Revision by Comparison as a Unifying Framework: Severe Withdrawal, Irrevocable Revision and Irrefutable Revision

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Abstract

Fermé and Rott (‘Revision by Comparison’, *Artificial Intelligence* 157, 2004) introduced a binary operation of ‘revision by comparison’ and pointed out that this method of changing epistemic states has both characteristics of belief contraction (with respect to the ‘reference sentence’) and characteristics of belief revision (with respect to the ‘input sentence’). Using revision by comparison as a unifying framework, the present paper studies the unary limiting cases of severe withdrawal, irrevocable revision and irrefutable revision. While variants of the first two operations are well-known from the literature, the last one is new.

Key words: belief revision, theory change, severe withdrawal, irrevocable revision, irrefutable revision

1 Introduction: Revision by comparison

Revision by comparison is a model introduced by Fermé and Rott [5, henceforth ‘F&R’] that offers a rational method of changing beliefs in response to inputs of the form

Accept β with a degree of plausibility that at least equals that of α

Belief states are represented either by systems \$ of spheres of possible worlds or by entrenchment orderings $\leq$ of the sentences in a given propositional language (Grove [10], Gärdenfors [8], Gärdenfors and Makinson [9]). An agent’s set K of beliefs can be retrieved from each of these structures:

(Def $K_\$$) $\quad K = \{\alpha : \alpha \text{ true in every world in } \bigcap \$\}$

(Def $K_\leq$) $\quad K = \{\alpha : \bot < \alpha\}$

It can be shown that sets of sentences K thus defined are closed under the Tarskian background logic Cn which we suppose to govern the object lan-

Preprint submitted to Elsevier Science *19 August 2004*

guage. Logically closed sets are also called *belief sets*.

If the agent in belief state \$ or $\leq$ receives the instruction to accept β at least as firmly as α, and if he conforms to it, then he will modify his belief state, and as a result obtain a new system of spheres $\$'$ or a new entrenchment ordering $\leq'$, respectively. Here is the essential construction of F&R in terms of systems of spheres:

(Def $\$'$ from \$)

$$\$' = (\{S \cap [\beta] : S \in \$ \text{ and } S \subseteq [\alpha]\} \cup \{S : S \in \$ \text{ and } S \not\subseteq [\alpha]\}) - \{\emptyset\}$$

for α and β such that not $\bigcup \$ \subseteq [\alpha] \cap [\beta]$. In the latter case, we put $\$' = \{\emptyset\}$. The same construction can be defined in terms of entrenchment relations:

(Def $\leq'$ from $\leq$)

$$\gamma \leq' \delta \ \text{ iff } \ \begin{cases} \alpha \wedge (\beta \to \gamma) \leq (\beta \to \delta) & \text{, if } \gamma \leq \alpha \\ \gamma \leq \delta & \text{, otherwise} \end{cases}$$

This paper builds crucially upon the model and results presented by F&R, to which the reader is referred to for extended explanation and motivation of the model of revision by comparison.

From $\$'$ or respectively from $\leq'$, a revised belief set can be retrieved that we denote by $K \circ_\alpha \beta$. One can plug together (Def $\leq'$ from $\leq$) and (Def $K_\leq$) and define $K \circ_\alpha \beta = K_{\leq'}$, which yields, after a few simplifying steps

(Def $\circ$ from $\leq$)

$$\gamma \in K \circ_\alpha \beta \ \text{ iff } \ \begin{cases} \neg\beta < \alpha \wedge (\beta \to \gamma) & \text{or} \\ \alpha < \gamma & \text{or} \\ \top \leq \alpha \wedge \neg\beta \end{cases}$$

This defines the revision-by-comparison function $\circ$. We skip the analogous definition of $\circ$ in terms of systems of spheres.

F&R give the following sound and complete axiomatization of one-step revisions by comparison:

(C1) $K \circ_\alpha \beta = Cn(K \circ_\alpha \beta)$. (Closure)

(C2) If $Cn(\alpha) = Cn(\gamma)$ and $Cn(\beta) = Cn(\delta)$, then $K \circ_\alpha \beta = K \circ_\gamma \delta$. (Extensionality)

(C3) If $\alpha \notin K \circ_\beta \bot$, then $K \circ_\beta \bot \subseteq K \circ_\alpha \bot$. (Strong Inclusion)

(C4) If $\alpha \in K \circ_\alpha \bot$, then $\alpha \in K \circ_\beta \gamma$. (Irrevocability)

(C5) If $\alpha \in K \circ_{\alpha\wedge\neg\beta} \bot$, then $K \circ_\alpha \beta = (K \circ_{\alpha\wedge\neg\beta} \bot) + \beta$. (Reduction 1)

(C6) If $\alpha \notin K \circ_{\alpha\wedge\neg\beta} \bot$, then $K \circ_\alpha \beta = K \circ_{\alpha\wedge\neg\beta} \bot$. (Reduction 2)

A single additional axiom (IT) captures the operation's behaviour in iterated applications. Unfortunately, this axiom is complicated and unintuitive. For the present paper, we need only the following two consequences of (IT):

(IT1) $\quad (K \circ_\alpha \beta) \circ_\alpha \gamma = K \circ_\alpha (\beta \wedge \gamma)$

(IT4) $\quad (K \circ_\alpha \bot) \circ_\beta \bot = \begin{cases} (K \circ_\alpha \bot) \cap (K \circ_\beta \bot) & \text{, if } K \circ_\alpha \bot \neq \mathcal{L} \neq K \circ_\beta \bot \\ \mathcal{L} & \text{, otherwise} \end{cases}$

The numbering is taken over from F&R.

Keeping fixed the prior belief set K, we can say that $\circ$ takes two arguments, the *reference sentence* α and the *input sentence* β. F&R [pp. 9, 21] point out there that this method of changing one's epistemic state has both characteristics of belief revision (with respect to the input sentence) and characteristics of belief contraction (with respect to the reference sentence).

In the present paper, this claim is fully substantiated. Notice first that the axioms (C5) and (C6) effectively reduce the binary operation $\circ_\alpha\beta$ to complex applications of the unary operation $\circ_\alpha\bot$. This operation will be investigated in the next section. Then we will consider two further special cases of revision by comparison that can be obtained by holding fixed one of the arguments of the binary operation $\circ$. Doing so specializes the dyadic belief change function $\circ$ to a monadic one in a much more direct way than (C5) and (C6) do. More precisely, we will consider the monadic operations

$$K \mathbin{\ddot{-}} \alpha \quad =_{\text{def}} \quad K \circ_\alpha \bot \qquad \textit{Severe withdrawal}$$

$$K * \alpha \quad =_{\text{def}} \quad K \circ_\top \alpha \qquad \textit{Irrevocable revision}$$

$$K \star \alpha \quad =_{\text{def}} \quad K \circ_\varepsilon \alpha \qquad \textit{Irrefutable revision}$$

In the last line, ε is supposed to denote a fixed reference sentence that expresses a belief which is not irrevocable, where a belief or sentence α is called *irrevocable* if it resists any attempt to remove it from the belief set K, or more formally, if $\alpha \in K \circ_\alpha \bot$.[1]

It is rather surprising that the above operations turn out to be specializations of a single more general belief change function. Intuitively, the operation of severe withdrawal is very different from the operation of irrevocable revision, which shares a number of important properties with irrefutable revision. While

[1] Alternatively, α is irrevocable iff $\alpha \in K \circ_\top \neg\alpha$. F&R (conditions Q5 and Q15) show that the two conditions are equivalent. If they are satisfied, then $K \circ_\alpha \bot$ and $K \circ_\top \neg\alpha$ are both inconsistent. In terms of systems of spheres and entrenchments, α is irrevocable iff $\bigcup \$ \subseteq [\alpha]$ and respectively $\top \leq \alpha$. – One can also think of the set of current beliefs as being determined by a given revision-by-comparison function. For instance, the equation $K = K \circ_\bot \bot$ can serve as a suitable definition (Def $K_\circ$). See F&R [p. 19].

variants of the first two operations are well-known from the literature, the last one is new.

