# Reapproaching Ramsey: Conditionals and Iterated Belief Change in the Spirit of AGM

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**Abstract**

According to the Ramsey Test, conditionals reflect changes of beliefs: $\alpha > \beta$ is accepted in a belief state iff $\beta$ is accepted in the minimal revision of it that is necessary to accommodate $\alpha$. Since Gärdenfors's seminal paper of 1986, a series of impossibility theorems ("triviality theorems") has seemed to show that the Ramsey test is not a viable analysis of conditionals if it is combined with AGM-type belief revision models. I argue that it is possible to endorse that Ramsey test for conditionals while staying true to the spirit of AGM. A main focus lies on AGM's condition of Preservation according to which the original belief set should be fully retained after a revision by information that is consistent with it. I use concrete representations of belief states and (iterated) revisions of belief states as semantic models for (nested) conditionals. Among the four most natural qualitative models for iterated belief change, two are identified that indeed allow us to combine the Ramsey test with Preservation in the language containing only flat conditionals of the form $\alpha > \beta$. It is shown, however, that Preservation for this simple language enforces a violation of Preservation for nested conditionals of the form $\alpha > (\beta > \gamma)$. In such languages, no two belief sets are ordered by strict subset inclusion. I argue that it has been wrong right from the start to expect that Preservation holds in languages containing nested conditionals.

## 1 Introduction: Remarks on the history of the AGM program

The AGM program of belief revision sprang from two sources. Peter Gärdenfors (1978) offered a semantics for Lewis's (1973) logic of conditionals in terms of belief revision models. Carlos Alchourrón and David Makinson (1981; 1982) studied the dynamics of normative systems. The three authors then discovered that the structures they used for these different purposes had much in common and jointly authored their celebrated paper "On the logic of theory change: Partial meet contraction and revision functions" (Alchourrón, Gärdenfors and Makinson, "AGM" 1985). In the present paper, I want to focus on the former strand of research which initiated belief revision research and seemed to be, with the appearance of AGM's paper, one of its most striking success stories. The epistemic or, as we will more precisely say, doxastic analysis suggested by Frank

Ramsey was used as the basis for a logic of conditionals by Robert Stalnaker.[1] Gärdenfors, however, did not want to accept the objective possible-worlds analysis given by Stalnaker and later elaborated by Lewis (1973), but attempted to found conditional logic on an alternative, purely doxastic semantics.[2]

The Ramsey test idea seemed to have come to its perfection in 1985, when AGM supplied a sophisticated model for belief change that was just what was needed in order to make full sense of Ramsey's idea. But only one year later, it was Gärdenfors (1986) himself who discovered that things were much more problematic. In a paper that became almost as famous as the seminal AGM piece, he seemed to establish that Ramsey's test for the acceptability of conditionals and AGM-style belief revision do not go together. His *impossibility* or *trivialization theorem* put everything in question again.

The reactions to Gärdenfors's result were mixed, and no consensus has emerged so far. First, there were people who thought that the right reaction is to say that conditionals should not be analyzed by AGM-style *revisions*, but rather by *updates* in the style of Katsuno and Mendelzon (1992). Members of this party include Ryan and Schobbens (1997), Grahne (1998) and Crocco and Herzig (2002).[3]

Another group of researchers suggested to adapt or restrict the Ramsey test. Solutions of this kind were discussed, amongst others, by Rott (1986), Gärdenfors (1987; 1988), Levi (1996, chapter 2), Lindström and Rabinowicz (1998) and Nute and Cross (2001). None of these adaptations of the Ramsey test has maintained the intuitive appeal of the original idea, and none of them has gained a large number of followers.

My plan for the present paper is to follow neither of these groups. The first group gives a solution for a certain class of conditionals, but the solution departs essentially from the doxastic intentions of AGM. The second group gives up a principle that seems just too intuitive to be of any harm. In the place of a conditional, one could imagine an assertive sentence like "On the supposition that $\alpha$ is true I believe that $\beta$ is true." How should this sentence engender a trivialization? It is just a *description* of the result of a hypothetical assumption. According to the Ramsey test, a conditional is an *expression* of the same thing, but the difference between description and expression does not seem to create

[1] "Epistemic" means "relating to knowledge", "doxastic" means "relating to belief". – Ramsey (1931, p. 247): "If two people are arguing 'If $p$ will $q$?' and are both in doubt as to $p$, they are adding $p$ hypothetically to their stock of knowledge and arguing on that basis about $q$". Stalnaker (1968, p. 102): "This is how to evaluate a conditional: First, add the antecedent (hypothetically) to your stock of beliefs; second, make whatever adjustments are required to maintain consistency (without modifying the hypothetical belief in the antecedent); finally, consider whether or not the consequent is then true." Both authors use the concept of *hypothetical addition*. Our task may thus be seen as one of finding out whether AGM offer the right sort of model for hypothetical addition.

[2] That it is a good idea to base just Lewis's (1973) logic VC on such a semantics is disputable. The idea was vigorously criticized by Isaac Levi (1996, Ch. 4).

[3] Updates in this sense are closely related to David Lewis's (1976) imaging. These operations blatantly violate all AGM-style Preservation conditions, but are very friendly to the Monotonicity condition. These conditions will be discussed below.

any special dangers. In the present paper I shall always take the Ramsey test for granted and see how it can be fit into the AGM paradigm. My claim will be that the problem of reconciling Ramsey with the spirit of AGM has a unique, intuitive solution.

The argument for this claim is based on a particular interpretation of the AGM program of belief change. With the benefit of hindsight, I think it is justified to describe the AGM theory as an essentially *semantic* one. AGM may have offered their theory without being aware of this, but their insistence on the logical closure of "belief sets" and the syntax-independence of the effects of new information indicated right from the beginning that it is the content and not the form of the sentences expressing beliefs that matters. However, it was only after Grove (1988) and Katsuno and Mendelzon (1991) had shown the equivalence of the AGM theory with a semantics using Lewis-style systems of spheres or pre-orderings of possible worlds that the essentially semantic nature of the AGM project became widely recognized. I will embrace this semantic reinterpretation of AGM, use possible worlds models as representations of belief states and consider a few (to be more precise: four) constructive models for the revision of such states. They will allow us to deal with iterated belief changes which are necessary for a Ramsey test interpretation of nested (to be more precise: right-nested) conditionals. Much of the plausibility of my argument will hinge on the plausibility of the claim that these four constructions are in fact the canonical ways of extending the original AGM approach to the problem of iterated belief change (about which AGM themselves said very little).

I believe that the focus on semantic structures and constructions for their transformation provides a solid kind of understanding that cannot easily be gained from the more postulational approaches that have dominated research in the AGM paradigm. The AGM postulates entail a condition of Preservation of accepted sentences on the acquisition or assumption of a sentence that is consistent with the prior beliefs. As we shall see, it is indeed possible to combine the Ramsey test with an AGM-style Preservation condition, if the language is restricted to non-nested conditionals. This combination, however, forces a violation of Preservation for more complex conditionals. I will argue that this is not a defect of the constructions, but rather shows that the idea that Preservation (should) be obeyed in languages admitting nested conditionals has simply been misguided. Even the preservation of flat (i.e., non-nested) conditionals has a rather weak intuitive basis. A general appeal to considerations of minimum mutilation of informational economy turns out to be too airy to bear up against the challenges presented by the reasoning with conditionals.

## 2 Belief states and belief revision models in general

Following the tradition of Darwiche and Pearl (1997), we use the concept of a *belief state* as a primitive. Belief states will be denoted by the letter $\mathcal{B}$, possibly decorated with primes and subscripts. We assume that there is a designated belief state $\mathcal{O}$ representing the "blank slate". $\mathcal{O}$ is the state of complete ignorance or of doxastic emptiness.[4]

A sentence $\alpha$ is either accepted or rejected in a belief state $\mathcal{B}$, or else a person in $\mathcal{B}$ suspends judgement with respect to $\alpha$. For a conditional, it is not determined by the current belief state alone whether or not it is accepted, rejected or neither. By the Ramsey test, this depends on a hypothetical change of the belief state.

Now let $\mathbb{B}$ be a set of belief states, and let $* : \mathbb{B} \times L \to \mathbb{B}$ be a *revision function* assigning to a pair consisting of a ("prior") belief state $\mathcal{B}$ and a sentence $\alpha$ from a language $L$ another ("posterior") belief state. $\alpha$ is a new piece of information or a hypothetical assumption.

A belief state $\mathcal{B}$ in a belief revision model $\langle \mathbb{B}, \mathcal{O}, * \rangle$ validates or supports a set of sentences. We write $\ulcorner \mathcal{B} \urcorner^*_L$ for the set of sentences so supported. These are the $L$-sentences that are believed to be true, or accepted, in the belief state $\mathcal{B}$. The asterisk in the superscript signals the dependence on the belief revision model's revision function, the $L$ in the subscript points to the fact that the claim is always to be restricted to a certain language. $\alpha \in \ulcorner \mathcal{B} \urcorner^*_L$ is read as follows: $\alpha \in L$ is accepted in belief state $\mathcal{B}$ in $\mathbb{B}$ with respect to revision method $*$; or alternatively, $\mathcal{B}$ in $\mathbb{B}$ supports $\alpha$ of $L$ with respect to $*$. Having made this clear, we shall drop subscripts and superscripts in contexts in which they are not immediately relevant.

Here are the famous *AGM postulates*, translated into a language that allows us to talk about belief states rather than just sets of belief (expressed by sentences):

(AGM*1) $\ulcorner \mathcal{B} * \alpha \urcorner$ is closed under $Cn$
(AGM*2) $\alpha \in \ulcorner \mathcal{B} * \alpha \urcorner$
(AGM*3) $\ulcorner \mathcal{B} * \alpha \urcorner \subseteq Cn(\ulcorner \mathcal{B} \urcorner \cup \{\alpha\})$
(AGM*4) If $\alpha$ is consistent with $\ulcorner \mathcal{B} \urcorner$, then $Cn(\ulcorner \mathcal{B} \urcorner \cup \{\alpha\}) \subseteq \ulcorner \mathcal{B} * \alpha \urcorner$
(AGM*5) If $\alpha$ is consistent under $Cn$, so is $\ulcorner \mathcal{B} * \alpha \urcorner$
(AGM*6) If $Cn(\alpha) = Cn(\beta)$, then $\ulcorner \mathcal{B} * \alpha \urcorner = \ulcorner \mathcal{B} * \beta \urcorner$
(AGM*7) $\ulcorner \mathcal{B} * (\alpha \wedge \beta) \urcorner \subseteq Cn(\ulcorner \mathcal{B} * \alpha \urcorner \cup \{\beta\})$
(AGM*8) If $\beta$ is consistent with $\ulcorner \mathcal{B} * \alpha \urcorner$, then $Cn(\ulcorner \mathcal{B} * \alpha \urcorner \cup \{\beta\}) \subseteq \ulcorner \mathcal{B} * (\alpha \wedge \beta) \urcorner$

Common names for these conditions are *Closure*, *Success*, *Inclusion*, *Vacuity*, *Consistency*, *Extensionality*, *Superexpansion* and *Subexpansion* (Hansson 1999, pp. 212, 216). Notice that in all postulates except Success, reference is made to some background logic or consequence operation $Cn$.[5] It is evident that for the proper appreciation of the AGM program much depends on the understanding of the background logic.

[4]The notion that the human mind is initially like a blank slate has had a long history in philosophy. It was most famously advocated, in varying versions, by Aristotle (*De anima* III, 4, 429b29-430a2) and Locke (*Essay Concerning Human Understanding*, I.2.15 and II.1.2).

[5]$Cn$ takes sets of sentences as arguments. We will abbreviate $Cn(\{\alpha\})$ by $Cn(\alpha)$.

## 3 The impossibility result

Gärdenfors's theorem in effect says that the Ramsey test is incompatible with a condition of Preservation regarding the case in which the input is consistent with the prior beliefs held by the person.

(RT) $\alpha > \beta$ is in $\ulcorner\mathcal{B}\urcorner$ iff $\beta$ is in $\ulcorner\mathcal{B} * \alpha\urcorner$

(Pres) If $\alpha$ is consistent with $\ulcorner\mathcal{B}\urcorner$, then $\ulcorner\mathcal{B}\urcorner \subseteq \ulcorner\mathcal{B} * \alpha\urcorner$

Both principles have very strong intuitions speaking in their favour. The Ramsey test (RT) is not offered here as an analysis of natural language conditionals, but as the defining acceptance condition for *Ramsey test conditionals*. Why shouldn't it be possible to record potential or hypothetical changes of belief by means of sentences in our language? There is no threat of self-reference here, and no constraint of assertibility that gets in our way. So (RT) seems fine. On the other hand, (Pres) looks like a plausible condition, too. If a piece of new information or a hypothetical assumption is consistent with one's beliefs, why should one give up any of them? There simply seems to be no reason to do so.

It is surprising that (RT) and (Pres) are not compatible. For the incompatibility result two additional background assumptions are needed that we do not want to challenge in this paper: The Success and the Consistency condition, (AGM*2) and (AGM*5).

Two other conditions are worth mentioning in the present context. First, the AGM condition (AGM*4) is clearly a strengthening of (Pres).[6] Second, (RT) entails a version of the following *Monotonicity condition*

(Mon) If $\ulcorner\mathcal{B}\urcorner \subseteq \ulcorner\mathcal{B}'\urcorner$, then $\ulcorner\mathcal{B} * \alpha\urcorner \subseteq \ulcorner\mathcal{B}' * \alpha\urcorner$

(Mon) has been frequently used in proofs of the impossibility theorem, and it is advantageous to their elegance. Still I think that from a heuristic point of view, the common shift in the attention from the Ramsey test to Monotonicity has been a mistake. It tends to cover up the fact that all the motivation for (Mon) derives from the Ramsey test for conditionals, and that it is really the behaviour of conditionals that should be studied. It also hides one level of the nesting of conditionals. In order to assess (Mon) properly, we have to have a closer look at the languages involved (see section 5 below). Finally, while (RT) is an interesting condition, (Mon) will turn out to be uninteresting in the context of languages permitting nested conditionals, because it is satisfied only vacuously in such contexts.

## 4 Proving the result

**4.1 Existing literature.** The crucial assumption in Gärdenfors's original proof (1986, 1988) relates revisions by a sentence $\alpha$ that is consistent with $\ulcorner\mathcal{B}\urcorner$

[6] Conversely, (AGM*4) follows from (Pres) in conjunction with Success and Closure. From an aesthetic point of view, (Pres) would be nicer as an AGM postulate than (AGM*4).

to revisions by disjunctions of $\alpha$ with some other sentence. Gärdenfors took the following to be self-evident:

$$Cn(K \cup \{\alpha \vee \beta\}) \subseteq Cn(K \cup \{\alpha\})$$

But what is $Cn$ in the context of Gärdenfors's powerful conditional language that allows for unrestricted embeddings of conditionals? Are we entitled to use the sort of monotonicity expressed in this condition?

