi = q\varphi\) or, letting \(I\) be the identity matrix, \((J^* - qI)\varphi = 0\). Non-trivial solutions \(\varphi \neq 0\) exist if and only if \(|J^* - qI| = 0\), that is

\[
0 = -q^3 + Tr J^* q^2 - B J^* q + Det J^* \equiv f(q),
\]

where:

\[
Tr J^* = \frac{\alpha}{1 - \alpha} \frac{g^*(g^* + m + \rho)}{g^* + m} + \rho - \beta
\]
is the determinant and

\[ \text{Det } J^* = (g^* + m + \rho) \left\{ \beta \left( \frac{g^*}{1 - \alpha} + m \right) + \frac{\alpha}{1 - \alpha} m \left[ g^* + \frac{(\beta + m)(g^* + m + \rho)}{g^* + m} \right] \right\}. \]

is the determinant and

\[ B J^* = -m(g^* + m) - g^*(g^* + m + \rho) - \rho(\beta + m) - (g^* + 2m + \beta)(g^* + m + \rho) \frac{g^*}{1 - \alpha g^* + m} \]
\[ - \frac{\alpha}{1 - \alpha} m(g^* + m + \rho)^2 \]

Proof:

\[ \text{Tr } J^* = J_{yy}^* - (g^* + m) - (\beta + m) \]
\[ = m + \frac{\alpha}{1 - \alpha} g^*(\rho + m + g^*) + \rho + m + g^* - (g^* + m) - (\beta + m) \]
\[ = \frac{\alpha}{1 - \alpha} g^*(g^* + m + \rho) + \rho - \beta. \]

\[ \text{Det } J^* = J_{yy}^* (g^* + m)(\beta + m) \]
\[ + J_{yL}^* \frac{m}{g^* + m}(\beta + m) \]
\[ + J_{yL}^* \frac{m}{g^* + m}(\beta + m) \]
\[ = (\beta + m) \left\{ m(g^* + m) + (g^* + m + \rho) \left( \frac{g^*}{1 - \alpha} + m \right) \right\} \]
\[ - [(\beta + m) + (g^* + m + \rho)]m(g^* + m) \]
\[ + \frac{\alpha}{1 - \alpha} (g^* + m + \rho)^2 \frac{m}{g^* + m} - m(g^* + m)(g^* + m + \rho) \]
\[ = (\beta + m) \left\{ (g^* + m + \rho) \left( \frac{g^*}{1 - \alpha} + m \right) + \frac{\alpha}{1 - \alpha} (g^* + m + \rho)^2 \frac{m}{g^* + m} \right\} \]
\[ - (g^* + m + \rho)m(g^* + m) \]
\[ = (g^* + m + \rho) \left\{ (\beta + m) \left[ \left( \frac{g^*}{1 - \alpha} + m \right) + \frac{\alpha}{1 - \alpha} m(g^* + m + \rho) \right] - m(g^* + m) \right\} \]
\[ = (g^* + m + \rho) \left\{ \beta \left( \frac{g^*}{1 - \alpha} + m \right) + m \frac{\alpha}{1 - \alpha} g^* + (\beta + m) \frac{\alpha}{1 - \alpha} m(g^* + m + \rho) \right\} \]
\[ = (g^* + m + \rho) \left\{ \beta \left( \frac{g^*}{1 - \alpha} + m \right) + \frac{\alpha}{1 - \alpha} m \left[ g^* + \frac{(\beta + m)(g^* + m + \rho)}{g^* + m} \right] \right\}. \]

\[ B J^* = \begin{vmatrix} J_{y\eta}^* & J_{y\xi}^* \\ \frac{m}{g^* + m} & -(g^* + m) \end{vmatrix} + \begin{vmatrix} J_{y\eta}^* & J_{yL}^* \\ ma & -(\beta + m) \end{vmatrix} + \begin{vmatrix} -(g^* + m) & 0 \\ 0 & -(\beta + m) \end{vmatrix}. \]
Result 3. If \( q_1 \) and \( q_2 \) are complex conjugates, the real part is not positive.