2 Iterable severe withdrawal

Severe withdrawal is a belief change operation that removes a belief from an existing belief system, without at the same time implanting any new beliefs into the belief system. Severe withdrawal is an alternative to the more widely known operation of contraction introduced by Alchourrón, Gärdenfors and Makinson, henceforth 'AGM' [1]. Its name is due to the fact that it incurs more loss of information than AGM's contraction operation. Severe withdrawal was studied by Rott [15, Section 5], by Levi [12,13] who calls the operation "mild contraction" and advances interesting philosophical arguments in its favour, by Pagnucco and Rott [14], by Fermé and Rodriguez [4] who call it "Rott contraction", and by Arló-Costa and Levi [2].

In this section we study the unary operation $\ddot{-}$ which is defined by putting

$$K \ddot{-} \alpha = K \circ_\alpha \bot$$

for every sentence α.[2] It turns out that the resulting operation is very close to the Pagnucco and Rott's model of severe withdrawal.

The semantics for this operation in terms of systems of spheres is discussed in Section 2 of F&R, and turns out to be very close to the semantics presented in [14]. The definition (Def $\$'$ from $\$$) for the operation $\ddot{-}\alpha = \circ_\alpha \bot$ reduces to

$$\$' = \begin{cases} \{S \in \$: S \not\subseteq [\alpha]\} & \text{, if } S \not\subseteq [\alpha] \text{ for some } S \in \$ \\ \{\emptyset\} & \text{, otherwise} \end{cases}$$

where α is the sentence to be withdrawn.[3]

The revised entrenchment relation corresponding to the operation $\ddot{-}\alpha$ can be derived from (Def $\leq'$ from $\leq$):

$$\beta \leq' \gamma \quad \text{iff} \quad \beta \leq \alpha \ \text{ or } \ \beta \leq \gamma$$

[2] More generally, $\bot$ could be replaced by any sentence the negation of which is irrevocable.

[3] F&R [p. 11] also consider a semantics of revision by comparison in terms of total pre-orderings of possible worlds. For severe contraction $K \ddot{-} \alpha = K \circ_\alpha \bot$, their definition (Def $\preceq'$ from $\preceq$) of the posterior ordering $\preceq'$ of worlds reduces to

$$w \preceq' w' \ \text{ iff } \begin{cases} w \preceq w' \text{ and } w'' \preceq w' \text{ for some } \neg\alpha\text{-world } w'' \quad \textit{or} \\ w \preceq w'' \text{ for all } \neg\alpha\text{-worlds } w'' \end{cases}$$

Loosely speaking, the operation is one on systems of spheres or on entrenchment relations (rather than on operation on belief sets), where "outer spheres" and, respectively, "higher entrenchments" are left untouched, and "inner spheres" and, respectively, "lower entrenchments" collapse into one single class.

As an operation of withdrawal of beliefs from belief sets which is defined in terms of entrenchments, we can take down

(Def $\ddot{-}$ from $\leq$) $\qquad \beta \in K \ddot{-} \alpha \quad \text{iff} \quad \alpha < \beta \ \text{ or } \ \top \leq \alpha$

Now we give an axiomatic definition of severe withdrawals which is close to the one given by Pagnucco and Rott [14, p. 512].

Definition 1 Let K be a belief set. A belief change function $\ddot{-}$ on K is an operation of *severe withdrawal* if and only if it satisfies the following postulates

($\ddot{-}$1) $K \ddot{-} \alpha = Cn(K \ddot{-} \alpha)$ (Closure)

($\ddot{-}$2) If $K \ddot{-} \alpha \neq \mathcal{L}$, then $K \ddot{-} \alpha \subseteq K$ (Restricted Inclusion)

($\ddot{-}$3) If $\alpha \notin K$, then $K \subseteq K \ddot{-} \alpha$ (Vacuity)

($\ddot{-}$4) If $K \ddot{-} \alpha \neq \mathcal{L}$, then $\alpha \notin K \ddot{-} \alpha$ (Restricted Success)

($\ddot{-}$5) If $\alpha \in K \ddot{-} \alpha$, then $\alpha \in K \ddot{-} \beta$ (Irrevocability)

($\ddot{-}$6) If $Cn(\alpha) = Cn(\beta)$, then $K \ddot{-} \alpha = K \ddot{-} \beta$ (Extensionality)

($\ddot{-}$7) If $K \ddot{-} \alpha \neq \mathcal{L}$, then $K \ddot{-} \alpha \subseteq K \ddot{-} (\alpha \wedge \beta)$ (Antitony)

($\ddot{-}$8) If $\alpha \notin K \ddot{-} (\alpha \wedge \beta)$, then $K \ddot{-} (\alpha \wedge \beta) \subseteq K \ddot{-} \alpha$ (Conjunctive Inclusion)

Half of these conditions, viz. ($\ddot{-}$1), ($\ddot{-}$3), ($\ddot{-}$6) and ($\ddot{-}$8), are well-known from the AGM belief contractions ([1,8]). ($\ddot{-}$2) and ($\ddot{-}$4) have new preconditions as compared with AGM. Condition ($\ddot{-}$5) is new. We need it because we allow for the possibility that a withdrawal of a non-tautology fails. Notice that our condition ($\ddot{-}$5) has got nothing to do with the fifth AGM postulate which is commonly called 'Recovery'. Recovery is a very controversial property, and it is invalid for severe withdrawal functions.

Finally, the Antitony Condition ($\ddot{-}$7) is new, too. It is much stronger than the seventh AGM condition. Both ($\ddot{-}$7) and the AGM condition ($\ddot{-}$8) depend crucially on the axiom (C3) for revision by comparison. The following conditions are specializations of the conditions (K$\dot{-}8^+$) and (K$\dot{-}$D) from F&R [p. 20]. They follow from the above axioms for severe withdrawals (the proofs of this claim and all other proofs are deferred to the appendix).