Makinson (1990) shows that allowing the logic to be non-monotonic does not necessarily block triviality. A variant of Gärdenfors's triviality theorem carries over to certain non-monotonic contexts. A closer inspection of Makinson's proof, however, reveals a crucial assumption that may, and indeed should, be questioned. Makinson makes it part of the meaning of the notion of "non-triviality" that for two classically unrelated factual sentences $\alpha$ and $\beta$, $Cn(\alpha)$ is consistent with $Cn(\beta)$. But the problem is again that we do not know what $Cn$ looks like in the context of languages having conditionals. Would it be implausible to expect that $\neg(\alpha \wedge \beta) > \alpha$ is in $Cn(\alpha)$, but $\neg(\alpha \wedge \beta) > \neg\alpha$ is in $Cn(\beta)$? And, as Ramsey himself suggested, these conditionals can be regarded as "contradictories". In sum, it seems that Makinson imposed unduly heavy demands on non-triviality and that his paper, though explicitly attending to the idea of non-monotonicity, proceeds still very much in the spirit of monotonic logic.

Lindström and Rabinowicz (1998, p. 151) discuss a possible way of finding two belief sets that are related by strict subset inclusion. They consider an agent who at first "has no opinion about" some $\alpha$ and $\beta$ and then comes to learn $\alpha$ in one scenario and alternatively $\alpha \wedge \beta$ in another scenario, where by "learning" they appear to mean consistent learning. Lindström and Rabinowicz take it for granted that the belief set resulting from the former is a subset of the belief set resulting from the latter information. This again may be doubted for reasons similar to the one we met above. We expect that $\neg(\alpha \wedge \beta) > \alpha$ is in $Cn(\alpha)$, but not in $Cn(\alpha \wedge \beta)$.[7]

In a recent paper on the Ramsey test, Bradley (2007, p. 8) considers

> a consistent belief set $K$ that contains the sentence $\alpha$ and two subsets $L$ and $M$ of $K$ respectively containing the sentences $\alpha \vee \beta$ and $\alpha \vee \neg\beta$. Then by K*4 and K*2 (the Preservation and Success conditions), $\alpha \vee \beta$, $\neg\alpha \in L * \neg\alpha$ and $\alpha \vee \neg\beta$, $\neg\alpha \in M * \neg\alpha$. Hence $\beta \in L * \neg\alpha$ and $\neg\beta \in M * \neg\alpha$. Then if belief revision is monotonic $K * \neg\alpha$ will contain both $\beta$ and $\neg\beta$. Since it should not, either RT or CDR [i.e., the Ramsey test] must be rejected.

[7] In the footnote on p. 151, Lindström and Rabinowicz warn us that appearances are deceptive here. In fact, later in the paper they claim to have presented a "*reductio* proof" of the negation of a Non-triviality condition requiring the subset condition mentioned in the text, and that "we have no reasons to believe [this sort of] Non-Triviality to be true" (p. 181). Although they reach this conclusion by the different route of an indexical interpretation of conditionals, their findings are consonant with the message of the present paper.

Bradley gives a diagnosis which is meant to defend the Ramsey test against Gärdenfors's attack. He criticizes this analysis, rightly I think, for failing "to consider whether the postulated subsets $L$ and $M$ really meet the conditions for epistemic states." (2007, p. 8) But Bradley says very little about what conditions for epistemic states he has in mind. The only logical principle he mentions is "Conditional Contradiction"

(CC) If $K$, $\beta$, $\gamma \vdash \bot$, then $K$, $\alpha > \beta$, $\alpha > \gamma \vdash \bot$

(CC), however, can hardly be valid. Consider, for instance, the case $\alpha = \beta = \gamma = p$ and the consistent belief set $K = Cn(\neg p)$. It seems clear that this is not a limiting-case problem, because $\beta$ and $\gamma$ may well be "counterfactual" from the point of view of $K$, but perfectly acceptable under the (counterfactual) assumption $\alpha$. Bradley argues that

> [t]he real source of the impossibility result would seem *thus* to lie not with RT or CDR [i.e., the Ramsey test] but PRES. *For* the fact is that PRES alone pretty much rules out any interpretation of the conditional connective $>$ other than the material conditional one ... (2007, p. 8, my emphasis)

Why "thus", why "for"? Why exactly does Preservation alone "pretty much" rule out anything but material conditionals? Bradley's (2007, pp. 8–9) argument appeals to the invalid principle (CC) and to Modus Ponens for nested conditionals, which has been called into question by McGee (1985). We will have a closer look on the postulate of Preservation below.

This short overview is certainly not exhaustive, but it does not seem unfair to conclude that the existing proofs in the literature are based on questionable assumptions and do not yield the kind of insight into the nature of the problem that one would like to gain.

**4.2 Two other attempts.** We have seen in our review of the proofs presented by Gärdenfors, Makinson, Lindström and Rabinowicz, and Bradley that it is not easy to make the idea of the trivialization theorem transparent. Let us try to do this on our own account. In both of the following proof sketches, we start from the blank slate $\mathcal{O}$. The first version compares two situations in which two factual sentences $\alpha$ and $\beta$ that are learned (or added hypothetically) in different order. $\alpha$ and $\beta$ are supposed to be independent in terms of classical propositional logic. At the end the results are used for checking the acceptance of two conditionals with the common antecedent $\neg(\alpha \wedge \beta)$. Given the Ramsey test, a simultaneous acceptance of both $\neg(\alpha \wedge \beta) > \alpha$ and $\neg(\alpha \wedge \beta) > \beta$ would contradict the principle of Consistency, (AGM*5) (see Fig. 1).

Without going into details, it seems plausible to assume that $\neg(\alpha \wedge \beta) > \alpha$ is accepted in $\mathcal{O} * \alpha$ and $\neg(\alpha \wedge \beta) > \beta$ is accepted in $\mathcal{O} * \beta$. Let us also suppose for the sake of argument that the steps from $\mathcal{O} * \alpha$ to $(\mathcal{O} * \alpha) * \beta$ and from $\mathcal{O} * \beta$ to $(\mathcal{O} * \beta) * \alpha$ are preservative, i.e., that nothing of the prior belief sets $\mathcal{O} * \alpha$

$$
\begin{array}{ccc}
\text{If } \neg(\alpha \wedge \beta)\text{, then } \alpha & & \text{If } \neg(\alpha \wedge \beta)\text{, then } \beta \\
\subseteq & & \subseteq \\
\ulcorner \mathcal{O} * \alpha \urcorner & & \ulcorner \mathcal{O} * \beta \urcorner \\
\subseteq & & \subseteq \\
\ulcorner (\mathcal{O} * \alpha) * \beta \urcorner & & \ulcorner (\mathcal{O} * \beta) * \alpha \urcorner
\end{array}
$$

*Fig. 1: First attempt*

and $\mathcal{O} * \beta$ gets lost in these transitions. Then still, if this proof sketch is to work, one has to assume that $\ulcorner (\mathcal{O} * \alpha) * \beta \urcorner = \ulcorner (\mathcal{O} * \beta) * \alpha \urcorner$. I find it hard to see any strong arguments that might support this assumption. So this attempt at trivializing the Ramsey test does not work.

We can try to circumvent the problem of a potential order-dependence in $(\mathcal{O} * \alpha) * \beta$ and $(\mathcal{O} * \beta) * \alpha$ by working with $\mathcal{O} * (\alpha \wedge \beta)$ instead for both branches of the proof (see Fig. 2).

$$
\begin{array}{ccc}
\text{If } \neg(\alpha \wedge \beta)\text{, then } \alpha & & \text{If } \neg(\alpha \wedge \beta)\text{, then } \beta \\
\subseteq & & \subseteq \\
\ulcorner \mathcal{O} * \alpha \urcorner & & \ulcorner \mathcal{O} * \beta \urcorner \\
\subseteq & & \subseteq \\
& \ulcorner \mathcal{O} * (\alpha \wedge \beta) \urcorner &
\end{array}
$$

*Fig. 2: Second attempt*

This proof sketch solves the identity problem in the bottom line.[8] However, there is a new problem now: How can we be sure that $\ulcorner \mathcal{O} * \alpha \urcorner$ and $\ulcorner \mathcal{O} * \beta \urcorner$ are subsets of $\ulcorner \mathcal{O} * (\alpha \wedge \beta) \urcorner$? It seems that we cannot. Our second attempt at trivializing the Ramsey test does not seem to work either.

So we still lack an instructive proof for Gärdenfors's notorious result. Too much has remained unclear about the attempts of proving the triviality theorem. We

[8]The following variant of the proof sketch in Fig. 2 may be even more illuminating:

$$
\begin{array}{ccc}
\text{If } \alpha\text{, then } \beta & & \text{If } \alpha\text{, then } \neg\beta \\
\subseteq & & \subseteq \\
\ulcorner \mathcal{O} * (\neg\alpha \vee \beta) \urcorner & & \ulcorner \mathcal{O} * (\neg\alpha \vee \neg\beta) \urcorner \\
\subseteq & & \subseteq \\
& \ulcorner \mathcal{O} * \neg\alpha) \urcorner &
\end{array}
$$

need to find a more principled approach to attack the problem. My proposal is to turn to constructive semantic modellings of potential changes of belief states rather than to continue focussing on postulates about which our intuitions have turned out to be shaky. Before doing that (in Section 7), a few preparatory distinctions between languages (Section 5) and logics (Section 6) are in order.

## 5 Languages

So far I have avoided an explicit specification of the languages involved. But for a finer analysis of the situation, we need to distinguish several layers of complexity in conditionals. Languages will be identified with the set of sentences that can be formed in them. Let $L_0$ be a propositional language capable of building *factual* sentences from atoms with the help of the usual connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$. The languages we are going to consider are *extensions of* $L_0$ by one extra rule for conditionals:

$L_1$ if $\alpha$ and $\beta$ are in $L_0$, then $\alpha > \beta$ is in $L_1$ (*flat conditionals*)

$L_2$ if $\alpha$ is in $L_0$ and $\beta$ is in $L_1$, then $\alpha > \beta$ is in $L_2$

$L_3$ if $\alpha$ is in $L_0$ and $\beta$ is in $L_3$, then $\alpha > \beta$ is in $L_3$ (*right-nested conditionals*)

$L_4$ unrestricted embedding of conditionals (free combinations of $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $>$)

Clearly, $L_0 \subseteq L_1 \subseteq L_2 \subseteq L_3 \subseteq L_4$. Gärdenfors used the strongest language $L_4$. We will stay clear of this language, because negated conditionals or conditionals embedded in material conditionals are very difficult to understand, and the important phenomena occur at the levels of $L_1$ and $L_2$. Using conditionals within the scope of a truth-functional connective seems to presuppose that conditionals (Ramsey test conditionals) express propositions, and I do not want to commit myself to such a position. Left-nested conditionals could also be considered if we possessed a good way of revising belief states by conditionals of the form $\alpha > \beta$ with $\alpha$ and $\beta$ from $L_0$. Unfortunately, there is little consensus on how such revisions should be performed,[9] so we are not going to talk about them.

In the following, we sometimes continue to use "$\ulcorner \mathcal{B} \urcorner$" even though our notation should now really be extended to "$\ulcorner \mathcal{B} \urcorner_{L_i}$", with $i$ appropriately replaced by some numeral. I will use the abbreviated notation only in contexts in which no confusion can arise.

We can now distinguish various levels of Preservation. That an $L_0$-sentence $\alpha$ is consistent with a belief state $\mathcal{B}$ is taken to mean that $\neg\alpha$ is not in $\ulcorner \mathcal{B} \urcorner_{L_0}$. The following conditions are possible precisifications of the unqualified Preservation condition.

($L_0$-Pres) If $\alpha \in L_0$ is consistent with $\mathcal{B}$ , then $\ulcorner \mathcal{B} \urcorner_{L_0} \subseteq \ulcorner \mathcal{B} * \alpha \urcorner_{L_0}$

[9] But see the interesting accounts of Boutilier and Goldszmidt (1995), Nayak et al. (1996) and Kern-Isberner (1999).

($L_1$-Pres) If $\alpha \in L_0$ is consistent with $\mathcal{B}$ , then $\ulcorner\mathcal{B}\urcorner_{L_1} \subseteq \ulcorner\mathcal{B} * \alpha\urcorner_{L_1}$
($L_2$-Pres) If $\alpha \in L_0$ is consistent with $\mathcal{B}$ , then $\ulcorner\mathcal{B}\urcorner_{L_2} \subseteq \ulcorner\mathcal{B} * \alpha\urcorner_{L_2}$

We will not have to go further than ($L_2$-Pres).

Strictly speaking, the Ramsey test involves a shift in the levels of our language hierarchy. This is indicated by the following scheme.

(RT$_i$) $\alpha > \beta$ is in $\ulcorner\mathcal{B}\urcorner_{L_{i+1}}$ iff $\beta$ is in $\ulcorner\mathcal{B} * \alpha\urcorner_{L_i}$

Similarly, we can now reconsider Monotonicity. The formulation of (Mon) in Section 3 is impeccable if the sets of sentences considered are from $L_3$ or $L_4$. In the restricted contexts of $L_1$ and $L_2$, we have to pay more attention. All we can derive from the Ramsey test is

($L_1$-$L_0$-Mon) If $\ulcorner\mathcal{B}\urcorner_{L_1} \subseteq \ulcorner\mathcal{B}'\urcorner_{L_1}$, then $\ulcorner\mathcal{B} * \alpha\urcorner_{L_0} \subseteq \ulcorner\mathcal{B}' * \alpha\urcorner_{L_0}$
($L_2$-$L_1$-Mon) If $\ulcorner\mathcal{B}\urcorner_{L_2} \subseteq \ulcorner\mathcal{B}'\urcorner_{L_2}$, then $\ulcorner\mathcal{B} * \alpha\urcorner_{L_1} \subseteq \ulcorner\mathcal{B}' * \alpha\urcorner_{L_1}$

and so on for more complex conditional nestings. The inclusion in the antecedent has always to be one level of nesting higher in order to guarantee the inclusion in the consequent. The proof of each of these variants from the Ramsey test is extremely simple: Let $\beta \in \ulcorner\mathcal{B} * \alpha\urcorner_{L_i}$ ($i = 0, 1, \ldots$). Then $\alpha > \beta \in \ulcorner\mathcal{B}\urcorner_{L_{i+1}}$ by (RT$_i$), so $\alpha > \beta \in \ulcorner\mathcal{B}'\urcorner_{L_{i+1}}$ by the antecedent of Monotonicity, thus $\beta \in \ulcorner\mathcal{B}' * \alpha\urcorner_{L_i}$ by (RT$_i$) again.

In Section 8.1, we shall also have a brief look at the extension of $L_0$ by a doxastic modal $\Diamond$.