**Proof:** Rewrite \( f(q) \) as

\[
f(q) = (q_1 - q)(q_2 - q)(q_3 - q) = -q^3 + (q_1 + q_2 + q_3)q^2 - [q_1q_2 + q_3(q_1 + q_2)]q + q_1q_2q_3.
\]

Suppose \( q_1 \) and \( q_2 \) are complex conjugates with positive real part: \( q_{1/2} = \gamma \mp \delta i \) with \( \gamma > 0 \). Then the coefficient of \( q \) in the characteristic equation is negative:

\[-[q_1q_2 + q_3(q_1 + q_2)] = -(\gamma^2 + \delta^2 + q_32\gamma) < 0.\]
This contradicts $B J^* < 0$ and thus proves that exactly two eigenvalues have negative real parts.\(^1\) Q.E.D.

To each eigenvalue $q_j$ corresponds a particular solution $\varphi(t) = b_j e^{q_j t}$ of (T.1), where $b_j = (b_{jq}, b_{jL}, b_{jLN})'$ is the eigenvector associated with the eigenvalue $q_j$. From $(J^* - q_j I)\varphi = 0$, it follows that $(J^* - q_j I)b_j = 0$ or, spelled out in detail,

$$\begin{pmatrix} J_{gg}^* - q_j & J_{gL}^* & J_{LN}^* \\ \frac{m}{\gamma + m} & -(\gamma + m + q_j) & 0 \\ ma & 0 & -(\beta + m + q_j) \end{pmatrix} \begin{pmatrix} b_{jq} \\ b_{jL} \\ b_{jLN} \end{pmatrix} = 0.$$

Eliminating $b_{jL}$ and $b_{jLN}$ yields

$$b_j = b_{jq} \begin{pmatrix} 1 \\ \frac{m}{(\gamma + m)(\gamma + m + q_j)} \\ \frac{ma}{\beta + m + q_j} \end{pmatrix}.$$ \hspace{1cm} (T.2)

The general solution of (T.1) is obtained by combining the particular solutions $\varphi(t) = b_j e^{q_j t}$ for $j = 1, 2, 3$ linearly: $\varphi(t) = \sum_{j=1}^{3} (B_j / b_{jq}) b_j e^{q_j t}$, where the $B_j$'s ($j = 1, 2, 3$) are arbitrary constants. Since we are interested in the behavior of the convergent growth path, the coefficient of the particular solution associated with the unstable eigenvalue $q_3$ has to be set equal to zero: $B_3 / b_{3q} = 0$ and $\varphi(t) = \sum_{j=1}^{2} (B_j / b_{jq}) b_j e^{q_j t}$. Evaluating this equation at $t = 0$ and inserting (T.2) yields:

$$\begin{pmatrix} \tilde{g}(0) \\ \tilde{\xi}(0) \\ \tilde{L}^N(0) \end{pmatrix} = B_1 \begin{pmatrix} 1 \\ \frac{m}{(\gamma + m)(\gamma + m + q_1)} \\ \frac{ma}{\beta + m + q_1} \end{pmatrix} + B_2 \begin{pmatrix} 1 \\ \frac{m}{(\gamma + m)(\gamma + m + q_2)} \\ \frac{ma}{\beta + m + q_2} \end{pmatrix}.$$

Since the initial values $\xi(0)$ and $L^N(0)$ are given, this is a system of algebraic equations in $\tilde{g}(0)$, $B_1$ and $B_2$. Solving for $\tilde{g}(0)$ yields the initial growth rate:

**Result 4.** The initial growth rate $\tilde{g}(0)$ satisfies

$$\tilde{g}(0) = \frac{1}{m(\beta - \gamma)} \left\{ \frac{2}{\prod_{j=1}^{2} (\beta + m + q_j)} \frac{\tilde{L}^N(0)}{a} - \frac{2}{\prod_{j=1}^{2} (\gamma + m + q_j)} (\gamma + m)\tilde{\xi}(0) \right\}.$$