($\ddot{-}8^+$) If $\alpha \notin K \ddot{-} (\alpha \wedge \beta)$, then $K \ddot{-} (\alpha \wedge \beta) = K \ddot{-} \alpha$

($\ddot{-}$D) $K \ddot{-} (\alpha \wedge \beta) = K \ddot{-} \alpha$ or $K \ddot{-} (\alpha \wedge \beta) = K \ddot{-} \beta$

There are three differences between severe withdrawal defined as revision by comparison with input sentence $\bot$ and severe withdrawal in the sense of [14]. In the Pagnucco-Rott model, we have that $K \dot{-} \alpha = K$ for $\alpha \in Cn(\emptyset)$. In the approach developed in this paper, we get $K \dot{-} \alpha = \mathcal{L}$ in this case. Secondly, according to Definition 1, $K \circ_\alpha \bot = \mathcal{L}$ is possible even for an α such that $\alpha \notin Cn(\emptyset)$, since there may be non-tautological sentences α that are irrevocable in the sense that $\alpha \in K \dot{-} \alpha$. For Pagnucco and Rott, only tautologies are irrevocable. For this reason, we now have the condition $K \dot{-} \alpha \neq \mathcal{L}$ in the axiomatization, wherever Pagnucco and Rott have the condition $\alpha \notin Cn(\emptyset)$. The restricting clause in the Success condition ($\dot{-}$2) is new.

There is a third, more fundamental difference. We will now add a condition for iterated severe withdrawal. This has not been done in the literature before, but the relevant idea is very straightforward. The one-step operation of severe withdrawal collapses the innermost rings of a system of spheres, or respectively, the lowest levels of entrenchment – and this is all the change that is being effected. Accordingly, we can say that a function $\dot{-}$ on K is an operation of *iterable severe withdrawal* if and only if it satisfies ($\dot{-}$1) – ($\dot{-}$8) and

$$(\dot{-}\text{it}) \quad (K \dot{-} \alpha) \dot{-} \beta \; = \; \begin{cases} (K \dot{-} \alpha) \cap (K \dot{-} \beta) & \text{if } K \dot{-} \alpha \neq \mathcal{L} \neq K \dot{-} \beta \\ \mathcal{L} & \text{otherwise} \end{cases}$$

The notation $(K \dot{-} \alpha) \dot{-} \beta$ is used here for iterated belief withdrawal. One may think of this as slightly misleading since considered as an operation on a belief set, the second occurrence of $\dot{-}$ does not denote "the same" contraction function as the first $\dot{-}$. But our notation should not conceal the fact that revision by comparison operates on belief states (systems of spheres or entrenchment relations) in the first place, and only in a derived sense on belief sets. Belief sets result only after applying (Def $K_\$$) or (Def $K_\leq$) respectively. [4]

It is evident how ($\dot{-}$it) can be used to construct any finite number of applications of severe withdrawal.

Taken together with the axioms for one-step severe withdrawal, ($\dot{-}$it) characterizes exactly the withdrawal functions defined through revision by comparison with input sentence $\bot$.

Theorem 1 (Soundness) Let $\circ$ be a revision-by-comparison operator on a belief set K that satisfies (C1) – (C6), and let $\dot{-}$ be defined by putting $K \dot{-} \alpha = K \circ_\alpha \bot$ for all sentences α. Then $\dot{-}$ is a severe withdrawal operator satisfying ($\dot{-}$1) – ($\dot{-}$8). If $\circ$ in addition satisfies (IT), then $\dot{-}$ satisfies ($\dot{-}$it).

[4] Another way of thinking about the matter is to insist that $\dot{-}$ is *not a two-place function*, taking ordered pairs as arguments with a belief set and a sentence as components, but a *one-place function* associated with one and the same initial belief set K. So $(K \dot{-} \alpha) \dot{-} \beta$, for instance, is really a contraction of this initial belief set K through a sequence of inputs $\langle \alpha, \beta \rangle$. For a detailed discussion of the conceptual problems besetting the representation of iterated belief changes, see [17].

Theorem 2 (Completeness) Let K be a belief set and $\dot{\overline{\ \ }}$ a severe withdrawal function on K that satisfies $(\dot{\overline{\ \ }}1)$ – $(\dot{\overline{\ \ }}8)$ and $(\dot{\overline{\ \ }}\mathrm{it})$. Then there is an entrenchment relation $\leq$ such that for all α, $K \dot{\overline{\ \ }} \alpha = K \circ_\alpha \bot$, where $\circ$ is defined by (Def $\circ$ from $\leq$), and moreover, for all α and β, $(K \dot{\overline{\ \ }} \alpha) \dot{\overline{\ \ }} \beta = (K \circ_\alpha \bot) \circ_\beta \bot$, where $\circ$ is defined by (Def $\leq'$ from $\leq$) and (Def $\circ$ from $\leq$).

3 Irrevocable belief revision

In contrast to severe withdrawal, irrevocable belief revision is an operation that has right from the beginning been invented for use in repeated belief revisions. Irrevocable revision is close in spirit to the non-repeatable revision operations introduced by Alchourrón, Gärdenfors and Makinson [1] because such revisions are *always successful* in the sense that the input sentence always gets accepted, but its results are *not always consistent*. In AGM, a repeatedly revised belief set is inconsistent if and only if the last input sentence is itself inconsistent. In irrevocable belief change, the revised belief set is inconsistent if and only if the conjunction of the sequence of input sentences is inconsistent. The operation of irrevocable revision was studied by Rott [15, Section 6], Friedman and Halpern [6,7], Segerberg [18] and Fermé [3].

In this section we study the unary operation $*$ which is defined by putting

$$K * \alpha = K \circ_\top \alpha$$

for every sentence α.[5] It turns out that the resulting operation is very close to irrevocable revision as defined in the literature. This is why we appropriate the name for our purposes.

The semantics for this operation in terms of systems of spheres is defined as a special case of definition (Def \$′ from \$) which reduces to

$$\$' \ = \begin{cases} \{S \cap [\alpha] : S \in \$\} \ - \ \{\emptyset\} & \text{, if } S \cap [\alpha] \neq \emptyset \text{ for some } S \in \$ \\ \{\emptyset\} & \text{, otherwise} \end{cases}$$

where α is the sentence to be accepted.[6]

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[5] More generally, $\top$ could be replaced by any irrevocable sentence.

[6] As regards the irrevocable revision semantics in terms of total pre-orderings of worlds (cf. footnote 3), F&R's definition (Def $\preceq'$ from $\preceq$) of the posterior ordering $\preceq'$ of worlds reduces to

$$w \preceq' w' \ \text{ iff } \begin{cases} w \preceq w' \text{ and } w \text{ and } w' \text{ are both } \alpha\text{-worlds} \ \ \textit{or} \\ w' \text{ is a } \neg\alpha\text{-world} \end{cases}$$

The revised entrenchment relation corresponding to the operation '$*\alpha$' defined above can be derived from (Def $\leq'$ from $\leq$):

$$\beta \leq' \gamma \quad \text{iff} \quad \alpha \to \beta \leq \alpha \to \gamma$$

After irrevocable revisions by α, the $\neg\alpha$-worlds become completely disregarded, or equivalently, sentences will always be compared in terms of their entrenchment conditional on α.