$L_\Diamond$ if $\alpha$ is in $L_\Diamond$, so is $\Diamond\alpha$

Here $\Diamond\alpha$ should be read as "for all I believe, it is possible that $\alpha$" or "for all I believe, it might be the case that $\alpha$".

## 6 Logics

The mutual relationship between logic and belief revision is troubled by a *circularity problem.* On the one hand, almost all of the belief revision postulates refer to some background logic *Cn*, and in this sense logic seems to be prior to belief revision. On the other hand, Gärdenfors used the Ramsey test for deriving some logic of conditionals from the systematic behaviour of belief revision models. In this sense, belief revision appears to be prior to logic. In any case, if our language has conditionals, then the logic should presumably include some logical principles involving conditionals to begin with.

Possible worlds are semantic items that are logically closed, by the semantic definition of logical entailment. I suggest exactly the same for belief states. The set $\ulcorner\mathcal{B}\urcorner$ of the sentences that are supported by a given belief state $\mathcal{B}$ is closed.[10]

[10]The set of sentences satisfied at a possible world is also consistent by definition. The same might be said for "good" belief states, but I don't want to restrict the concept of a belief

Belief states are always parts of belief revision models. As mentioned above, a belief state doesn't support sentences *per se*, if the language used contains conditionals. A reference to other belief states is essential. Given such a doxastic semantics, different definitions of logical consequence are conceivable.[11] Let us first introduce a *belief state logic*:

(Def $Cn^-$) $\quad \alpha \in Cn^-(M)$ iff for all BRMs $\langle \mathbb{B}, \mathcal{O}, * \rangle$ and all $\mathcal{B}$ in $\mathbb{B}$ , if $\beta \in \ulcorner \mathcal{B} \urcorner$ for all $\beta$ in $M$, then $\alpha \in \ulcorner \mathcal{B} \urcorner$.

Here, $\alpha$ and $M$ may be from any language $L_i$ as specified above. This definition is less determinate than it looks, because in various contexts one will want to place various constraints on $\mathbb{B}$, $\mathcal{O}$ and $*$. We will not pursue this topic systematically in this paper, but we will soon propose special formats for $\mathbb{B}$ (Section 7.1) and particular instantiations of $*$ (Section 7.2).

The *belief change logic* is defined only for single premises $\beta$ from $L_0$:

(Def $Cn^+$) $\quad \alpha \in Cn^+(\beta)$ iff for all BRMs $\langle \mathbb{B}, \mathcal{O}, * \rangle$, $\alpha \in \ulcorner \mathcal{O} * \beta \urcorner$.

The belief change logic makes reference to the distinguished belief state $\mathcal{O}$. It is a lot bolder than the belief state logic that quantifies over all belief states. Due to the Success condition (AGM*2), $Cn^-(\beta) \subseteq Cn^+(\beta)$ for every $\beta$ in $L_0$. The latter set only requires its elements to be in $\ulcorner \mathcal{O} * \beta \urcorner$ (which contains $\beta$), the former that its elements be in all $\ulcorner \mathcal{B} \urcorner$ containing $\beta$.

As the following example bears out, it is intuitively quite plausible that the logic of conditionals should be *non-monotonic* in the sense that a strengthening of the premise(s) of an inference may well lead to the loss of some conclusions. The following inference seems paradigmatically desirable for doxastic conditionals:

$$\beta \vee \gamma \;\mid\!\sim \neg\beta > \gamma$$

If *all we believe* is a disjunction then the conditional with a negated disjunct in the antecedent and the other disjunct unnegated in the consequent is very hard to deny. But the next inference seems paradigmatically undesirable:

$$\beta \;\mid\!\sim \neg\beta > \gamma$$

Although logically stronger according to the canons of propositional logic, the premise here gives no reason whatsoever for accepting the conclusion. How is this possible?[12] It is $Cn^+$ which is to account for these aspects of reasoning with conditionals. But as we also need the monotonic notion of $Cn^-$, we keep

state to consistent states here. On the other hand, the postulation of closure indicates that we are not talking about occurrent or professed beliefs here, but rather about ascribed beliefs or beliefs as commitments. In the context of the present paper, closure is guaranteed by the integrity of the semantic models. They represent belief states in equilibrium.

[11] Compare Veltman (1996, pp. 224–225) for more and more general notions of logical consequence ('validity of arguments' in his terminology).

[12] Notice how important it is that $\beta \vee \gamma$ is *all* one believes in the first inference pattern. Only-believing is of course not closed under any $Cn$. If $\alpha$ logically entails $\beta$ and I only-believe that $\alpha$, it does not follow that I only-believe that $\beta$. On the logic of only-believing, compare Levesque (1990), the title of which mentions, somewhat confusingly, only-*knowing*.

both as we move along. We now turn to more specific modellings of belief states and their changes.

## 7 Belief states and belief revision models: Specific representations

**7.1 Particular modellings of belief states.** In AGM's theory, doxastic states were identified with sets of sentences of the propositional language $L_0$, so-called *belief sets*, that is, sets of sentences of a propositional language. But it was soon noticed that for iterated belief changes one needs changes of more complex *belief states*. We will follow a tradition in the belief revision literature and represent a *doxastic state* or *belief state* $\mathcal{B}$ by a preordering of possible worlds, where a *possible world* is an interpretation of $L_0$.[13] We shall use a strict relation $<$ between possible worlds and require that it be *asymmetric* ($w < w'$ implies $w' \not< w$) and *modular* ($w < w'$ implies $w < w''$ or $w'' < w'$).[14] Asymmetry and modularity taken together imply transitivity. Intuitively, $<$ represents comparative plausibility as seen from the person's doxastic point of view. $w < w'$ expresses the fact that $w$ is more plausible or, in a pretheoretical sense, more likely to be the real world than $w'$.[15] As a second component of a belief state, we include an explicit specification of the field $W$ of $<$. We do not require $W$ to be the universal set $W_{\mathrm{all}}$ of all interpretations of $L_0$.[16] Intuitively, $W$ denotes the set of worlds that are *doxastically possible* for (i.e., *considered possible* by) a person in belief state $\mathcal{B} = \langle W, < \rangle$. A sentence $\alpha$ from $L_0$ is called *doxastically possible* with respect to $\langle W, < \rangle$ iff $[\alpha] \cap W$ is non-empty, where $[\alpha]$ is the set of interpretations that evaluate $\alpha$ as true. For $w, w' \in W$, we write $w \sim w'$ iff neither $w < w'$ nor $w' < w$, and $w \leq w'$ iff either $w < w'$ or $w \sim w'$. By the asymmetry of $<$, $w \leq w'$ is equivalent with $w' \not< w$. Notice that both $\sim$ and $\leq$ are transitive, due to the modularity of $<$. With a slight abuse of the notation, we will sometimes write $w \ll w'$ if $w \in W$ and $w' \notin W$. To keep things mathematically simple, we assume throughout this paper that all sets involved are finite.

Alternatively and equivalently, we can represent a belief state by a Lewis-Grove style *system of spheres* of possible worlds for $L_0$. A system of spheres \$ is a non-empty set of non-empty sets of possible worlds that is linearly ordered by the subset relation, i.e., for all $S$ and $S'$ in \$, either $S \subseteq S'$ or $S' \subseteq S$.[17]

---

The soft $\mid\!\sim$ in the place of the hard $\vdash$ is supposed to indicate that the inferences are of a defeasible kind. The same paradigmatic pair of defeasible inferences was the starting point of the foundationalist approach of Rott (1991). Due to its focus on semantics, the present approach is a coherentist one. For the application of the epistemological distinction between *coherentism* and *foundationalism* to problems of belief change, see Rott (2001, chapter 3).

[13] See, for instance, Katsuno and Mendelzon (1991).

[14] Modularity is also called *virtual connectedness* or *negative transitivity*.

[15] Notice the slightly counter-intuitive reversal of the relation here. Traditionally, 'more plausible' has been read as 'less far-fetched' or 'less deviating from one's beliefs and expectations'.

[16] This is in order to make room for the model of 'radical revision' below, but it is in tension with (AGM*5).

[17] See Lewis (1973) and Grove (1988). The connection between preorderings and systems of

We say that a world $w$ is *doxastically preferred* in the state represented by $\langle W, < \rangle$ if $w$ is minimal in $W$ under $<$, or if $w \in \bigcap \$$ in the system of spheres modelling. The set of sentences that are true at all doxastically preferred worlds is the set of sentences supported by the belief state, $\ulcorner \langle W, < \rangle \urcorner$ or $\ulcorner \$ \urcorner$. Let us call a belief state $<$ *opinionated* or *dogmatic* if it has only one doxastically preferred world. Call it *undogmatic* if it has at least two doxastically preferred worlds, i.e., if for all $w \in W$ there is a $w' \neq w$ such that $w' \leq w$.[18] Only notorious know-it-alls have an opinion about every single proposition. Every reasonable person should suspend judgment about some atom $p$. We can think of non-dogmatism as a condition of *non-triviality*.

A belief state $\mathcal{B}'$ is a refinement of another belief state $\mathcal{B}$ if it keeps all the doxastic distinctions recognized by the latter. A proper refinement $\mathcal{B}'$ of $\mathcal{B}$ keeps all of $\mathcal{B}$'s distinctions and adds new ones, so that $\mathcal{B}$ is not a refinement of $\mathcal{B}'$. Refinements can elegantly be defined in terms of strict preorderings and systems of spheres. The preordering $<'$ is a *refinement* of the preordering $<$ (within the subfield $V \subseteq W$) iff for all worlds $w$ and $w'$ (in $V$), if $w < w'$ then $w <' w'$ (since the preorderings are assumed to be modular, this is equivalent to saying that if $w' \leq' w$ then $w' \leq w$). The system of spheres $\$'$ is a *refinement* of the system of spheres $\$$ iff all the spheres $S$ in $\$$ are contained in $\$'$. $\$'$ is a *refinement* of $\$$ within the subfield $V \subseteq W$ iff all the spheres $S_V$ in $\$ \cap V = \{S \cap V\colon\ S \text{ is in } \$\}$ are contained in $\$' \cap V$.

There is another, more genearal relationship between systems of spheres we will have reason to consider. The preordering $<'$ is a *nonreversal* of the preordering $<$ (within the field $V \subseteq W$) iff for all worlds $w$ and $w'$ (in $V$), if $w < w'$ then $w \leq' w'$ (this is equivalent to saying that not both $w < w'$ and $w' <' w$). The system of spheres $\$'$ is a *nonreversal* of the system of spheres $\$$ iff $\$$ and $\$'$ have a common refinement, or more succinctly, if $\$ \cup \$'$ is a system of spheres. $\$'$ is a *nonreversal* of $\$$ within the field $V \subseteq W$ iff $\$$ and $\$'$ have a common refinement within $V$, or more succinctly, if $(\$ \cap V) \cup (\$' \cap V)$ is a system of spheres.

The blank slate $\mathcal{O}$ knows of no distinctions. In our particular modellings, it is represented by the empty strict preordering $\langle W_{\text{all}}, \emptyset \rangle$ or the singleton system of spheres $\$ = \{W_{\text{all}}\}$.

Preorderings of possible worlds and systems of spheres can be considered as projections of possible belief states. Belief sets clearly don't capture everything

spheres is as follows: An ordering $<$ is obtained from a system of spheres $\$$ by defining $w < w'$ iff there is an $S \in \$$ such that $w \in S$ but $w' \notin S$; the field $W$ of $<$ is $\bigcup \$$. Conversely, a system of spheres $\$ = \{S_0, \ldots, S_n\}$ is obtained from a preordering $<$ of $W \subseteq W_{\text{all}}$ by defining $S_0 := \{w \in W\colon w \text{ is } <\text{-minimal in } W\}$ and $S_{i+1} := S_i \cup \{w \in W\colon w \text{ is } <\text{-minimal in } W - S_i\}$. Because we assume that $W_{\text{all}}$ is finite, this process is guaranteed to finish. What is called the universality of a system of spheres $\$$ by Lewis (1973), $\bigcup \$ = W_{\text{all}}$, cannot be expressed by a preordering alone. This is why we need to specify the field $W$ of $<$. One may wonder if we also need to make room for inconsistent belief states $\$$ containing the empty sphere $\emptyset$ as an element. If a sphere in $\$$ is empty, we can just drop it, with the same preordering $<$ corresponding, in the sense just explained, to $\$$ and to $\$ - \{\emptyset\}$. I neglect this case in the present paper and require all spheres to be non-empty.

[18] Compare this with Gärdenfors's (1986, p. 85) original *non-triviality condition*, which essentially required three worlds in $\min_<$ for some belief state $<$ in a belief revision model.

that is relevant in belief states. It is evident that even the more complex structures $<$ and \$ do not capture all important features of real belief states.[19] The only thing we need for our purposes, however, is that these structures capture the aspects that are relevant for the analysis of conditionals. Given that we have restricted our attention to *Ramsey test conditionals* and have not discussed whether Ramsey test conditionals come anywhere close to the conditionals used in ordinary language, this seems to be a very modest aim. Still the task we have set us is not completely trivial.

**7.2 Particular belief revision models.** The famous paper by AGM used the method of partial meet contraction and revision which refers to intersections of some maximal subsets of a belief set that don't imply a given sentence. It was only Adam Grove who highlighted the fact that the AGM theory can be more graphically presented in terms of changes of systems of spheres. Since the publication of Grove's insight, it has been justified, I think, to conceive of the AGM theory as a *semantic* modelling. We can describe the work of a revision function in two equivalent ways, as effecting transitions between preorderings, from $\langle W, < \rangle$ to $\langle W * \alpha, <^*_\alpha \rangle$, or between systems of spheres, from \$ to $\$ * \alpha$. Here and in the following, the input $\alpha$ is always a factual sentence from $L_0$.

Grove's result essentially said that every revision function that satisfies the eight AGM postulates can be represented as a revision function based on a system of spheres. The crucial fact is that if the prior belief state is represented by \$, then the set of models of the revised belief set $\ulcorner \$ * \alpha \urcorner$ is precisely the set of most plausible $\alpha$-models according to \$, that is, the intersection of $[\alpha]$ with the smallest sphere in \$ that has a non-empty intersection with $[\alpha]$. It is easy to rephrase this constraint in terms of strict preorderings of possible worlds.

(AGM) If the prior belief state is represented by $\langle W, < \rangle$, then for every doxastically possible sentence $\alpha \in L_0$, the revised belief set is determined by $\min_{<^*_\alpha}(W * \alpha) = \min_<([\alpha])$.

We may call belief state revision functions satisfying the constraint that (AGM) places on $\langle W * \alpha, <^*_\alpha \rangle$ *AGM-style revision functions*. In AGM's own theory, $W$ always equals $W_{\text{all}}$, a fact that is essentially mirrored by (AGM*5). While we do not want (AGM) to depend on this, we shall allow ourselves to speak briefly of a 'belief state $<$' rather than of a 'belief state $\langle W, < \rangle$'.