**Proof:** First eliminate $B_2 = \tilde{g}(0) - B_1$:

$$\tilde{\xi}(0) = \frac{mB_1}{(\gamma + m)(\gamma + m + q_1)} - \frac{m[B_1 - \tilde{g}(0)]}{(\gamma + m)(\gamma + m + q_2)},$$

$$\tilde{L}^N(0) = \frac{maB_1}{\beta + m + q_1} + \frac{ma[\tilde{g}(0) - B_1]}{\beta + m + q_2}.$$

\(^1\)This can also be proved by applying the Routh-Hurwitz Theorem, which states that the number of eigenvalues with positive real parts is equal to the number of sign changes in the scheme $-1 || Tr J^* || - B J^* + Det J^* || Det J^*$. If $Tr J^* > 0$ the sign scheme is $-|| + || +||+$, so there is one sign change and hence one unstable eigenvalue. If $Tr J^* < 0$, the sign scheme is $-|| - || -||+$. Again there is one sign change and one unstable eigenvalue.
The adjustment of the rate of innovation obeys the three variables displayed damped oscillations. For instance:

\[ z \] where the other hand, the stable eigenvalues are complex conjugates, that is \( q = \text{real} \). So

If the stable eigenvalues \( q_1 \) and \( q_2 \) are real, then the elements of \( b_1 \) and \( b_2 \) as well as \( B_1 \) and \( B_2 \) are real. So \( \varphi(t) = \sum_{j=1}^{2} (B_j/b_{j\gamma})b_j e^{q_j t} \) implies that \( \tilde{g}, \tilde{\xi} \) and \( \tilde{L}^N \) converge monotonically to zero. If, on the other hand, the stable eigenvalues are complex conjugates, that is \( q_{1/2} = \gamma \mp \delta i \) with \( \gamma < 0 \), then the three variables display damped oscillations. For instance:

**Result 5.** The adjustment of the rate of innovation obeys

\[
\tilde{g}(t) = e^{\gamma t} \left[ \tilde{g}(0) \cos(\delta t) - \frac{z - \gamma \tilde{g}(0)}{\delta} \sin(\delta t) \right],
\]

where \( z \equiv \left[ \prod_{j=1}^{2} (g^* + m + q_j) \right] (g^* + m) \tilde{\xi}(0)/g^* - (g^* + m) \tilde{g}(0) \) is a (real-valued) constant.

*Proof:* Given \( \tilde{g}(0) \), either one of the two formulas for \( \tilde{\xi}(0) \) and \( \tilde{L}^N(0) \) can be used to solve for \( B_1 \). Taking the first one, one obtains:

\[
B_1 = \frac{1}{q_2 - q_1} \left\{ \prod_{j=1}^{2} (g^* + m + q_j) \right\} \frac{g^* + m}{m} \tilde{\xi}(0) - (g^* + m) \tilde{g}(0)
\]

where

\[
z \equiv \left[ \prod_{j=1}^{2} (g^* + m + q_j) \right] \frac{g^* + m}{m} \tilde{\xi}(0) - (g^* + m) \tilde{g}(0)
\]

is a real-valued constant. Moreover,

\[
B_2 = \frac{\tilde{g}(0) - B_1}{q_2 - q_1} = \frac{q_1 - q_2}{q_1 - q_2} \tilde{g}(0) - B_1 = \frac{z - \tilde{g}(0)q_2}{q_1 - q_2}.
\]

The convergent growth path obeys \( \varphi(t) = \sum_{j=1}^{2} (B_j/b_{j\gamma})b_j e^{q_j t} \), hence \( \tilde{g}(t) = B_1 e^{\gamma_1 t} + B_2 e^{\gamma_2 t} \). Inserting the formulas for \( B_1 \) and \( B_2 \) derived above, we have

\[
\tilde{g}(t) = \frac{z - \tilde{g}(0)q_1}{q_2 - q_1} e^{\gamma_1 t} + \frac{z - \tilde{g}(0)q_2}{q_1 - q_2} e^{\gamma_2 t}.
\]
Now suppose the stable eigenvalues \( q_1 \) and \( q_2 \) are complex conjugates: \( q_{1/2} = \gamma \mp \delta i \) (with \( \gamma < 0 \)). Then