As an operation of revision of belief sets by new input sentences which is defined in terms of entrenchments, we take down

(Def $*$ from $\leq$) $\qquad \beta \in K * \alpha \quad \text{iff} \quad \neg\alpha < \alpha \to \beta \;\text{ or }\; \top \leq \neg\alpha$

Now we give an axiomatic definition of irrevocable revision which is similar to the ones given in Segerberg [18] and Fermé [3], with a few additional elements from [16]. Both Segerberg's and Fermé's axiomatizations make use of an explicit representation of the set of irrevocable sentences, which is used to specify a consistency condition for $*$. Starting from revision by comparison, however, we can work with a much simpler format which leaves the set of irrevocable sentences implicit.

Definition 2 Let K be a belief set. A belief change function $*$ on K is an operation of *irrevocable belief revision* if and only if $*$ satisfies:

($*1$) $\quad K * \alpha = Cn(K * \alpha)$ (Closure)

($*2$) $\quad \alpha \in K * \alpha$ (Success)

($*3$) $\quad K * \alpha \subseteq K + \alpha$ (Inclusion)

($*4$) $\quad$ If $\neg\alpha \notin K$, then $K \subseteq K * \alpha$ (Preservation)

($*5$) $\quad$ If $\alpha \in K * \neg\alpha$, then $\alpha \in K * \beta$ (Irrevocability)

($*6$) $\quad$ If $Cn(\alpha) = Cn(\beta)$, then $K * \alpha = K * \beta$ (Extensionality)

($*7$) $\quad K * (\alpha \wedge \beta) \subseteq (K * \alpha) + \beta$ (Superexpansion)

($*8$) $\quad$ If $\neg\beta \notin K * \alpha$, then $K * \alpha \subseteq K * (\alpha \wedge \beta)$ (Conjunctive Preservation)

A function $*$ on K is an operation of *iterable irrevocable revision* if and only if it satisfies ($*1$) – ($*8$) and

($*$it) $\quad (K * \alpha) * \beta = K * (\alpha \wedge \beta)$

Conditions ($*1$) – ($*8$) are very close to AGM's postulates for belief revision. Conditions ($*4$) and ($*8$) are slightly weakened as compared to AGM's postulates with the same label, but are equivalent given ($*1$) and ($*2$). The remaining change is that AGM's fifth postulate called 'Consistency preservation' is not there any more. We want to make room for the possibility that the revision by a non-contradiction can make the agent's beliefs collapse into $\mathcal{L}$, and condition ($*5$) expresses that such non-contradictions are the negations of irrevocable sentences (where irrevocability now means $\alpha \in K * \neg\alpha$). In fact, any revision by the negation of an irrevocable sentence will lead to an epistemic collapse, due to ($*$2). Our condition ($*$5) is effectively weaker than AGM's Consistency preservation which is invalid for irrevocable revision functions. But it is worth keeping in mind that the above axiomatization includes all the other AGM postulates for belief revision.

In AGM revision, the problem of the epistemic collapse always arises if the negation of the input sentence is irrevocable (which in the particular case of AGM means that the input sentence is a contradiction). In revision by comparison, it arises only in the rather peculiar case where both the reference sentence and the negation of the input sentence are irrevocable. If only the latter condition is satisfied, then we obtain a special kind of belief withdrawal function: Severe withdrawal.

As in AGM theory, we can equivalently use the following condition instead of the pair ($*$7) and ($*$8): [7]

($*$7&8) Either $K * (\alpha \vee \beta) = K * \alpha$ or $K * (\alpha \vee \beta) = K * \beta$
or $K * (\alpha \vee \beta) = K * \alpha \ \cap \ K * \beta$. (Disjunctive factoring)

In condition ($*$it), we use the notation $(K * \alpha) * \beta$ for iterated belief change. One may think of this as slightly misleading since considered as an operation on a belief set, the second occurrence of $*$ does not denote "the same" revision function as the first $*$. But the reply is the same as in the case of severe withdrawal. Revision by comparison operates on belief states in the first place, and only secondarily on belief sets. It is evident how ($*$it) can be used to construct any finite number of applications of irrevocable revision. By ($*$2) and ($*$it), it is clear that once a belief set K gets revised by α, the sentence α is never going to be lost under irrevocable revisions any more, and thus assumes an irrevocable status in the posterior belief set.

The axiom set listed in Definition 2 characterizes exactly the revision functions defined through revision by comparison with reference sentence $\top$.

Theorem 3 (Soundness) Let $\circ$ be a revision-by-comparison operator on a belief set K that satisfies (C1) – (C6), and let $*$ be defined by putting $K * \alpha = K \circ_\top \alpha$ for all sentences α. Then $*$ is an irrevocable revision operator satisfying ($*$1) – ($*$8). If $\circ$ in addition satisfies (IT), then $*$ satisfies ($*$it).

Theorem 4 (Completeness) Let K be a belief set and $*$ an irrevocable revision function on K that satisfies ($*$1) – ($*$8) and ($*$it). Then there is an entrenchment relation $\leq$ such that for all α, $K * \alpha = K \circ_\top \alpha$, where $\circ$ is defined by (Def $\circ$ from $\leq$), and moreover, for all α and β, $(K * \alpha) * \beta = (K \circ_\top \alpha) \circ_\top \beta$, where $\circ$ is defined by (Def $\leq'$ from $\leq$) and (Def $\circ$ from $\leq$).

[7] See property (K$*$7&8) of F&R.

4 Irrefutable belief revision

Like irrevocable revision, irrefutable belief revision is an operation that is designed to deal with repeated belief revision operations. It is different in spirit from the non-repeatable revision operations introduced by Alchourrón, Gärdenfors and Makinson: Irrefutable revisions are *always consistent* (the resulting belief set is always free of contradictions) but *not always successful* (the input sentence is not always accepted). The success condition of AGM is weakened in irrefutable belief revision, in order to pay unrestricted respect to the requirement of consistency. For instance, while AGM require $\bot$ to be included in $K * \bot$, irrefutable belief revision models agents as being less gullible and refusing to accept any inconsistencies.

In irrevocable belief change, α is invariably accepted in $K * \alpha$, and it can never be lost in subsequent belief changes through revision by comparison (whence the name). In *irrefutable belief change*, α is not necessarily accepted in $K \star \alpha$. The revised belief set $K \star \alpha$ is invariably consistent even though sometimes $\neg\alpha$ is retained. However, *if* $\neg\alpha$ is eliminated in $K \star \alpha$, then $\neg\alpha$ will never be regained in any subsequent belief change through further applications of irrefutable revision. In this sense, α will never be refuted again (whence the name).

In this section we study the unary operation $\star$ which is defined by putting

$$K \star \alpha = K \circ_\varepsilon \alpha$$

for every sentence α. Here ε is supposed to be a fixed reference sentence in K which is *not* irrevocable, that is, $\varepsilon \notin K \circ_\varepsilon \bot$. This latter requirement is precisely what distinguishes irrefutable from irrevocable belief revision. As far as I know, this operation has not been studied in the literature before.

An intended application of belief revision with a fixed reference sentence is the case where all incoming information stems from the same source and the source's credibility is understood to be specified by the fixed reference sentence. Sources should better be reliable, however. It is a weakness of irrefutable belief change that once a revision fails to be successful (i.e., $\alpha \notin K \star \alpha$), the agent is caught in a belief set from which he can never escape by further irrefutable revisions. It will turn out that $K \star \bot$, though consistent, is this strange attractor. That belief set can be left only if the agent decides to raise the standard set by the reference sentence, or else to change his revision method.