(AGM) is essentially a constraint on the kinds of changes of the prior belief state \$ or $<$ that may be effected by an input or assumption $\alpha$. But (AGM) does not fully determine the shape of the posterior belief state $\$ * \alpha$ or $<^*_\alpha$, since there are many different change operations that satisfy (AGM). This is illustrated with the help of the system of spheres model in Fig. 3. Here the numeral '1' indicates the set of doxastically preferred worlds after an AGM-style revision. AGM made no commitment regarding the ordering of worlds outside $\min_<([\alpha])$. Notice that (AGM) entails

[19] For the diversity of problems pertinent to the philosophy of belief, see Schwitzgebel (2006).

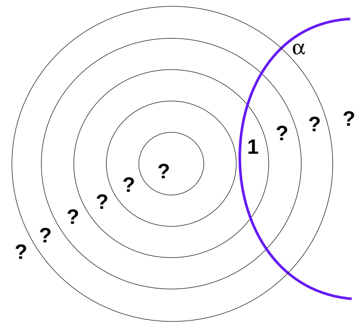



*Fig. 3: AGM revision(in system of sphere representation)*

($L_0$-expansion) If $\alpha$ is consistent with $\langle W, < \rangle$, that is, if $[\alpha] \cap \min_<(W) \neq \emptyset$, then
$\min_{<^*_\alpha}(W * \alpha) = \min_<(W) \cap [\alpha]$

The theory of AGM thus determines a revised belief *set* on the basis of the prior belief *state*. This serves well for one-shot belief revision, and there has in fact been no controversy revolving around the (AGM) constraint. However, if iterated changes of belief are to be effected, the AGM theory is not enough. Metaphorically speaking, a belief set is not capable of guiding its own revisions. The more inclusive structure of a belief state is necessary for performing this task. What was needed was a theory that determined a revised belief *state* on the basis of the prior belief state. This was no longer the business of AGM themselves.

Within the qualitative framework based on orderings or systems of spheres of possible worlds, four canonical constructions have been advocated since the 1990s: Radical revision (Rad), moderate revision (Mod), restrained revision (Rest) and conservative revision (Cons).[20] I list them in order of increasing conservatism, as measured by the "weight" given to (the contents of) the prior belief state in relation to (the contents of) the input.[21]

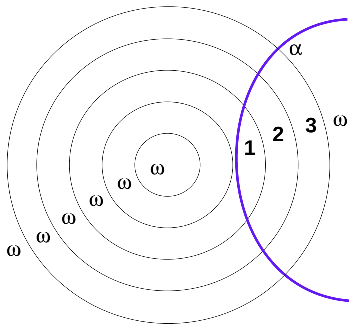
Radical belief revision (also known as "irrevocable" belief revision, Fig. 4):

(Rad)[22] $W * \alpha = W \cap [\alpha]$
$v <^*_\alpha w$ iff $v < w$ , i.e., $<^*_\alpha = < |_{[\alpha]}$

[20]With the exception of (Rest), the names of these constructions are mine. Here is a short list of the most relevant literature: For (Rad), see Segerberg (1998); for (Mod), see Nayak (1994); for (Rest), see Booth and Meyer (2006); for (Cons), see Boutilier (1996). A more general approach to iterated belief revision is taken in the seminal paper of Darwiche and Pearl (1997). Also compare Konieczny and Pino Pérez (2000), Rott (2003; 2009) and, for a recent more philosophical discussion, Stalnaker (2009).

[21]It is worth pointing out that even though everything is qualitative here, the four methods can be meaningfully ordered in terms of their conservatism. For quantitative parameters of conservativeness and boldness, cf. Carnap's (1952) $\lambda$ and Levi's (1967, p. 107) $\alpha$.

[22]Notation: In the following definitions, vertical strokes denote domain restrictions of functions, horizontal strokes denote set complements. – (Rad) is slightly deviant because it does



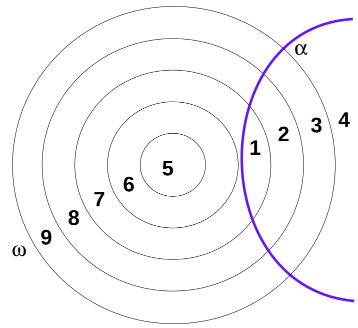




*Fig. 4: Radical revision*
*(= "irrevocable revision")*

*Fig. 5: Moderate revision*
*(= "lexicographic revision")*

Moderate belief revision (also known as "lexicographic" belief revision, Fig. 5):

(Mod) $$W * \alpha = W$$

$$v <^*_\alpha w \text{ iff } \begin{cases} v < w \text{ and } (v, w \in [\alpha] \text{ or } v, w \notin [\alpha]) & \text{or} \\ v \in [\alpha] \text{ and } w \notin [\alpha] \end{cases}$$

$$\text{i.e., } <^*_\alpha \, = \, < |_{[\alpha]} \cup < |_{[\neg\alpha]} \cup ([\alpha] \times [\neg\alpha])$$

Note that this model makes previously unaccessible $\alpha$-worlds accessible.

Restrained belief revision (Fig. 6):

(Rest) $$W * \alpha = W$$

$$v <^*_\alpha w \text{ iff } \begin{cases} v < w \text{ and } v, w \notin \min_<([\alpha]) & \text{or} \\ v \sim w \text{ and } v \in [\alpha] \text{ and } w \notin [\alpha] & \text{or} \\ v \in \min_<([\alpha]) \text{ and } w \notin \min_<([\alpha]) \end{cases}$$

$$\text{i.e., } <^*_\alpha \, = \, < |_{\overline{\min_<([\alpha])}} \cup (\sim \cap ([\alpha] \times [\neg\alpha])) \cup (\min_<([\alpha]) \times \overline{\min_<([\alpha])})$$

Conservative belief revision (also known as "natural" belief revision, Fig. 7):

(Cons) $$W * \alpha = W$$

$$v <^*_\alpha w \text{ iff } \begin{cases} v < w \text{ and } v, w \notin \min_<([\alpha]) & \text{or} \\ v \in \min_<([\alpha]) \text{ and } w \notin \min_<([\alpha]) \end{cases}$$

$$\text{i.e., } <^*_\alpha \, = \, < |_{\overline{\min_<([\alpha])}} \cup (\min_<([\alpha]) \times \overline{\min_<([\alpha])})$$

---

not square with the Consistency condition (AGM*5). Alternatively, we could put $W * \alpha = W$ for (Rad). This would spare us the mentioning of the field $W$, because we could invariably stick with the set $W_{\text{all}}$ of all logically possible worlds, in accordance with (AGM*5). Then we could define radical revision by

$$v <^*_\alpha w \text{ iff } \begin{cases} v < w \text{ and } v, w \in [\alpha] & \text{or} \\ v \in [\alpha] \text{ and } w \notin [\alpha] \end{cases}$$

i.e., $<^*_\alpha \, = \, < |_{[\alpha]} \cup ([\alpha] \times [\neg\alpha])$. This is all fine, but it is not truly radical (and not truly suited for hypothetical reasoning).

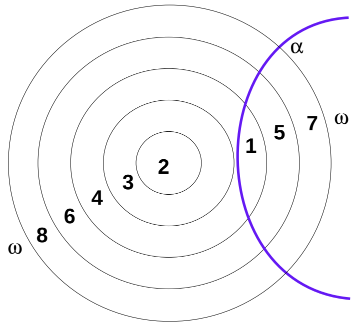



*Fig. 6: Restrained revision*

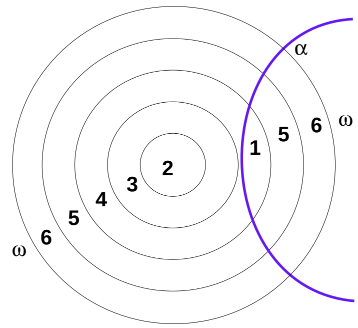



*Fig. 7: Conservative revision (= "natural revision")*

Using the equivalent systems of spheres model again, Figures 4–7 show what happens according to the four canonical methods of revising belief states. The numbers indicate the relative position of the relevant possible worlds in the posterior strict preordering or system of spheres. Intuitively speaking, lower numbers mean smaller distances from the person's actual beliefs, i.e., higher plausibility. These numbers do not indicate any distances, and they are not suitable for addition, subtraction and multiplication. The symbol '$\omega$' indicates inaccessibility, i.e., exclusion from $\bigcup(\$ * \alpha)$.

If the belief revision function $*$ is fixed as one of the canonical methods, we assume that the field $\mathbb{B}$ in a belief revision model $\langle \mathbb{B}, \mathcal{O}, * \rangle$ is closed under the relevant transformations. A foolproof way of guaranteeing such closure is to take $\mathbb{B}$ as the class of all possible belief states $\langle W, < \rangle$ or \$.

If $\alpha$ is a contradiction, all canonical constructions except (Rad) result in $<^*_\alpha$ $= <$.[23] So contradictory inputs have no effects at all. While reasonable in itself, this does not conform with the unrestricted Success condition (AGM*2). We neglect this limiting case problem here. More importantly, all canonical constructions except (Rad) satisfy the postulates of Darwiche-Pearl (1997) for iterated revisions. Rather than delving into this topic, we will have a look at an illustration of how these constructions work.

**Example 1** We reconsider the attempted proofs of Section 4.2 and substitute the atoms $p$ and $q$ for the schematic letters $\alpha$ and $\beta$. Since we are interested only in the language with these two propositional atoms, we restrict our attention to the four interpretations ("possible worlds") $pq$, $p\overline{q}$, $\overline{p}q$ and $\overline{pq}$. When an atom is mentioned in the name of a possible world, this means that the atom is true at this world; when an atom occurs overlined, the atom is false at this world. As already mentioned, the blank slate $\mathcal{O}$ can be represented by the empty strict preordering $pq \sim p\overline{q} \sim \overline{p}q \sim \overline{pq}$. Table 1 shows what happens if the revisions are guided by one of the four canonical methods.

Each of the canonical revision methods (Mod), (Rest) and (Cons) leads to the result that $\ulcorner \mathcal{O} * p * q \urcorner_{L_1}$ , $\ulcorner \mathcal{O} * q * p \urcorner_{L_1}$ and $\ulcorner \mathcal{O} * (p \wedge q) \urcorner_{L_1}$ are three distinct

[23] (Rad) gives the absurd belief state $\langle W * \bot, <^*_\bot \rangle = \langle \emptyset, \emptyset \rangle$.

| | Conservative revision | Restrained revision | Moderate revision | Radical revision |
|---|---|---|---|---|
| $\mathcal{O} * p$ | $pq \sim p\overline{q} < \overline{p}q \sim \overline{pq}$ | | | $pq \sim p\overline{q} \ll \overline{p}q \sim \overline{pq}$ |
| $\mathcal{O} * p * q$ | $pq < p\overline{q} < \overline{p}q \sim \overline{pq}$ | $pq < p\overline{q} < \overline{p}q < \overline{pq}$ | $pq < \overline{p}q < p\overline{q} < \overline{pq}$ | $pq \ll p\overline{q} \sim \overline{p}q \sim \overline{pq}$ |
| $\mathcal{O} * q$ | $pq \sim \overline{p}q < p\overline{q} \sim \overline{pq}$ | | | $pq \sim \overline{p}q \ll p\overline{q} \sim \overline{pq}$ |
| $\mathcal{O} * q * p$ | $pq < \overline{p}q < p\overline{q} \sim \overline{pq}$ | $pq < \overline{p}q < p\overline{q} < \overline{pq}$ | $pq < p\overline{q} < \overline{p}q < \overline{pq}$ | $pq \ll p\overline{q} \sim \overline{p}q \sim \overline{pq}$ |
| $\mathcal{O} * (p \wedge q)$ | $pq < p\overline{q} \sim \overline{p}q \sim \overline{pq}$ | | | $pq \ll p\overline{q} \sim \overline{p}q \sim \overline{pq}$ |
| $\neg(p \wedge q) > p$ | $\in \mathcal{O} * p, \mathcal{O} * p * q$<br>$\notin \mathcal{O} * q, \mathcal{O} * q * p,$<br>$\mathcal{O} * (p \wedge q)$ | exactly as in conservative revision | $\in \mathcal{O} * p, \mathcal{O} * q * p$<br>$\notin \mathcal{O} * q, \mathcal{O} * p * q,$<br>$\mathcal{O} * (p \wedge q)$ | $\in \mathcal{O} * p$ non-vacuously,<br>$\in \mathcal{O} * p * q, \mathcal{O} * q * p,$<br>$\mathcal{O} * (p \wedge q)$ vacuously<br>$\notin \mathcal{O} * q$ |
| $\neg(p \wedge q) > q$ | $\in \mathcal{O} * q, \mathcal{O} * q * p$<br>$\notin \mathcal{O} * p, \mathcal{O} * p * q,$<br>$\mathcal{O} * (p \wedge q)$ | exactly as in conservative revision | $\in \mathcal{O} * q, \mathcal{O} * p * q$<br>$\notin \mathcal{O} * p, \mathcal{O} * q * p,$<br>$\mathcal{O} * (p \wedge q)$ | $\in \mathcal{O} * q$ non-vacuously,<br>$\in \mathcal{O} * p * q, \mathcal{O} * q * p,$<br>$\mathcal{O} * (p \wedge q)$ vacuously<br>$\notin \mathcal{O} * p$ |
| 1st proof attempt | $\ulcorner \mathcal{O} * p * q \urcorner_{L_1} \neq \ulcorner \mathcal{O} * q * p \urcorner_{L_1}$ | | | |
| 2nd proof attempt | $\ulcorner \mathcal{O} * p \urcorner_{L_1} \not\subseteq \ulcorner \mathcal{O} * (q \wedge p) \urcorner_{L_1}$ , $\ulcorner \mathcal{O} * q \urcorner_{L_1} \not\subseteq \ulcorner \mathcal{O} * (q \wedge p) \urcorner_{L_1}$ | | | |

Table 1: The canonical constructions (Cons), (Rest) and (Mod) make visible why the attempted proofs fail

sets.[24] Inspecting what the canonical models do, we can now understand why our attempted proofs in Section 4.2 are not successful. And this is, I think, just as it ought to be.[25]

## 8 The subset problem

**8.1 Doxastic possibility.** For AGM it was easy to say when a person believes more than another person: Just when the first person's belief set is a superset of the second person's belief set. When belief states came to be perceived as more fundamental than belief sets in the 1990s, there was no need to change anything on the level of $L_0$. In semantical terms, believing more just means having a smaller set of doxastically preferred worlds. To say that $\ulcorner \mathcal{B} \urcorner_{L_0} \subseteq \ulcorner \mathcal{B}' \urcorner_{L_0}$ means, in the modelling using a strict preordering of possible worlds, that $\min_{<'}(W) \subseteq \min_{<}(W)$ or, in the system of spheres modelling, that $\bigcap \$' \subseteq \bigcap \$$.