\[
\begin{align*}
\tilde{y}(t) &= \frac{z - \tilde{g}(0)q_1}{q_2 - q_1} e^{q_1 t} + \frac{z - \tilde{g}(0)q_2}{q_1 - q_2} e^{q_2 t} \\
&= \frac{z - (\gamma - \delta i)\tilde{g}(0)}{2\delta i} e^{(\gamma - \delta i)t} - \frac{z - (\gamma + \delta i)\tilde{g}(0)}{2\delta i} e^{(\gamma + \delta i)t} \\
2\delta i e^{-\gamma t} \tilde{y}(t) &= \frac{|z - (\gamma - \delta i)\tilde{g}(0)|}{2\delta i} - \frac{|z - (\gamma + \delta i)\tilde{g}(0)|}{2\delta i} \\
&= \frac{\{z - (\gamma - \delta i)\tilde{g}(0)\} - |z - (\gamma + \delta i)\tilde{g}(0)| i \sin(\delta t)}{\delta i} \\
&= 2\delta i \tilde{g}(0) \cos(\delta t) - 2[z - \gamma \tilde{g}(0)] i \sin(\delta t) \\
\tilde{y}(t) &= e^{\gamma t} \left[ \tilde{g}(0) \cos(\delta t) - \frac{z - \gamma \tilde{g}(0)}{\delta} \sin(\delta t) \right].
\end{align*}
\]

Q.E.D.

To prove Theorem 2, it remains to show that the stable eigenvalues \( q_1 \) and \( q_2 \) may in fact be real (as illustrated in Figure T.1) or complex. This is done by way of example. Let \( q_- \) denote the value of \( q \) for which \( f(q) \) attains its local minimum (\( f'(0) = -BJ^* > 0 \) implies that a minimum exists). \( q_1 \) and \( q_2 \) are real if \( f(q_-) < 0 \) and complex otherwise.

**Result 6.** \( q_- \) is given by

\[
q_- = \frac{Tr J^*}{3} - \sqrt{\left(\frac{Tr J^*}{3}\right)^2 - \frac{BJ^*}{3}}.
\]

**Proof:** Equating the derivative of \( f(q) \) to zero yields \( f'(0) = -3q^2 + 2Tr J^* q - B J^* = 0 \) or

\[
q^2 - 2Tr J^* q + \frac{B J^*}{3} = 0.
\]

\( q_- \) is the smaller solution of this quadratic equation. Since \( B J^* < 0 \), the smaller solution is given by:

\[
q_- = \frac{Tr J^*}{3} - \sqrt{\left(\frac{Tr J^*}{3}\right)^2 + \frac{B J^*}{3}}.
\]

Q.E.D.

**Result 7.** The negative eigenvalues \( q_1 \) and \( q_2 \) may be real or complex.

**Proof:** Examples for both cases are easily found. Suppose \( \alpha = 0.5, \beta = 1, \rho = 0.02 \). Further, assume that \( L^N/a \) is such that \( q^* = 0.03 \). Finally, let \( m = 0.1 \). Then \( Tr J^* = -0.94538, \ Det J^* = 0.04349 \) and \( B J^* = -0.09938 \). Hence \( q_- = -0.67904 \) and \( f(q_-) = -0.14680 \). In this example, \( q_1 \) and \( q_2 \) are real.

Now, everything else equal, let \( m = 1 \). Then \( Tr J^* = -0.94942, \ Det J^* = 3.28528 \) and \( B J^* = -2.26455 \), so
that \( q_- = -1.24114 \) and \( f(q_-) = 0.92403 > 0 \). Here, the stable eigenvalues are complex and the dynamics are cyclical. In both examples constructed above, the consistency requirement (10) is satisfied if \( L^S \) is sufficiently great. For the sake of completeness, it may be noticed that, from (9), \( \tilde{L}^N/a = 0.06808 \) in the first example and \( \tilde{L}^N/a = 0.09117 \) in the second one. Q.E.D.

This completes the proof of Theorem 2. Q.E.D.