We thus assume as given (and fixed) a parameter sentence ε such that $\varepsilon \notin K \circ_\varepsilon \bot$. In terms of entrenchment, the condition on ε means that $\varepsilon < \top$. Notice that if ε is not irrevocable before the revision, then it is not made irrevocable by application of the revision operation $\circ_\varepsilon$ either. This is guaranteed by (IT1) which implies that $(K \circ_\varepsilon \alpha) \circ_\varepsilon \bot = K \circ_\varepsilon \bot$. If ε is not contained in K, then

$K \circ_{\varepsilon} \alpha = K$ (by the Vacuity property of $\circ$ mentioned in F&R, Lemma 5), that is, $K \star \alpha = K$ for any arbitrary sentence α.

The system of spheres and entrenchment semantics for this revision operation are given by the general forms of (Def $\$'$ from $\$$) and (Def $\leq'$ from $\leq$) respectively, as given in Section 1. This time there is no way to reduce these definitions.

Now we introduce a set of properties characterizing the monadic revision functions that we want to call irrefutable from now on.

Definition 3 Let K be a consistent belief set. A belief change function $\star$ on K is an operation of *irrefutable belief revision* if and only if $\star$ satisfies:

($\star$1) $K \star \alpha = Cn(K \star \alpha)$ (Closure)

($\star$2c) If $K \star \alpha \neq K \star \bot$, then $\alpha \in K \star \alpha$ (Conditional Success)

($\star$3) $K \star \alpha \subseteq K + \alpha$ (Inclusion)

($\star$4) If $\neg\alpha \notin K$, then $K \subseteq K \star \alpha$ (Preservation)

($\star$5u) $K \star \alpha \neq \mathcal{L}$ (Unconditional Consistency)

($\star$6) If $Cn(\alpha) = Cn(\beta)$, then $K \star \alpha = K \star \beta$ (Extensionality)

($\star$7) $K \star (\alpha \wedge \beta) \subseteq (K \star \alpha) + \beta$ (Superexpansion)

($\star$8) If $\neg\beta \notin K \star \alpha$, then $K \star \alpha \subseteq K \star (\alpha \wedge \beta)$ (Conjunctive Preservation)

($\star$9) If $K \star \bot = K$, then $K \star \alpha = K$ (Vacuity)

($\star$10) If $K \star \bot \neq K$, then $K \star \bot \subsetneqq \bigcap\{K \star \alpha : \alpha \in K \star \alpha\}$ (Fallback)

An function $\star$ on K is an operation of *iterable irrefutable revision* if and only if $\star$ satisfies ($\star$1), ($\star$2c), ($\star$3), ($\star$4), ($\star$5u), ($\star$6) – ($\star$10) and

($\star$it) $(K \star \alpha) \star \beta = K \star (\alpha \wedge \beta)$.

This axiomatization for $\star$ includes versions of all the AGM postulates, but it weakens AGM's unconditional Success postulate ($*$2) to the Conditional Success postulate ($\star$2c), and it strengthens AGM's conditional Consistency preservation postulate (which has the antecedent $\neg\alpha \notin Cn(\emptyset)$) to the unconditional ($\star$5u). In ordinary AGM revision, inconsistent input leads the agent into the inconsistent belief set $\mathcal{L}$. In irrefutable belief change, in contrast, inconsistent input leads to a special fallback theory $K \star \bot$ which normally is a rather weak (and in any case consistent) belief set. Our characterization includes two additional axioms concerning this fallback theory. ($\star$9) states that if a revision by a contradiction does not affect the prior belief set, then no revision whatsoever does. This case applies if and only if $\varepsilon \notin K$ (by ($\dot{-}$2)–($\dot{-}$4), remembering that $K \star \bot = K \dot{-} \varepsilon$). On the other hand, ($\star$10) makes clear that $K \star \bot$ represents a state of serious puzzlement: If this fallback position does differ from the original belief set K, then it contains less information than any successful revision; it even loses sentences that are contained in all successful revisions.

In condition ($\star$it), we use the notation $(K \star \alpha) \star \beta$ for iterated belief change. This presents the same notational problem as in the case of severe withdrawal and irrevocable revision, from which no confusion should arise by now.

The following lemma lists two useful facts concerning irrefutable revision:

Lemma 5 Let $\star$ satisfy ($\star$1), ($\star$2c), ($\star$3) – ($\star$4), ($\star$5u) and ($\star$6) – ($\star$10).

(i) If $\alpha \in K \star \alpha$ and $\alpha \vdash \beta$ then $\beta \in K \star \beta$ (Strict Improvement[8])

(ii) $K \star \bot \subseteq K \star \alpha$ (Fallback Inclusion)

In Lemma 5, condition (i) says that if a revision by α is successful, then any revision by a weaker sentence is also successful. Condition (ii) says that the fallback theory is a subset of any irrefutable revision of K, whether successful or not.

Theorem 6 (Soundness) Let K be a consistent belief set, let $\circ$ be a revision-by-comparison operator on K that satisfies (C1) – (C6), and let ε be such that $\varepsilon \notin K \circ_\varepsilon \bot$. Define $\star$ by putting $K \star \alpha = K \circ_\varepsilon \alpha$ for all sentences α. Then $\star$ is an irrefutable belief revision function satisfying ($\star$1), ($\star$2c), ($\star$3)–($\star$4), ($\star$5u) and ($\star$6)–($\star$10). If $\circ$ in addition satisfies (IT), then $\star$ satisfies ($\star$it).

Theorem 7 (Completeness) Let K be a consistent belief set and $\star$ an irrefutable revision function on K that satisfies ($\star$1), ($\star$2c), ($\star$3)–($\star$4), ($\star$5u) and ($\star$6)–($\star$10). Then there is an entrenchment relation $\leq$ and a sentence ε such that for all α, $K \star \alpha = K \circ_\varepsilon \alpha$, where $\circ$ is defined by (Def $\circ$ from $\leq$), and moreover, for all α and β, $(K \star \alpha) \star \beta = (K \circ_\varepsilon \alpha) \circ_\varepsilon \beta$, where $\circ$ is defined by (Def $\leq'$ from $\leq$) and (Def $\circ$ from $\leq$).

Irrefutable belief revision is close in spirit to *credibility-limited belief revision*, or more precisely, to sphere-based and entrenchment-based credibility limited revision in the sense of Hansson, Fermé, Cantwell and Falappa [11]. Both models put a very strong emphasis on the consistency of revised belief sets, and in exchange for this, they do not place their trust in unconditional "success" ($\ast$2). They achieve this by providing for the possibility that the agent refuses to accept certain "incredible" input sentences. Irrefutable and credibility-limited revision differ only in the case where the agent is confronted with an incredible input sentence α. The case in point here arises when all spheres of possible worlds that are covered by $[\varepsilon]$ are also covered by $[\neg\alpha]$, or equivalently in terms of entrenchment, when $\varepsilon \leq \neg\alpha$. In this case credibility-limited belief revision recommends that the agent does nothing (keeps the same belief set K[9]), whereas irrevocable belief revision leads the subject into a state of great

[8] Compare Hansson et al. [11, p. 1582].