Can this idea of capturing an increase in beliefs by a subset relation be upheld in more complex languages? A hint that there may be a serious *subset problem* here comes from the possibility operator. The operator $\Diamond$ in $\Diamond\alpha$, read as "for all I believe, it might be the case that $\alpha$", is a *negative static (categorical) doxastic modal*.[26] In contrast to the usual accounts in modal logic, possibility is here viewed from a first-person perspective: The point of reference is the agent's own current set of beliefs. Here is the doxastic possibility test, or simply, the *might test*:

($\Diamond$T) $\Diamond\alpha$ is in $\ulcorner \mathcal{B} \urcorner$ iff $\neg\alpha$ is not in $\ulcorner \mathcal{B} \urcorner$

Since there are no negations of *might*-sentences in $L_\Diamond$, let us suppose that ($\Diamond$T) applies only to $L_0$-sentences $\alpha$. The following simple observation serves as a motivation for looking for analogous problems in languages containing conditionals.

**Observation 1** *Given ($\Diamond$T), there are no belief states $\mathcal{B}$ and $\mathcal{B}'$ such that their belief sets $\ulcorner \mathcal{B} \urcorner_{L_\Diamond}$ and $\ulcorner \mathcal{B}' \urcorner_{L_\Diamond}$ are related by strict subset inclusion.*

*Proof.* Suppose that the $L_\Diamond$-belief set $\ulcorner \mathcal{B} \urcorner$ is a proper subset of the $L_\Diamond$-belief set $\ulcorner \mathcal{B}' \urcorner$. Take some sentence $\alpha$ in $L_\Diamond$ that is in $\ulcorner \mathcal{B}' \urcorner$ but not in $\ulcorner \mathcal{B} \urcorner$. Suppose first that $\alpha$ is in $L_0$. Then $\neg\neg\alpha$ is in $\ulcorner \mathcal{B}' \urcorner$ but not in $\ulcorner \mathcal{B} \urcorner$, and by ($\Diamond$T) we

[24] (Rad) is special. After a number of revision steps it tends to produce belief states that validate very many conditionals vacuously, just because the antecedent is not doxastically possible. Recall that this is not in line with (AGM*5).

[25] As noted before, trivially $\ulcorner \mathcal{O} * p \urcorner_{L_1} = Cn^-(\ulcorner \mathcal{O} * p \urcorner_{L_1})$, and similarly for all other belief states. – We identify $\ulcorner \mathcal{O} * (p \wedge q) \urcorner_{L_1}$ with $Cn^+(p \wedge q)|_{L_1}$, i.e., $Cn^+(p \wedge q)$ restricted to $L_1$. But it is doubtful, to say the least, whether we can identify $\ulcorner \mathcal{O} * p * q \urcorner_{L_1}$ with any $Cn^+$-set at all.

[26] The term 'doxastic modal' is etymologically more precise than the more common term 'epistemic modal'. That the operator is *static* means that it does not refer to any belief changes. That it is *negative* means that its test ($\Diamond$T) below makes essential use of a negation. That it is *categorical* means that it does not involve hypothetical assumptions or conditionals.

get that $\Diamond\neg\alpha$ is in $\ulcorner\mathcal{B}\urcorner$ but not in $\ulcorner\mathcal{B}'\urcorner$. Suppose secondly that $\alpha$ is not in $L_0$. Then it is of the form $\Diamond\beta$. Then by ($\Diamond$T) we get that $\neg\beta$ is in $\ulcorner\mathcal{B}\urcorner$ but not in $\ulcorner\mathcal{B}'\urcorner$. So either way we get that $\ulcorner\mathcal{B}\urcorner$ is not a subset of $\ulcorner\mathcal{B}'\urcorner$, and we have found a contradiction. $\Box$

It can immediately be understood why there are no proper subsets, if a possibility operator $\Diamond$ governed by ($\Diamond$T) is available. For any factual sentence $\alpha$ one gains, one loses $\Diamond\neg\alpha$, and for any sentence $\Diamond\alpha$ one gains, one loses $\neg\alpha$. Proper expansions do not exist in $L_\Diamond$. There is nothing wrong or even paradoxical about this, it is just a consequence of the meaning of the diamond $\Diamond$.

**8.2 Conditional doxastic necessity.** The conditional $\alpha > \beta$ may be read as "for all I believe, if $\alpha$, it is necessary that $\beta$". In the same sense in which $\Diamond$ is a negative static modal, the conditional connective $>$ can be understood as a *positive dynamic (conditional) doxastic modal.* Having become suspicious by the observation concerning doxastic possibility, we may wonder whether similar problems might arise in the context of conditional necessity. No negation is present here that causes any trouble, but perhaps the (iterable) reference to other belief states may create similar problems? So we ask the same question for this sort of modal as we asked in the previous section: Are there any belief states $\mathcal{B}$ and $\mathcal{B}'$ such that their $L_1$-belief sets $\ulcorner\mathcal{B}\urcorner$ and $\ulcorner\mathcal{B}'\urcorner$ are related by strict subset inclusion? We shall show that the answer is positive, and confirm that *in* $L_1$, we can indeed have the Ramsey test and Preservation at the same time.

But can we go any further? Are there any belief states $\mathcal{B}$ and $\mathcal{B}'$ such that their $L_2$-belief sets $\ulcorner\mathcal{B}\urcorner$ and $\ulcorner\mathcal{B}'\urcorner$ are related by strict subset inclusion? As we shall see, the answer here is negative. *In* $L_2$, we cannot have Preservation together with the Ramsey test, on pain of triviality. I try to show where exactly Preservation fails if we continue to use the Ramsey test. Let us put the following

**Questions** *Given (RT), are there any belief states $\mathcal{B}$ and $\mathcal{B}'$ such that their belief sets*
*(i)* $\ulcorner\mathcal{B}\urcorner_{L_1}$ *and* $\ulcorner\mathcal{B}'\urcorner_{L_1}$
*(ii)* $\ulcorner\mathcal{B}\urcorner_{L_2}$ *and* $\ulcorner\mathcal{B}'\urcorner_{L_2}$
*are related by strict subset inclusion?*

As we will see, a positive answer can be given on the basis of the constraint (AGM) alone on the level of $L_1$. For $\ulcorner\mathcal{B}\urcorner_{L_i}$ with $i \geq 2$, however, the AGM context is not rich enough. We will have recourse to specific properties of the relevant revision functions $*$ for an informative answer.

Let us collect the basic assumptions that we need in order to proceed. We start with three rather unproblematic conditions: a condition concerning the acceptance of conditionals in general belief states, a condition concerning the acceptance of $L_0$-sentences in particular belief states, and a condition on the revision of particular belief states:

(RT) For $\alpha \in L_0$, $\alpha > \beta \in \ulcorner\mathcal{B}\urcorner$ iff $\beta \in \ulcorner\mathcal{B} * \alpha\urcorner$

($L_0$-Bel) For $\alpha \in L_0$, $\alpha \in \ulcorner<\urcorner$ iff $\min_<(W) \subseteq [\alpha]$

(AGM) For doxastically possible $\alpha \in L_0$, $\min_{<^*_\alpha}(W * \alpha) = \min_<([\alpha])$

From this basis we can derive a condition concerning the acceptance of conditionals in particular belief states.

**Lemma 2** *Given (RT), ((*$L_0$*-Bel)) and (AGM), the acceptance condition for conditionals with a doxastically possible antecedent in a belief state* $\langle W, < \rangle$ *is*

*(*$L_1$*-Bel)* *For* $\alpha, \beta \in L_0$*,* $\alpha > \beta \in \ulcorner < \urcorner$ *iff* $\min_<([\alpha]) \subseteq [\beta]$

Proof. Let $\alpha$ and $\beta$ be in $L_0$ and $\alpha$ be doxastically possible in $\langle W, < \rangle$. Then we get

$\alpha > \beta \in \ulcorner < \urcorner$ iff (by (RT))
$\beta \in \ulcorner <^*_\alpha \urcorner$ iff (by ($L_0$-Bel))
$\min_{<^*_\alpha}(W * \alpha) \subseteq [\beta]$ iff (by (AGM))
$\min_<([\alpha]) \subseteq [\beta]$ □

Now we can clarify what it means that two belief sets in $L_1$ are related by subset inclusion. In order to keep things reasonably simple, we will from now on assume, for the remainder of this section, that the field of all the preorderings $<$ of possible worlds is the universal set $W_{all}$.

**Observation 3** *Let (*$L_1$*-Bel) be given and let* $L_0$ *be finite, so that every set of possible worlds can be expressed by a single sentence. Then the following conditions are equivalent:*[27]

*(i)* $\ulcorner < \urcorner_{L_1} \subseteq \ulcorner <' \urcorner_{L_1}$

*(ii)* $< \subseteq <'$*, that is,* $<'$ *is a refinement of* $<$*.*

*Proof.* As always in this paper, we consider only the finite case. Assume ($L_1$-Bel) as given. Then we get the following equivalences:

$\ulcorner < \urcorner_{L_1} \subseteq \ulcorner <' \urcorner_{L_1}$ iff (by ($L_1$-Bel))
for all $\alpha, \beta \in L_0$, if $\min_<([\alpha]) \subseteq [\beta]$, then $\min_{<'}([\alpha]) \subseteq [\beta]$ iff
(finite case, see remark (i) below)
for all $\alpha \in L_0$, if $\min_{<'}([\alpha]) \subseteq \min_<([\alpha])$ iff
(finite case, see remark (ii) below)
for all sets $\{w, w'\}$, $w < w'$ implies $w <' w'$ iff
$< \subseteq <'$

Two remarks on the finite case: (i) In the finite case, $\min_<([\alpha])$ is expressible by a single sentence $\beta$. (ii) For all worlds $w$ and $w'$, the set $\{w, w'\}$ is expressible by a single sentence $\gamma$. So if $w < w'$ but $w \sim' w'$, then $w' \in \min_{<'}([\gamma])$ but not $w' \in \min_<([\gamma])$. □

[27] A similar observation was made independently by Hannes Leitgeb (2010, pp. 29–30). An analogous result for doxastic states that are represented by entrenchment orderings is given in Rott (1991, Observation 5).

We now turn to the context of revision functions for belief states. The following observations are immediate consequences of Observation 3.

**Corollary 4** *Let (RT), ($L_0$-Bel) and (AGM) be given.*
*(i) If* $\ulcorner < \urcorner_{L_1} = \ulcorner <' \urcorner_{L_1}$*, then* $< \, = \, <'$
*(ii) If* $\ulcorner < \urcorner_{L_1} = \ulcorner <' \urcorner_{L_1}$ *and a function* $*$ *for the revision of belief states is fixed, then* $\ulcorner < \urcorner_{L_3} = \ulcorner <' \urcorner_{L_3}$
*(iii) The following two conditions are equivalent*

($L_1$-Pres) *If* $\alpha \in L_0$ *is consistent with* $<$ *, then* $\ulcorner < \urcorner_{L_1} \subseteq \ulcorner <^*_\alpha \urcorner_{L_1}$
(Refine) *If* $\alpha \in L_0$ *is consistent with* $<$*, then* $<^*_\alpha$ *is a refinement of* $<$ *.*

Thus Preservation within $L_1$ is equivalent with a refinement operation for factual assumptions consistent with the prior beliefs. The condition (Refine) will play a central role in what follows. Let us also consider three variants of it. First, it can be weakened to

(NoReverse) If $\alpha \in L_0$ is consistent with $<$, then $<^*_\alpha$ does not reverse $<$: $w \leq^*_\alpha w'$ holds whenever $w < w'$ [28]

Second, both the constraints (Refine) and (NoReverse) can be generalized to the case where $\alpha$ is inconsistent with $<$. In this case, due to the shifting of the worlds in $\min_<([\alpha])$ prescribed by (AGM), the posterior preordering cannot be a refinement of the prior preordering. However, *apart from these minimal $\alpha$-worlds*, the ideas of Refinement and Nonreversal can be upheld.

(GenRefine) Within the subfield $W - \min_<([\alpha])$ of $W$, $<^*_\alpha$ is a refinement of $<$ [29]

(GenNoReverse) Within the subfield $W - \min_<([\alpha])$ of $W$, $<^*_\alpha$ does not reverse $<$

Let us comment on the pair (GenRefine)/(Refine), the relevant comments on (GenNoReverse)/(NoReverse) are similar. Taken together with (AGM), (GenRefine) implies (Refine). Intuitively it is quite a lot stronger than the latter. On the one hand, (GenRefine) seems to derive all its motivation from (Refine) which is in turn motivated by $L_1$-Preservation. It is hard to think of anything else that might come to its support. On the other hand, (GenRefine) is not much less plausible than (Refine), and any principled justification for (Refine) appears to carry over to (GenRefine). If it is desirable to preserve all conditionals in the consistent case (i.e., where the input or assumption $\alpha$ is consistent with one's beliefs), then there should be similar reasons for preserving conditionals "as far as possible" in the belief-contravening case as well.

[28] The corresponding condition for conditionals is this: If $\neg\alpha \notin \ulcorner < \urcorner_{L_0}$ and $\beta > \gamma \in \ulcorner < \urcorner_{L_1}$, then $\beta > \neg\gamma \notin \ulcorner <^*_\alpha \urcorner_{L_1}$. — If joined with (AGM*5) for $L_0$-sentences, ($L_1$-Pres) implies this condition.

[29] The corresponding condition for conditionals is this: If $\alpha > \neg\beta \in \ulcorner < \urcorner_{L_1}$ and $\beta > \gamma \in \ulcorner < \urcorner_{L_1}$, then $\beta > \gamma \in \ulcorner <^*_\alpha \urcorner_{L_1}$. — If joined with (AGM*3) and (AGM*4) for $L_0$-sentences, this condition implies ($L_1$-Pres).

Of our four canonical constructions, exactly two, namely conservative and restrained revision, (Cons) and (Rest), satisfy (Refine). They also satisfy (GenRefine), (NoReverse) and (GenNoReverse). The other two canonical constructions, (Mod) and (Rad) satisfy none of these conditions. In my opinion, the canonical constructions, as operations on semantic models of belief states, carry much more conviction themselves than the less intuitive postulates of (Generalized) Refinement and (Generalized) Non-reversal. Nevertheless, for the sake of generality, we will work with these postul

While Observation 3 is good news for the supporters of both the Ramsey test and the idea of Preservation, bad news is bound to come. Here is the result that sets the limits.