[9] Or, potentially, that he keeps the same system of spheres $ or the same entrenchment relation $\leq$. But notice that irrevocable belief change is in itself suitable for

ignorance, into the belief set $K \star \bot$.

5 Conclusion

We have seen that by some special fixations of input or reference sentences, the binary operation of revision by comparison can be reduced to three of interesting unary operations for iterated belief change. Using revision by comparison as a unifying framework, we uncovered a common superstructure of severe withdrawal and irrevocable revision – two well-known operations that have not been thought of as in any way related so far. Revision by comparison has also helped us to design irrefutable revision, a new variant of belief revision that values the consistency of belief sets higher than the unconditional acceptance of the input sentence. The axiomatization of revision by comparison served as a convenient means for setting up streamlined axiomatizations of the three unary operations mentioned.

However, the cautionary remark on revision by comparison at the end of Fermé and Rott [5, p. 27] also applies to the unary limiting cases: The operations in question tend to make belief states (i.e., systems of spheres or entrenchment relations) coarser and coarser along a series of repeated belief changes of these types, and the reasoner may find himself ending up in a belief state that is not going to be changed any more by any operation of revision by comparison. Therefore the operations studied in the present paper have to be combined with other methods – preferably methods that tend to refine belief states – in order to serve as a practical and realistic model of the dynamics of epistemic states.

Acknowledgements

I would like to thank Eduardo Fermé for many intense and pleasurable discussions of revision by comparison, and an anonymous referee for helpful comments. The support of this research by the Argentinean Antorchas Foundation and the German Academic Exchange Service (DAAD), project number 1521/710218 ("Revision by Comparison"), is gratefully acknowledged.

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Appendix: Proofs

The following proofs make free use not only of the conditions (C1) – (C6) and (IT1) and (IT4) mentioned in Section 1, but also of the following derived properties that were motivated and proved to hold for revision by comparison in Fermé and Rott [5, pp. 19–21, 28].

Lem. 0(d) If $\alpha \in K \circ_\alpha \bot$, then $K \circ_\alpha \bot = K_\bot$.

(Vacuity) If $\alpha \notin K$, then $K \circ_\alpha \beta = K$.

(Q2) $\alpha \in K \circ_\alpha \neg\beta$ iff $\alpha \in K \circ_\beta \neg\alpha$.

(Q16) If $\alpha \notin K \circ_\alpha \bot$, then $K \circ_\alpha \bot \subseteq K \circ_{\alpha\wedge\beta} \bot$.

(Q17) If $\alpha \notin K \circ_\alpha \bot$, then $K \circ_\alpha \bot \subseteq K \circ_\alpha \beta$.

(K∗2) If $\alpha \in K \circ_\alpha \beta$ then $\beta \in K \circ_\alpha \beta$ (Weak Success)

(K∗3) $K \circ_\alpha \beta \subseteq K + \beta$ (Inclusion)

(K∗4) If $\neg\beta \notin K$, then $K \subseteq K \circ_\alpha \beta$ (Preservation)

(K∗5) $\alpha \wedge \neg\beta \in K \circ_{\alpha\wedge\neg\beta} \bot$ iff $K \circ_\alpha \beta = K_\bot$ (Consistency)

(K∗7) $K \circ_\alpha (\beta \wedge \gamma) \subseteq (K \circ_\alpha \beta) + \gamma$ (Superexpansion)

(K∗8) If $\neg\gamma \notin K \circ_\alpha \beta$, then $K \circ_\alpha \beta \subseteq K \circ_\alpha (\beta \wedge \gamma)$ (Conj. Preservation)

We also need the following properties of entrenchment relations (Gärdenfors and Makinson [9]):

(E1) If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$ (Transitivity)

(E2) If $\alpha \vdash \beta$, then $\alpha \leq \beta$ (Dominance)

(E3) $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$ (Conjunctiveness)

Proof of ($\ddot{-}8^+$) and ($\ddot{-}$D) from the axioms for severe withdrawal

($\ddot{-}8^+$) Let $\alpha \notin K \mathbin{\ddot{-}} (\alpha \wedge \beta)$. Then $K \mathbin{\ddot{-}} (\alpha \wedge \beta) \subseteq K \mathbin{\ddot{-}} \alpha$ is immediate from ($\ddot{-}8$). On the other hand, we can conclude from $\alpha \notin K \mathbin{\ddot{-}} (\alpha \wedge \beta)$ with the help of ($\ddot{-}$5) that $K \ddot{-} \alpha \neq \mathcal{L}$. So $K \ddot{-} \alpha \subseteq K \ddot{-} (\alpha \wedge \beta)$ by ($\ddot{-}$7), which gives us the desired identity.

($\ddot{-}$D) Suppose that not $K \ddot{-} (\alpha \wedge \beta) = K \ddot{-} \alpha$. We need to show that in this case $K \ddot{-} (\alpha \wedge \beta) = K \ddot{-} \beta$. From the hypothesis we know that either $K \ddot{-} (\alpha \wedge \beta) \not\subseteq K \ddot{-} \alpha$ or $K \ddot{-} \alpha \not\subseteq K \ddot{-} (\alpha \wedge \beta)$. By ($\ddot{-}$7) and ($\ddot{-}$8), either $\alpha \in K \ddot{-} (\alpha \wedge \beta)$ or $K \ddot{-} \alpha = \mathcal{L}$. By ($\ddot{-}$5), we get in any case that $\alpha \in K \ddot{-} (\alpha \wedge \beta)$. Suppose first that $\beta \in K \ddot{-} (\alpha \wedge \beta)$ as well. Then we have $\alpha \wedge \beta \in K \ddot{-} (\alpha \wedge \beta)$, whence $K \ddot{-} (\alpha \wedge \beta) = \mathcal{L} = K \ddot{-} \beta$, by ($\ddot{-}$4) and ($\ddot{-}$5). Suppose, secondly, that $\beta \notin K \ddot{-} (\alpha \wedge \beta)$. Then $K \ddot{-} (\alpha \wedge \beta) = K \ddot{-} \beta$, by ($\ddot{-}8^+$), and we are done. ■

Proof of Theorem 1 (Soundness of Severe Withdrawal)

($\ddot{-}$1) follows from (C1).

($\ddot{-}$2)

Let $K \circ_\alpha \bot \neq \mathcal{L}$. Then by Lemma 0(d) of part I of this paper, $\alpha \notin K \circ_\alpha \bot$, so by applying (Q16), (K$*$3) and (C1) we get that $K \circ_\alpha \bot \subseteq K \circ_\alpha \top \subseteq K + \top = K$.

($\ddot{-}$3) follows from Vacuity.

($\ddot{-}$4) is Lemma 0(d) of part I of this paper.

($\ddot{-}$5) follows from (C4).

($\ddot{-}$6) follows from (C2).

($\ddot{-}$7) Let $K \circ_\alpha \bot \neq \mathcal{L}$. Then by Lemma 0(d) of part I of this paper, $\alpha \notin K \circ_\alpha \bot$. By (C1), $\alpha \wedge \beta \notin K \circ_\alpha \bot$. So by (C3), $K \circ_\alpha \bot \subseteq K \circ_{\alpha \wedge \beta} \bot$.