**Observation 5** *Let (RT), ($L_0$-Bel) and (AGM) be given, and let $\ulcorner < \urcorner_{L_1}$ be a proper subset of $\ulcorner <' \urcorner_{L_1}$, i.e. let $<'$ be an proper refinement of $<$. If either*

*(i) $\ulcorner < \urcorner_{L_0}$ is a proper subset of $\ulcorner <' \urcorner_{L_0}$, the latter set is undogmatic and $*$ satisfies (NoReverse),*

*or*

*(ii) $\ulcorner < \urcorner_{L_0}$ is identical with $\ulcorner <' \urcorner_{L_0}$ and $*$ satisfies (GenNoReverse),*

*then $\ulcorner < \urcorner_{L_2}$ is not a subset of $\ulcorner <' \urcorner_{L_2}$.*

*Proof.* We talk only about the finite case in which every proposition is expressible by some sentence from $L_0$. For the sake of simplicity, we identify propositions (sets of possible worlds) with sentences in the following proof, trusting that this does not cause any confusion. Let $<'$ be a proper refinement of $<$, i.e., for all $w$ and $w'$, if $w < w'$ then $w <' w'$, and there are $w1$ and $w2$ such that $w1 <' w2$ but not $w1 < w2$. Since $<'$ refines $<$ and $<'$ is asymmetric, we can conclude that $w1 \sim w2$.

(i) Let $<'$ be undogmatic, and suppose that $\ulcorner < \urcorner_{L_0}$ is a proper subset of $\ulcorner <' \urcorner_{L_0}$. The latter means that $\min_{<'}(W')$ is a proper subset of $\min_{<}(W)$. The set $\min_{<'}(W')$ is non-empty. So we can choose $w1 \in \min_{<'}(W')$ and $w2 \in \min_{<}(W) - \min_{<'}(W')$. Given such $w1$ and $w2$, we can pick a world $w3 \neq w1$ with $w3 \leq' w1$. Such a $w3$ exists, because $<'$ is undogmatic. Since $<'$ is a refinement of $<$, $w3 \leq w1$. In short, we have $w3 \leq w1 \sim w2$ and $w3 \leq' w1 <' w2$.

Provided that the worlds $w1$, $w2$ and $w3$ satisfy these constraints, we consider the conditional

(+) $\quad \{w1, w2\} > (\{w2, w3\} > \{w2\})$

We show that the conditional (+) is accepted in the belief state represented by $<$, but not in that represented by $<'$.

Let $*$ satisfy the nonreversal condition (NoReverse).

The revision of $<$ by $\{w1, w2\}$ gives $w1 \sim^*_{\{w1,w2\}} w2 <^*_{\{w1,w2\}} w3$, by (AGM). Further revision by $\{w2, w3\}$ gives $\min_{<^{*}_{\{w1,w2\}}{}^{*}_{\{w2,w3\}}}(W*\{w1, w2\}*\{w2, w3\}) = \{w2\}$, by (AGM) again. Hence (+) is accepted in $<$, by (RT).

The revision of $<'$ by $\{w1, w2\}$ gives $w1 <'^*_{\{w1,w2\}} w3$ by (AGM). Now notice that $\{w1, w2\}$ is consistent with $\ulcorner <' \urcorner$, because $w1 \in \min_{<'}(W')$. Thus we can use (NoReverse) and get that $w3 \leq'^*_{\{w1,w2\}} w2$. Further revision by $\{w2, w3\}$ yields $w3 \in \min_{<'^*_{\{w1,w2\}}{}^*_{\{w2,w3\}}}(W' * \{w1, w2\} * \{w2, w3\})$, by (AGM) again. Hence this latter set is not a subset of the proposition $\{w2\}$, and (+) is not accepted in $<'$, by (RT).

Case (ii). Suppose that $\ulcorner < \urcorner_{L_0}$ is identical with $\ulcorner <' \urcorner_{L_0}$. This means that $\min_{<'}(W') = \min_{<}(W)$. So neither $w1$ nor $w2$ as defined above are in $\min_{<}(W)$. Given $w1$ and $w2$, we can pick a world $w3$ from $\min_{<}(W)$ with $w3 < w1$ and $w3 <' w1$. In short, we have $w3 < w1 \sim w2$ and $w3 <' w1 <' w2$.

We again consider the conditional (+) above and show that it is accepted in the belief state represented by $<$, but not in that represented by $<'$.

Let $*$ satisfy the generalized nonreversal condition (GenNoReverse).

The revision of $<$ by $\{w1, w2\}$ and then by $\{w2, w3\}$ has exactly the same effects as in case (i), so (+) is again accepted in $<$, by (RT).

The revision of $<'$ by $\{w1, w2\}$ gives $w1 <'^*_{\{w1,w2\}} w3$ by (AGM). Now notice that this time $\{w1, w2\}$ is not consistent with $\ulcorner <' \urcorner$, so we need to use (GenNoReverse) rather than the weaker condition (NoReverse) in order to get that $w3 \leq'^*_{\{w1,w2\}} w2$. The rest is exactly as in case (i), hence again (+) is not accepted in $<'$, by (RT).

We have seen that both in case (i) and in case (ii), the conditional (+) is accepted in the belief state represented by $<$ but not in the belief state represented by $<'$. Since (+) is in $L_2$, we can conclude that $\ulcorner < \urcorner_{L_2}$ is not a subset of $\ulcorner <' \urcorner_{L_2}$. $\square$

**Corollary 6** *Let (RT), ($L_0$-Bel) and (AGM) be given. If $*$ produces some undogmatic belief states for an assumption consistent with, but not already accepted in a prior belief state $<$, then ($L_1$-Pres) implies a violation of ($L_2$-Pres).*

*Proof.* Let (RT), ($L_0$-Bel), (AGM) and ($L_1$-Pres) be given. Take a belief state $<$ and an $\alpha \in L_0$ that is consistent with, but not supported by $<$, and for which $<^*_\alpha$ is undogmatic. We have $\emptyset \neq \min_{<}(W) \cap [\alpha] \subset \min_{<}(W)$, so by (AGM), we can conclude that $\min_{<^*_\alpha}(W * \alpha)$ is a proper subset of $\min_{<}(W)$. Since the minima of $<$ and $<^*_\alpha$ are different, we know that $< \neq <^*_\alpha$. By ($L_1$-Pres), $\ulcorner < \urcorner_{L_1} \subseteq \ulcorner <^*_\alpha \urcorner_{L_1}$. Thus by Observation 3, $< \subseteq <^*_\alpha$. So $< \subset <^*_\alpha$. By ($L_1$-Pres) and Observation 3, we get (Refine) which implies (NoReverse). So by Observation 5, part (i), not $\ulcorner < \urcorner_{L_2} \subseteq \ulcorner <^*_\alpha \urcorner_{L_2}$, which finishes the proof. $\square$

The corollary tells us that Preservation on the level of $L_1$ forces a violation of Preservation of $L_2$. This result is not a bug, it is a feature of belief revision modellings in the spirit of AGM. The violation of $L_2$-Preservation is at least as plausible as the canonical methods (Rest) and (Cons) are. These are conservative methods that are designed to retain many conditionals. On the other hand, in so far as (Mod) is a reasonable method, it is doubtful whether $L_1$-Preservation should be adopted as a rationality assumption for belief change.

We now also see that the shift of attention from the Ramsey test (RT) to the Monotonicity condition (Mon) in much of the relevant literature was an infelicitous move. ($L_2$-$L_1$-Mon), which is the version of Monotonicity one needs for the triviality proofs, is correct, but only vacuously so: It is correct just because its antecedent is satisfied only in trivial cases. So the problem with (Mon) is not just that it does not have all that much to recommend itself intuitively. In contrast to the Ramsey test, the Monotonicity condition is itself trivial.

Let us illustrate the message of Observation 5 with the help of an example. In order to make the case more conspicuous, let us assume that the revision function satisfies the stronger conditions (Refine) and (GenRefine), as notably the canonical constructions (Cons) and (Rest) do.

**Example 2** We consider two belief states $\mathcal{B}$ and $\mathcal{B}'$ for which $\ulcorner\mathcal{B}\urcorner_{L_1} \subseteq \ulcorner\mathcal{B}'\urcorner_{L_1}$, but not $\ulcorner\mathcal{B}\urcorner_{L_2} \subseteq \ulcorner\mathcal{B}'\urcorner_{L_2}$.

Let $\mathcal{B}$ be $pq \sim p\overline{q} \sim \overline{p}q \sim \overline{pq}$ and $\mathcal{B}'$ be $p\overline{q} \sim \overline{p}q < pq \sim \overline{pq}$. Intuitively the belief sets involved are $\ulcorner\mathcal{B}\urcorner = \ulcorner\mathcal{O}\urcorner = Cn^+(\top)$ and $\ulcorner\mathcal{B}'\urcorner = \ulcorner\mathcal{O} * (p \dot{\vee} q)\urcorner = Cn^+(p \dot{\vee} q)$, where $\dot{\vee}$ is the exclusive disjunction. Clearly, $\mathcal{B}'$ is a (proper) refinement of $\mathcal{B}$, whence $\ulcorner\mathcal{B}\urcorner_{L_1} \subseteq \ulcorner\mathcal{B}'\urcorner_{L_1}$, by Observation 3. Applying ($L_1$-Bel), we can verify that $\mathcal{B}'$ satisfies the conditional $p > \neg q$ which $\mathcal{B}$ does not satisfy. So we have $\ulcorner\mathcal{B}\urcorner_{L_1} \subset \ulcorner\mathcal{B}'\urcorner_{L_1}$.

With the question mark '?' indicating that the relation between the adjacent worlds is open, we get

$$
\begin{array}{lll}
\mathcal{B} * p & = pq \sim p\overline{q} < \overline{p}q \;?\; \overline{pq} & \text{, by (AGM), and} \\
\mathcal{B} * p * q & = pq < p\overline{q} < \overline{p}q \;?\; \overline{pq} & \text{, again by (AGM) and (Refine).} \\
\mathcal{B}' * p & = p\overline{q} < \overline{p}q < pq \;?\; \overline{pq} & \text{, by (AGM) and (Refine), and} \\
\mathcal{B}' * p * q & = \overline{p}q < p\overline{q} < pq \;?\; \overline{pq} & \text{, by (AGM) and (GenRefine).}
\end{array}
$$

Notice that the conditional $q > \neg p$ accepted in $\mathcal{B}'$ is preserved in $\mathcal{B}' * p$ and $\mathcal{B}' * p * q$. Intuitively speaking, the exclusive disjunction of the initial belief state retains its force throughout the iterated revision process. In $\mathcal{B} * p$ and $\mathcal{B} * p * q$, the contrary conditional $q > p$ is accepted. Since $\mathcal{B}$ supports, but $\mathcal{B}'$ does not support $p > (q > p)$, we see that not $\ulcorner\mathcal{B}\urcorner_{L_2} \subseteq \ulcorner\mathcal{B}'\urcorner_{L_2}$. While one may certainly wonder whether (Refine) and (GenRefine) are too conservative, there is nothing unnatural involved here.

Can we elaborate on the result contained in Observation 5? We recall from section 8.1 that in the presence of the doxastic possibility operator $\Diamond$ it was particularly easy to show that there are no two belief sets in $L_\Diamond$ that are related by proper subset inclusion (see Observation 1). The point was basically that for every $\alpha$ in $\ulcorner\mathcal{B}'\urcorner - \ulcorner\mathcal{B}\urcorner$ it is very easy to find a sentence that is in $\ulcorner\mathcal{B}\urcorner - \ulcorner\mathcal{B}'\urcorner$, namely $\Diamond\neg\alpha$. It is natural to ask whether a sentence can be found in the language using only the positive dynamic modal $>$ (instead of negative static modal $\Diamond$) that similarly witnesses the non-elementship in the belief set. The answer is positive, provided that the belief state in question is undogmatic.

**Observation 7** *Let (RT), ($L_0$-Bel) and (AGM) be given. Let the belief state $<$ be undogmatic, and choose a propositional atom $p$ such that neither $p$ nor $\neg p$*

*is in* $\ulcorner < \urcorner$. *Then for all* $L_0$*-sentences* $\alpha$,

*(i)* *If* $\alpha$ *is not in* $\ulcorner < \urcorner$, *then* $(\neg\alpha \vee p) > ((\neg\alpha \vee \neg p) > \neg\alpha)$ *is in* $\ulcorner < \urcorner$

*If in addition* $*$ *satisfies the condition (NoReverse), we get:*

*(ii)* *If* $\alpha$ *is in* $\ulcorner < \urcorner$, *then* $(\neg\alpha \vee p) > ((\neg\alpha \vee \neg p) > \neg\alpha)$ *is not in* $\ulcorner < \urcorner$

*If in addition* $*$ *satisfies (Refine), we get:*

*(iii)* *If* $\alpha$ *is in* $\ulcorner < \urcorner$, *then* $(\neg\alpha \vee p) > ((\neg\alpha \vee \neg p) > \alpha)$ *is in* $\ulcorner < \urcorner$

*Proof.* Let $<$ be undogmatic. Then there is a propositional atom $p$ such that $min_<(W)$ is neither a subset of $[p]$ nor of $[\neg p]$, so one can indeed find an atom as required by the observation.

If we are interested in the nested conditional $(\neg\alpha \vee p) > ((\neg\alpha \vee \neg p) > \neg\alpha)$ for a given $L_0$-sentence $\alpha$, then (RT) tells us that we should be interested in the set

$$min_{(<^*_{\neg\alpha\vee p})^*_{\neg\alpha\vee\neg p}}(W)$$

By (AGM),

$$min_{(<^*_{\neg\alpha\vee p})^*_{\neg\alpha\vee\neg p}}(W) = min_{<^*_{\neg\alpha\vee p}}([\neg\alpha \vee \neg p])$$

We first determine $min_{<^*_{\neg\alpha\vee p}}(W)$, which by (AGM) is identical with $min_<([\neg\alpha \vee p])$. Since $min_<(W)$ intersects with $[p]$, it does so with $[\neg\alpha \vee p]$, and we get that

$$min_{<^*_{\neg\alpha\vee p}}(W) = min_<([\neg\alpha \vee p]) = min_<(W) \cap [\neg\alpha \vee p] \neq \emptyset$$

Now we start with the crucial case distinction.

Case (i). Suppose that $\alpha$ is not believed in $<$, that is $\alpha \notin \ulcorner < \urcorner$, or equivalently, $min_<(W)$ intersects $[\neg\alpha]$.

Then $min_{<^*_{\neg\alpha\vee p}}(W) = min_<(W) \cap [\neg\alpha \vee p]$ intersects $[\neg\alpha \vee \neg p]$, and we get

$$\begin{aligned} min_{(<^*_{\neg\alpha\vee p})^*_{\neg\alpha\vee\neg p}}(W) &= min_{<^*_{\neg\alpha\vee p}}([\neg\alpha \vee \neg p]) \\ &= min_{<^*_{\neg\alpha\vee p}}(W) \cap [\neg\alpha \vee \neg p] \\ &= min_<(W) \cap [\neg\alpha \vee p] \cap [\neg\alpha \vee \neg p] \\ &= min_<(W) \cap [\neg\alpha] \\ &\subseteq [\neg\alpha] \end{aligned}$$

Cases (ii) and (iii). Suppose that $\alpha$ is believed in $<$, that is $\alpha \in \ulcorner < \urcorner$, or equivalently $min_<(W)$ is a subset of $[\alpha]$.

Since $min_<(W)$ intersects with $[p]$, it does so with $[\neg\alpha \vee p]$. Thus $min_{<^*_{\neg\alpha\vee p}}(W) = min_<([\neg\alpha \vee p]) = min_<(W) \cap [\neg\alpha \vee p]$ is a subset of $[\alpha \wedge p]$, and we know that $min_{<^*_{\neg\alpha\vee p}}([\neg\alpha \vee \neg p])$ is not a subset of $min_{<^*_{\neg\alpha\vee p}}(W)$.

Since $min_<(W)$ intersects with $[\neg p]$, it does so with $[\neg\alpha \vee \neg p]$. And clearly, for all worlds $w1$ in the non-empty set $[\neg\alpha \vee \neg p] \cap min_<(W)$ and all worlds $w2$ not in $min_<(W)$, $w1 < w2$.

Case (ii). Suppose that $*$ satisfies (NoReverse). Since $\neg\alpha \vee p$ is consistent with $<$, we can conclude by (NoReverse) that for all worlds $w1$ in the non-empty set $[\neg\alpha \vee \neg p] \cap min_{<}(W)$ and all worlds $w2$ not in $min_{<}(W)$, $w1 \leq^{*}_{\neg\alpha\vee p} w2$. So we get

$$\emptyset \neq min_{<^{*}_{\neg\alpha\vee p}}([\neg\alpha \vee \neg p]) \cap min_{<}(W) \subseteq [\alpha]$$

Thus $min_{<^{*}_{\neg\alpha\vee p}}([\neg\alpha \vee \neg p]$ is not a subset of $[\neg\alpha]$.

Case (iii). Suppose that $*$ satisfies (Refine). Since $\neg\alpha \vee p$ is consistent with $<$, we can conclude by (Refine) that for all worlds $w1$ in the non-empty set $[\neg\alpha \vee \neg p] \cap min_{<}(W)$ and all worlds $w2$ not in $min_{<}(W)$, $w1 <^{*}_{\neg\alpha\vee p} w2$. Hence we get

$$min_{<^{*}_{\neg\alpha\vee p}}([\neg\alpha \vee \neg p]) \subseteq min_{<}(W) \subseteq [\alpha]$$

In sum, what we have demonstrated is that the non-empty set $min_{(<^{*}_{\neg\alpha\vee p})^{*}_{\neg\alpha\vee\neg p}}(W)$ is a subset of $[\neg\alpha]$ in case (i), no subset of $[\neg\alpha]$ in case (ii), and a subset of $[\alpha]$ in case (iii). Therefore, we have:

Given (AGM) alone,

(i) if $\alpha$ is not in $\ulcorner<\urcorner$, then $(\neg\alpha\vee p) > ((\neg\alpha\vee\neg p) > \neg\alpha)$ is in $\ulcorner<\urcorner$

Given in addition (NoReverse),

(ii) if $\alpha$ is in $\ulcorner<\urcorner$, then $(\neg\alpha\vee p) > ((\neg\alpha\vee\neg p) > \neg\alpha)$ is not in $\ulcorner<\urcorner$

Given in addition (Refine),

(iii) if $\alpha$ is in $\ulcorner<\urcorner$, then $(\neg\alpha\vee p) > ((\neg\alpha\vee\neg p) > \alpha)$ is in $\ulcorner<\urcorner$ □

Notice that the converse of (iii) follows from (i) and the fact that $(\neg\alpha\vee p) > ((\neg\alpha\vee\neg p) > \neg\alpha)$ and $(\neg\alpha\vee p) > ((\neg\alpha\vee\neg p) > \alpha)$ are not both in $\ulcorner<\urcorner$, due to (RT) and the consistency of the belief set $\ulcorner(<^{*}_{\neg\alpha\vee p})^{*}_{\neg\alpha\vee\neg p}\urcorner$.

Parts (i) and (ii) of Observation 7 specify a (somewhat complex) $L_2$-sentence the presence of which can serve as an indicator of the absence of an $L_0$-sentence $\alpha$ in a belief state $<$, provided the state revision function $*$ obeys the conservative idea of (NoReverse). Are there similar indicators for the absence of $L_1$-sentences of the form $\alpha > \beta$? It turns out that there are.

**Observation 8** *Let (RT), ($L_0$-Bel) and (AGM) be given, and consider the unnested conditional $\alpha > \beta$ with $\alpha$ and $\beta$ from $L_0$.*

*(i) Suppose the conditional is* open *in the sense that its antecedent $\alpha$ is consistent with the belief state $<$. If $<$ is undogmatic, and the propositional atom $p$ is such that neither $p$ nor $\neg p$ is in $\ulcorner<\urcorner$, then*

> *If $\alpha > \beta$ is not in $\ulcorner<\urcorner$, then $((\alpha\wedge\neg\beta)\vee p) > (((\alpha\wedge\neg\beta)\vee\neg p) > (\alpha\wedge\neg\beta))$ is in $\ulcorner<\urcorner$*

*If in addition $*$ satisfies the condition (NoReverse), we get:*

> *If $\alpha > \beta$ is in $\ulcorner<\urcorner$, then $((\alpha\wedge\neg\beta)\vee p) > (((\alpha\wedge\neg\beta)\vee\neg p) > (\alpha\wedge\neg\beta))$ is not in $\ulcorner<\urcorner$*

*(ii) Suppose the conditional is* belief-contravening *in the sense that its antecedent $\alpha$ is inconsistent with the belief state $<$. Then*

*If $\alpha > \beta$ is not in $\ulcorner < \urcorner$, then $\alpha > ((\neg\alpha \vee \neg\beta) > \alpha)$ is in $\ulcorner < \urcorner$*

*If in addition $*$ satisfies the condition (GenNoReverse), we get:*

*If $\alpha > \beta$ is in $\ulcorner < \urcorner$, then $\alpha > ((\neg\alpha \vee \neg\beta) > \alpha)$ is not in $\ulcorner < \urcorner$*

*Proof.* (i) Note that a conditional $\alpha > \beta$ that is open in belief state $<$, i.e., one for which $\min_<(W) \cap [\alpha] \neq \emptyset$, is equivalent with respect to its acceptance in $<$ to the $L_0$-sentence $\neg\alpha \vee \beta$. This follows from (AGM), since $\min_{<^*_\alpha}(W) = \min_<([\alpha]) = \min_<(W) \cap [\alpha] \subseteq [\beta]$ if and only if $\min_<(W) \subseteq [\neg\alpha \vee \beta]$. This means that we can simply apply parts (i) and (ii) of Observation 7.

(ii) Let the conditional $\alpha > \beta$ now be belief-contravening, i.e., $\min_<(W) \cap [\alpha] = \emptyset$. If we are interested in the nested conditional $\alpha > ((\neg\alpha \vee \neg\beta) > \alpha)$, then (RT) tells us that we should be interested in the set

$$min_{(<^*_\alpha)^*_{\neg\alpha\vee\neg\beta}}(W)$$

By (AGM),

$$min_{(<^*_\alpha)^*_{\neg\alpha\vee\neg\beta}}(W) = min_{<^*_\alpha}([\neg\alpha\vee\neg\beta])$$

Suppose first that $\alpha > \beta$ is not in $\ulcorner < \urcorner$. Then $[\neg\beta]$ intersects $min_{<^*_\alpha}(W)$, and so does $[\neg\alpha \vee \neg\beta]$. Hence $min_{<^*_\alpha}([\neg\alpha\vee\neg\beta]) = min_{<^*_\alpha}(W) \cap [\neg\alpha\vee\neg\beta] = min_<([\alpha]) \cap [\neg\alpha\vee\neg\beta]$, by (AGM) again. But this latter set is a subset of $[\alpha]$. So $\alpha > ((\neg\alpha \vee \neg\beta) > \alpha)$ is in $\ulcorner < \urcorner$.

Suppose second that $\alpha > \beta$ is in $\ulcorner < \urcorner$, and that $*$ satisfies (GenNoReverse). Then $min_{<^*_\alpha}(W)$, which is equal to $min_<([\alpha])$ by (AGM), is a subset of $[\beta]$, and so $[\neg\alpha \vee \neg\beta]$ does not intersect it. Trivially, for all worlds $w1$ in $min_<(W)$ and all worlds $w2$ not in $min_<(W)$, $w1 < w2$. By (GenNoReverse), $<^*_\alpha$ does not reverse $<$ in $W - min_<([\alpha])$. Hence for all worlds $w1$ in $min_<(W)$ and all worlds $w2$ not in $min_<(W) \cup min_<([\alpha])$, $w1 \leq^*_\alpha w2$, or more simply, for all worlds $w1$ in $min_<(W)$ and all worlds $w2$ not in $min_<([\alpha])$, $w1 \leq^*_\alpha w2$. Because we have already shown that $[\neg\alpha \vee \neg\beta]$ does not intersect $min_{<^*_\alpha}(W) = min_<([\alpha])$, but it does intersect $min_<(W)$, we can conclude that all elements of $min_<(W)$ are included in $min_{<^*_\alpha}([\neg\alpha\vee\neg\beta])$. But since no element of $min_<(W)$ is in $[\alpha]$, we get that $min_{<^*_\alpha}([\neg\alpha\vee\neg\beta])$ is not a subset of $[\alpha]$. So we have finally found that $\alpha > ((\neg\alpha \vee \neg\beta) > \alpha)$ is not in $\ulcorner < \urcorner$. □

As the proof shows, open conditionals with antecedents that are not in conflict with the current belief state can already be taken care of by parts (i) and (ii) of Observation 7. For belief-contravening conditionals, we needed a new argument, but the indicator clause for their absence is simpler than the one mentioned in rule (>>T) for factual sentences.

We can now take down analogues to the might test in the language $L_2$ and have more direct proofs of the existence of our subset problems.

**Corollary 9** *Let (RT), ($L_0$-Bel), (AGM) and (NoReverse) be given.*

*(i) Let the belief state $<$ be undogmatic, and choose a propositional atom $p$ such that neither $p$ nor $\neg p$ is in $\ulcorner < \urcorner$. Then for all $L_0$-sentences $\alpha$,*

(>>T$_0$) $\alpha$ *is not in* $\ulcorner < \urcorner$ *iff* $(\neg\alpha \vee p) > ((\neg\alpha \vee \neg p) > \neg\alpha)$ *is in* $\ulcorner < \urcorner$

*(ii) For all belief states $<$ and $<'$, if $\ulcorner < \urcorner_{L_0}$ is a proper subset of $\ulcorner <' \urcorner_{L_0}$ and $<'$ is undogmatic, then $\ulcorner < \urcorner_{L_2}$ is not a subset of $\ulcorner <' \urcorner_{L_2}$.*

*(iii) For any $L_0$-sentence $\alpha$ that is not in $\ulcorner < \urcorner$, there is no undogmatic belief state $<'$ that satisfies $\ulcorner < \urcorner_{L_2} \cup \{\alpha\}$.*

*Let in addition (GenNoReverse) be given. Then*

*(iv) For all $L_0$-sentences $\alpha$ inconsistent with $<$ and all $L_0$-sentences $\beta$,*

(>>T$_1$) $\alpha > \beta$ *is not in* $\ulcorner < \urcorner$ *iff* $\alpha > ((\neg\alpha \vee \neg\beta) > \alpha)$ *is in* $\ulcorner < \urcorner$

*(v) For all belief states $<$ and $<'$, if $\ulcorner < \urcorner_{L_1}$ is a proper subset of $\ulcorner <' \urcorner_{L_1}$ then $\ulcorner < \urcorner_{L_2}$ is not a subset of $\ulcorner <' \urcorner_{L_2}$.*

*Proof.* (i) just summarizes parts (i) and (ii) of Observation 7. For (ii), suppose that the belief set $\ulcorner < \urcorner_{L_0}$ is a proper subset of the belief set $\ulcorner < \urcorner_{L_0}$ and $<'$ is undogmatic. Take some sentence $\alpha$ in $L_0$ that is in $\ulcorner <' \urcorner$ but not in $\ulcorner < \urcorner$. Notice that neither $p$ nor $\neg p$ is in $\ulcorner < \urcorner$. So by (>>T$_0$), we get that $(\neg\alpha \vee p) > ((\neg\alpha \vee \neg p) > \neg\alpha)$ is in $\ulcorner < \urcorner$ but not in $\ulcorner <' \urcorner$. Since this sentence belongs to $L_2$, we have a counterexample to $\ulcorner < \urcorner_{L_2} \subseteq \ulcorner <' \urcorner_{L_2}$, as desired. For (iii), let $\alpha$ be not in $\ulcorner < \urcorner$. Notice that if an undogmatic belief state $<'$ satisfied $\ulcorner < \urcorner_{L_2} \cup \{\alpha\}$, then $\ulcorner <' \urcorner_{L_2}$ would be a superset of $\ulcorner < \urcorner_{L_2}$ which is excluded by (ii). (iv) just summarizes part (ii) of Observation 8. For (v) suppose that the belief set $\ulcorner < \urcorner_{L_1}$ is a proper subset of the belief set $\ulcorner < \urcorner_{L_1}$. Take some sentence $\alpha > \beta$ in $L_1$ that is in $\ulcorner <' \urcorner$ but not in $\ulcorner < \urcorner$. So by (>>T$_1$), we get that $\alpha > ((\neg\alpha \vee \neg\beta) > \alpha)$ is in $\ulcorner < \urcorner$ but not in $\ulcorner <' \urcorner$. Since this sentence belongs to $L_2$, we have a counterexample to $\ulcorner < \urcorner_{L_2} \subseteq \ulcorner <' \urcorner_{L_2}$, as desired. □

We have established two claims, viz., that under certain preconditions,

($\not\subset$0) $\qquad \ulcorner < \urcorner_{L_0} \subset \ulcorner <' \urcorner_{L_0}$ entails $\ulcorner < \urcorner_{L_2} \not\subseteq \ulcorner <' \urcorner_{L_2}$

and that under certain other, slightly different preconditions,

($\not\subset$1) $\qquad \ulcorner < \urcorner_{L_1} \subset \ulcorner <' \urcorner_{L_1}$ entails $\ulcorner < \urcorner_{L_2} \not\subseteq \ulcorner <' \urcorner_{L_2}$

The preconditions are essentially Non-dogmatism for belief states and Non-reversal of $*$ for the former, and Generalized Non-reversal of $*$ for the latter condition. It is worth pointing out that the claims ($\not\subset$0) and ($\not\subset$1) are logically independent, because neither antecedent implies the other. The antecedent of ($\not\subset$0) does not say anything about conditionals in $L_1$. The antecedent of ($\not\subset$1) leaves open the possibility that the belief sets in $L_0$ are identical. Now recall from Corollary 4(ii) that as long as a belief revision function $*$ is fixed, two belief sets cannot be identical in our modelling within $L_1$ and yet diverge in $L_2$ or $L_3$. This taken together with ($\not\subset$1) implies that under the preconditions mentioned,

($\not\subset$2) There are no two belief states $<$ and $<'$ such that $\ulcorner < \urcorner_{L_2} \subset \ulcorner <' \urcorner_{L_2}$.