($\ddot{-}$8) Let $\alpha \notin K \circ_{\alpha \wedge \beta} \bot$. Then by (C3), $K \circ_{\alpha \wedge \beta} \bot \subseteq K \circ_\alpha \bot$, as desired.

($\ddot{-}$it) follows from (IT4). ■

Proof of Theorem 2 (Completeness of Severe Withdrawal)

We use (Def $\ddot{-}$ from $\leq$) which is a special case of (Def $\circ$ from $\leq$), and as its "converse"

(Def $\leq$ from $\ddot{-}$) $\qquad \alpha \leq \beta \quad$ iff $\quad \alpha \notin K \ddot{-} \beta$ or $\beta \in K \ddot{-} \beta$

which is just a rewriting of (Def $\leq$ from $\circ$) of Fermé and Rott [5].

First we show that (Def $\leq$ from $\ddot{-}$) defines an entrenchment relation $\leq$ that satisfies (E1)–(E3).

As a preparation, we prove that

(†) $\qquad$ If $\alpha \notin K \ddot{-} \beta$, then $\alpha \notin K \ddot{-} (\alpha \wedge \beta)$

We show the contraposition. Suppose that $\alpha \in K \ddot{-} (\alpha \wedge \beta)$. Then by ($\ddot{-}$1), either $\alpha \wedge \beta \in K \ddot{-} (\alpha \wedge \beta)$ or $\beta \notin K \ddot{-} (\alpha \wedge \beta)$. In the former case, we get from ($\ddot{-}$5) that $\alpha \wedge \beta \in K \ddot{-} \beta$ and thus, by ($\ddot{-}$1), that $\alpha \in K \ddot{-} \beta$. In the latter case, we get from ($\ddot{-}$8) that $K \ddot{-} (\alpha \wedge \beta) \subseteq K \ddot{-} \beta$. Since $\alpha \in K \ddot{-} (\alpha \wedge \beta)$, we get $\alpha \in K \ddot{-} \beta$ as well. This proves (†).

$\leq$ satisfies (E1): Let $\alpha \leq \beta$ and $\beta \leq \gamma$, i.e., by (Def $\leq$ from $\ddot{-}$), ($\alpha \notin K \ddot{-} \beta$ or $\beta \in K \ddot{-} \beta$) and ($\beta \notin K \ddot{-} \gamma$ or $\gamma \in K \ddot{-} \gamma$). We want to show that $\alpha \leq \gamma$. i.e., that ($\alpha \notin K \ddot{-} \gamma$ or $\gamma \in K \ddot{-} \gamma$).

If $\gamma \in K \ddot{-} \gamma$, then we are done. Suppose that $\gamma \notin K \ddot{-} \gamma$. Thus $\beta \notin K \ddot{-} \gamma$. By ($\ddot{-}$5), $\beta \notin K \ddot{-} \beta$. Thus $\alpha \notin K \ddot{-} \beta$. From $\beta \notin K \ddot{-} \gamma$, we get by (†) that $\beta \notin K \ddot{-} (\beta \wedge \gamma)$, which implies, by ($\ddot{-}$8), $K \ddot{-} (\beta \wedge \gamma) \subseteq K \ddot{-} \beta$. Hence $\alpha \notin K \ddot{-} (\beta \wedge \gamma)$. From $\gamma \notin K \ddot{-} \gamma$, on the other hand, we get that $K \ddot{-} \gamma \neq \mathcal{L}$ and thus, by ($\ddot{-}$7), $K \ddot{-} \gamma \subseteq K \ddot{-} (\beta \wedge \gamma)$. Hence $\alpha \notin K \ddot{-} \gamma$. This proves (E1).

$\leq$ satisfies (E2): Let $\alpha \vdash \beta$. We want to show that $\alpha \leq \beta$, that is, by (Def $\leq$ from $\ddot{-}$), $\alpha \notin K \ddot{-} \beta$ or $\beta \in K \ddot{-} \beta$. Suppose that $\alpha \in K \ddot{-} \beta$. Then, by ($\ddot{-}$1), we get $\beta \in K \ddot{-} \beta$, and we are done.

$\leq$ satisfies (E3): We want to show that either $\alpha \leq (\alpha \wedge \beta)$ or $\beta \leq (\alpha \wedge \beta)$, that is, by (Def $\leq$ from $\ddot{-}$), either ($\alpha \notin K \ddot{-} (\alpha \wedge \beta)$ or $\alpha \wedge \beta \in K \ddot{-} (\alpha \wedge \beta)$) or ($\beta \notin K \ddot{-} (\alpha \wedge \beta)$ or $\alpha \wedge \beta \in K \ddot{-} (\alpha \wedge \beta)$). But this follows immediately from ($\ddot{-}$1).

Second, we show that $\ddot{-}$ is identical with the operation $\ddot{-}' = \circ_{\dots} \bot$ defined from $\ddot{-}$ by successive application of (Def $\leq$ from $\ddot{-}$) and (Def $\ddot{-}$ from $\leq$).

$\beta \in K \ddot{-}' \alpha$ iff (by Def $\ddot{-}$ from $\leq$)

$\alpha < \beta$ or $\top \leq \alpha$ iff (by Def $\leq$ from $\ddot{-}$)

($\beta \in K \ddot{-} \alpha$ and $\alpha \notin K \ddot{-} \alpha$) or ($\top \notin K \ddot{-} \alpha$ or $\alpha \in K \ddot{-} \alpha$) iff (by $\ddot{-}$1)

$\beta \in K \ddot{-} \alpha$ or $\alpha \in K \ddot{-} \alpha$.

That $\beta \in K \ddot{-} \alpha$ implies $\beta \in K \ddot{-}' \alpha$ is obvious.

For the converse, suppose that $\beta \in K \ddot{-}' \alpha$. Then either $\beta \in K \ddot{-} \alpha$ straight away, or $\alpha \in K \ddot{-} \alpha$. But in the latter case, by ($\ddot{-}$4), $K \ddot{-} \alpha = \mathcal{L}$, and so $\beta \in K \ddot{-} \alpha$, too.

Hence, $K \ddot{-} \alpha = K \ddot{-}' \alpha$, for arbitrary α.

Third, a comparative inspection of ($\ddot{-}$it) and (IT4) shows that $(K \ddot{-} \alpha) \ddot{-} \beta = (K \circ_\alpha \bot) \circ_\beta \bot$ for all α and β. ■

Proof of Theorem 3 (Soundness of Irrevocable Revision)

(∗1) follows from (C1).

(∗2) follows from (C1) and (K∗2).

(∗3) follows from (K∗3).

(∗4) follows from (K∗4).

(∗5). Let $\alpha \in K \circ_\top \neg\alpha$. Then by (Q2), $\alpha \in K \circ_\alpha \neg\top = K \circ_\alpha \bot$. So by (C4), $\alpha \in K \circ_\top \beta$.

(∗6) follows from (C2).

(∗7) follows from (K∗7).

(∗8) follows from (K∗8).