Observations 7 and 8 tell us that for undogmatic belief states and revision functions $*$ satisfying the (Generalized) Non-reversal condition, the language $L_2$ provides for non-elementhood tests ($\gg$T$_0$) and ($\gg$T$_1$) for $L_0$-sentences and, respectively, $L_1$-sentences that are similar to the *might* test ($\Diamond$T) within the language $L_\Diamond$. The doxastic possibility statement $\Diamond\alpha$ has a close counterpart in the nested conditional[30]

$$(\alpha\vee p) > ((\alpha\vee\neg p) > \alpha)$$

and something like a belief-contravening *might* conditional 'If $\alpha$ were the case, it might be the case that $\beta$' is expressible by

$$\alpha > ((\neg\alpha\vee\beta) > \alpha)$$

*provided that* the revision method is non-reversing in the ordinary or, repectively, generalized way.

The test sentence corresponding to $\Diamond\alpha$ rather complex. Supposing that ordinary language users can understand it at all and that natural language conditionals are captured by the Ramsey test, we may ask how plausible the result just obtained is. Unfortunately, it does not look very plausible. Many people may (using, at least implicitly, the Import-Export Rule for conditionals) feel tempted to read it as equivalent with the flat conditional $((\alpha\vee p) \wedge (\alpha\vee\neg p)) > \alpha$ which can be reduced to $\alpha > \alpha$. This conditional should certainly be acceptable, even if $\alpha$ is believed to be false! This finding does not speak against part (i) of Observation 7, but parts (ii) and (iii) look objectionable now. If this line of thought is considered to be convincing, it may be taken as an argument against (NoReverse), and hence also against (Refine) and $L_1$-Preservation. We will return to this topic in Section 10.

## 9 The AGM postulates reconsidered

What becomes of the AGM postulates if we accept the Ramsey test in the context of $L_2$ and more complex languages? As Gärdenfors showed, the AGM postulates must be invalid for conditionals that go by the Ramsey test. We restrict our discussion to inputs from $L_0$. Still the situation is somewhat complex.

Remember that *a belief set (in $L_i$)* is the set of $L_i$-sentences validated by some belief state in some given belief revision model $\langle \mathbb{B}, \mathcal{O}, * \rangle$. All postulates except the Success postulate (AGM*2) refer to the background logic $Cn$, and we have seen that there are different ways of interpreting $Cn$ in the context of belief revision models. The meaning and validity of almost all the AGM postulates thus depends on the meaning that we attach to $Cn$. Since we are not claiming

[30] Incidentally, we might also regard $(\neg\alpha \vee p) > ((\neg\alpha \vee \neg p) > \alpha)$ as a counterpart to the doxastic necessity statement $\Box\alpha$, provided the revision function is refining. But this is not really an achievement, since we could have a simpler conditional representation of $\Box\alpha$ by $\top > \alpha$. It is the possibility operator that is interesting here.

that conditionals express propositions, we cannot use a truth conditional semantics but stick to the semantics in terms of the acceptability in belief states described in Section 6.

The Closure condition (AGM*1) is automatically satisfied for $Cn^-$, but it is and should be invalid for $Cn^+$. For instance, $\ulcorner \mathcal{O} * p \urcorner$ contains $p \vee q$, and $\neg p > q$ is in $Cn^+(p \vee q)$, but $\ulcorner \mathcal{O} * p \urcorner$ does not and should not contain $\neg p > q$. The Consistency condition (AGM*5) is satisfied in all belief states represented by finitistic strict preorderings $<$ since there are always minimal worlds under $<$.[31] The Extensionality condition (AGM*6) is unproblematic. Since we have little grip on $Cn^+$ yet, we focus on $Cn^-$ in the following.

The main issue lies with the pairs (AGM*3)/(AGM*4) and (AGM*7)/(AGM*8) that ask for subset inclusions. We comment on the fist pair, the case of the latter is similar. All canonical construction of course satisfy (AGM*3) and (AGM*4) within $L_0$.

Let us now look at languages containing conditionals. The first case we consider for is the one in which $\alpha$ is in $\ulcorner \mathcal{B} \urcorner$. Then (AGM*3) and (AGM*4) basically say that $\ulcorner \mathcal{B} * \alpha \urcorner \subseteq \ulcorner \mathcal{B} \urcorner$ and $\ulcorner \mathcal{B} \urcorner \subseteq \ulcorner \mathcal{B} * \alpha \urcorner$, respectively. Both are satified if $\mathcal{B} = \mathcal{B} * \alpha$, a result given by conservative revision, but by none of the other canonical constructions. Restrained revision, a refining model, satisfies (AGM*4), but not (AGM*3) within $L_1$. Moderate and radical revision violate both (AGM*3) and (AGM*4) even in this first case.

Now consider the more interesting case in which $\alpha$ is not in $\ulcorner \mathcal{B} \urcorner$. We need to consider $Cn^-(\ulcorner \mathcal{B} \urcorner \cup \{\alpha\}) \subseteq \ulcorner \mathcal{B} * \alpha \urcorner$. But we know from the last section (see in particular Corollary 9(iii)), that within $L_2$ no belief state includes the set $\ulcorner \mathcal{B} \urcorner \cup \{\alpha\}$. So $Cn^-(\ulcorner \mathcal{B} \urcorner \cup \{\alpha\})$ is the inconsistent set of all sentences, which makes (AGM*3) trivially true and (AGM*4) trivially false. Can these conditions be satisfied, in a more interesting way, within $L_1$? Here the postulate (AGM*4), according to which $Cn^-(\ulcorner \mathcal{B} \urcorner \cup \{\alpha\}) \subseteq \ulcorner \mathcal{B} * \alpha \urcorner$ for $\alpha$ consistent with $\mathcal{B}$, is fulfilled for conservative and restrained revision, and indeed all successful revision methods $*$ satisfying Refinement. But (AGM*3), according to which $\ulcorner \mathcal{B} * \alpha \urcorner \subseteq Cn^-(\ulcorner \mathcal{B} \urcorner \cup \{\alpha\})$, does not hold for any of the canonical constructions, not even for the refining methods (Cons) and (Rest). To see this, consider the blank slate $\mathcal{O}$ (any belief state that "knows nothing" about $p$ and $q$ would do). $\mathcal{O} * p$ satisfies the conditional $q > p$. But this conditional is not in $Cn^-(\ulcorner \mathcal{O} \urcorner \cup \{p\})$, because there is a belief state satisfying the $L_1$-part of $\ulcorner \mathcal{O} \urcorner$ and $\{p\}$ which does not satisfy $q > p$. For instance, $\mathcal{O} * (p \wedge \neg q)$ is such a belief state, with $*$ denoting conservative revision.[32]

Summing up, little is left of the AGM postulates if we move from $L_0$ to $L_2$ and more complex languages. Restricting the postulates to a language of low

[31] The situation is slightly different for systems of spheres which may be called *inconsistent* if they include the empty set. But we neglect this case since dynamically speaking, the inconsistent system of spheres $\$ \cup \{\emptyset\}$ is equivalent to the consistent system of spheres $\$ - \{\emptyset\}$.

[32] But *should* $q > p$ perhaps be in $Cn(\ulcorner \mathcal{O} \urcorner \cup \{p\})$? When $\alpha$ from $L_0$ is consistent with $\ulcorner \mathcal{B} \urcorner$, it looks indeed very natural to identify $Cn(\ulcorner \mathcal{B} \urcorner \cup \{\alpha\})$ with $\ulcorner \mathcal{B} * \alpha \urcorner$, where $*$ is conservative revision. This operation $Cn$ is stronger than the operation $Cn^-$, and it might be an appropriate multiple-premise extension of $Cn^+$. Such questions have to be left for later work.

level is an obvious way out. Restricting them to $L_0$ is the safe option and brings us back to the propositional language of AGM. Restricting the AGM postulates to $L_1$ is possible in quite some cases, but lacks an independent motivation, once one has understood that any way of claiming that the postulates are plausible for $L_2$ is blocked. If we are prepared to include conditionals in the language, then we should at the same time be prepared to give up the postulates (AGM*3)/(AGM*4) concerning assumptions that are consistent with one's beliefs, as well as the postulates (AGM*7)/(AGM*8) dealing with revisions by conjunctions.

## 10 Conclusion

I have argued that the spirit of AGM is semantic. This is not a historical claim about the origin and early development of the AGM program, but with the benefit of hindsight, it seems to me systematically correct. The semantic reinterpretation of AGM was first worked out by Grove and later confirmed in the influential work of Katsuno and Mendelzon.

Throughout this paper I have presupposed that we use the Ramsey test as an interpretation of conditionals – a certain kind of conditionals that we called "Ramsey test conditionals". Once one adopts a resolutely semantic point of view, there is a very natural resolution of the riddle surrounding the applicability of the Ramsey test in the context of AGM style belief change. The key to it lies in the inspection of constructive revision operations on simple semantic models of belief states, or more briefly, on semantic methods for iterated belief change. It does *not* lie in any firm intuition about the various postulates that have been discussed over and over again in the literature.

The spirit of AGM is encoded in the constraint (AGM) for the revision of orderings of possible worlds or systems of spheres. One of our results has been that one can very well keep both the Ramsey Test (RT) and a model for belief state revision (or iterated belief set revision) that is fully in the spirit of AGM. We have questioned neither (RT) nor (AGM) in this paper.

Following from the Ramsey test, the Monotonicity condition is true as well. But it is important to distinguish two cases. ($L_1$-$L_0$-Mon) is non-vacuously satisfied by some revision functions, notably by conservative and restrained revision. But ($L_2$-$L_1$-Mon) can only be satisfied vacuously, due to the fact that its antecedent, which requires subset inclusion between belief sets in $L_2$, is only fulfilled in trivial cases.

Like Monotonicity, the Preservation condition has been more finely analyzed by distinguishing different levels of conditional nestedness. As AGM argued, Preservation seems compelling for the factual sentences of $L_0$. Preservation for $L_1$-sentences is the first and mildest extrapolation from $L_0$-Preservation. We have seen that it has a semantic equivalent in the Refinement property. We then considered other refinement and non-reversal properties which are similarly motivated as Refinement. Significantly, two out of our four canonical constructions for iterated belief revision, viz., conservative and restrained revision, satisfy Refinement, and exactly these revision methods also satisfy the other refinement and non-reversal properties.

So Preservation for non-nested conditionals in $L_1$ has turned out to be implementable together with the Ramsey test. We have seen, however, that Preservation cannot be obeyed for the language $L_2$. Given the canonicity of the four constructions that we considered, it is not awkward, but perfectly natural that Preservation for $L_2$ should fail. We can understand why it fails by inspecting constructive processes of belief state change. In a second proof of this result, we identified ($>>\mathrm{T}_0$), a kind of *might* test in a language that does not have the diamond $\Diamond$, but allows the simplest nestings of conditionals.

The case for Preservation is strong for factual sentences in $L_0$ and hopeless for nested Ramsey test conditionals in $L_2$. The merits of Preservation are most difficult to evaluate for $L_1$. While technically feasible, it seems to me that Preservation for $L_1$ is badly in need of independent motivation. Once Preservation has been discredited in the context of $L_2$, it is no longer self-evident that it should hold for $L_1$. I think it has been a philosophical error to transfer, as a matter of course, the intuition of Preservation from truth-functional propositional language to languages containing doxastic conditionals. The complexities involved in nested conditionals (and iterated revisions) have simply been underappreciated in the literature. Preservation for $L_2$ and more complex languages has been a wrong desideratum in the first place.

This paper has addressed a logical problem, not a linguistic one. I did not claim that the meaning of natural language conditionals, indicative or subjunctive, is captured by the Ramsey test. My topic here have been Ramsey test conditionals themselves, as objects of logical analysis, not as possible elements of common parlance. Our discussion of the Ramsey test in the AGM framework has emphasized the idea of preservation, and we have seen that the preservation of flat conditionals recommends conservative methods of belief change: (Cons) and (Rest) are favoured, while moderate and radical revision, (Mod) and (Rad), are dismissed as unnecessarily losing flat conditionals.

I believe that a discussion of conditionals from the linguistic point of view tends to lead to the opposite conclusion. Let us illustrate this by McGee's (1985) notorious counterexample to Modus Ponens. The polls conducted before the American election in 1980 predicted that the Republican Reagan would end up first, the Democrat Carter would be second by a wide margin, and finally the Republican Anderson would come in as a distant third. The polls gave excellent justification for believing that Reagan wins ($r$), the second most plausible scenario was that Carter wins ($c$) and the least plausible (but not completely impossible) scenario was that Anderson wins ($a$). A rational person then was in a belief state characterizable by $\overline{ac}r < \overline{a}c\overline{r} < a\overline{cr}$. In this state she accepts $r$ and $\neg r > c$. Preservative revision methods like conservative and restrained revision prescribe that $\neg r > c$ be retained when she learns or hypothetically assumes that it is a Republican who will win. But this is counterintuitive. It seems hard to deny that in such a situation she would and should rather switch to accepting $\neg r > a$. So the two preservative methods seem inadequate for the Ramsey test on intuitive grounds. The two remaining canonical constructions, radical and moderate revision, predict the intuitively right result and thus seem more suitable for the analysis of natural language conditionals.[33] Both (Rad) and (Mod) ensure the validity of the Import-Export Law, according to which $\alpha > (\beta > \gamma)$ is equivalent with $(\alpha \wedge \beta) > \gamma$, provided that $\alpha \wedge \beta$ is consistent. According to (Rad) and (Mod), but not according to (Cons) and (Rest), McGee is right in maintaining that $\neg r > a$ is not in $Cn\{(a \vee r) > (\neg r > a), a \vee r\}$, hence that Modus Ponens is not valid within $L_2$.[34] All of this suggests that Preservation is not a reasonable requirement even within the language of flat conditionals, $L_1$.

## 11 Acknowledgments

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[33] Note that in this example the person learns or hypothetically assumes something that she believes to be true anyway (viz., $r \vee a$). Still it is the *non-conservative* methods that get things intuitively right here! The hypothetical character of the antecedent is captured better by methods that give precedence to the antecedent over (the structure of) the prior beliefs.

[34] Here $Cn$ is $Cn^-$.

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