(∗it) follows from (IT1). ■

Proof of Theorem 4 (Completeness of Irrevocable Revision)

We use (Def $*$ from $\leq$) which is a special case of (Def $\circ$ from $\leq$), and as its "converse"

(Def $\leq$ from $*$) $\quad \alpha \leq \beta \quad$ iff $\quad \alpha \notin K * \neg(\alpha \wedge \beta)$ or $\beta \in K * \neg(\alpha \wedge \beta)$

Observation 7(e) of Fermé and Rott [5] shows that this condition is equivalent with (Def $\leq$ from $\circ$).

First we show that (Def $\leq$ from $*$) defines an entrenchment relation $\leq$ that satisfies (E1)–(E3).

$\leq$ satisfies (E1): Let $\alpha \leq \beta$ and $\beta \leq \gamma$, i.e., by (Def $\leq$ from $*$), ($\alpha \notin K*\neg(\alpha\wedge\beta)$ or $\alpha, \beta \in K * \neg(\alpha \wedge \beta)$) and ($\beta \notin K * \neg(\beta \wedge \gamma)$ or $\beta, \gamma \in K * \neg(\beta \wedge \gamma)$). We want to show that $\alpha \leq \gamma$. i.e., that $\alpha \notin K * \neg(\alpha \wedge \gamma)$ or $\gamma \in K * \neg(\alpha \wedge \gamma)$.

Suppose for reductio that this is not true, i.e. that $\alpha \in K * \neg(\alpha \wedge \gamma)$ and $\gamma \notin K*\neg(\alpha\wedge\gamma)$. If $\beta\wedge\gamma \in K*\neg(\beta\wedge\gamma)$, then by (∗5) and (∗1) $\gamma \in K*\neg(\alpha\wedge\gamma)$, and we are done. So suppose that $\beta \notin K * \neg(\beta \wedge \gamma)$. Then, by (∗5) and (∗1), $\alpha \wedge \beta \notin K * \neg(\alpha \wedge \beta)$. So by hypothesis $\alpha \notin K * \neg(\alpha \wedge \beta)$.

Now we show that

(†) If $\alpha \in K * \neg(\alpha \wedge \gamma)$, then $\alpha \in K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$.

Let $\alpha \in K * \neg(\alpha \wedge \gamma)$ =(by * 6) $K * (\neg\alpha \vee \neg\beta \vee \neg\gamma) \wedge (\neg\alpha \vee \neg\gamma)$. Then by (∗7) and (∗1) $(\neg\alpha \vee \neg\gamma) \rightarrow \alpha \in K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$, and by (∗1) again, $\alpha \in K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$.

On the other hand, we can prove that

(‡) If $\alpha \notin K * \neg(\alpha \wedge \beta)$, then either $\alpha \wedge \beta \wedge \neg\gamma \in K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$ or $\alpha \notin K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$.

Let $\alpha \notin K*\neg(\alpha\wedge\beta)$. By (∗6), this is equivalent with $\alpha \notin K*(\neg\alpha\vee\neg\beta\vee\neg\gamma)\wedge(\neg\alpha \vee \neg\beta \vee \gamma)$. Then, by (∗8), either $\neg(\neg\alpha \vee \neg\beta \vee \gamma) \in K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$ or $\alpha \notin K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$. So (‡) follows by (∗1).

Now the situation is that we have both $\alpha \in K*\neg(\alpha\wedge\gamma)$ and $\alpha \notin K*\neg(\alpha\wedge\beta)$. So we can apply both (†) and (‡). Taking together the consequences of both, we find that $\alpha \wedge \beta \wedge \neg\gamma \in K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$.

Next we take the fact that $\beta \notin K * \neg(\beta \wedge \gamma)$ and apply exactly the same reasoning as we have in (‡), obtaining the conclusion that either $\neg\alpha \wedge \beta \wedge \gamma \in K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$ or $\beta \notin K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$. But we also know that

$\alpha \wedge \beta \wedge \neg\gamma \in K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$. Using ($*1$), we finally find that our assumptions lead us to conclude that $\alpha \wedge \beta \wedge \gamma \in K * (\neg\alpha \vee \neg\beta \vee \neg\gamma)$. By ($*5$) and ($*1$), this finally implies that $\alpha \wedge \beta \wedge \gamma \in K * \neg(\alpha \wedge \gamma)$ and $\gamma \in K * \neg(\alpha \wedge \gamma)$, and we have a contradiction. This proves (E1).

$\leq$ satisfies (E2): Let $\alpha \vdash \beta$. We want to show that $\alpha \leq \beta$, that is, by (Def $\leq$ from $*$), $\alpha \notin K * \neg(\alpha \wedge \beta)$ or $\beta \in K * \neg(\alpha \wedge \beta)$. Suppose that $\alpha \in K * \neg(\alpha \wedge \beta)$. Then, by ($*1$), we get $\beta \in K * \neg(\alpha \wedge \beta)$, and we are done.

$\leq$ satisfies (E3): We want to show that either $\alpha \leq (\alpha \wedge \beta)$ or $\beta \leq (\alpha \wedge \beta)$, that is, by (Def $\leq$ from $*$), either ($\alpha \notin K * \neg(\alpha \wedge \alpha \wedge \beta)$ or $\alpha \wedge \beta \in K * \neg(\alpha \wedge \alpha \wedge \beta)$) or ($\beta \notin K * \neg(\beta \wedge \alpha \wedge \beta)$ or $\alpha \wedge \beta \in K * \neg(\beta \wedge \alpha \wedge \beta)$). Using ($*6$), this can be simplified to $\alpha \notin K * \neg(\alpha \wedge \beta)$ or $\beta \notin K * \neg(\alpha \wedge \beta)$ or $\alpha \wedge \beta \in K * \neg(\alpha \wedge \beta)$. But this follows immediately from ($*1$).

Second, we show that $*$ is identical with the operation $*' = \circ_\top \ldots$ defined from $*$ by the successive application of (Def $\leq$ from $*$) and (Def $*$ from $\leq$).

$\beta \in K *' \alpha$ iff (by Def $*$ from $\leq$)

$\neg\alpha < \alpha \to \beta$ or $\top \leq \neg\alpha$ iff (by Def $\leq$ from $*$)

($\alpha \to \beta \in K * \neg(\neg\alpha \wedge (\alpha \to \beta))$ and $\neg\alpha \notin K * \neg(\neg\alpha \wedge (\alpha \to \beta))$) or ($\top \notin K * \neg(\top \wedge \neg\alpha)$ or $\neg\alpha \in K * \neg(\top \wedge \neg\alpha)$ iff (by $*1$ and $*6$)

$\alpha \to \beta \in K * \alpha$ or $\neg\alpha \in K * \alpha$ iff (by $*1$)

$\alpha \to \beta \in K * \alpha$ iff (by $*1$ and $*2$)

$\beta \in K * \alpha$.

Hence, $K *' \alpha = K * \alpha$, for arbitrary α.

Third, a comparative inspection of ($*$it) and (IT1) shows that $(K * \alpha) * \beta = (K \circ_\top \alpha) \circ_\top \beta$ for all α and β. ■

Proof of Lemma 5

(i): If $\alpha \in K \star \alpha$ and $\alpha \vdash \beta$, then by ($\star 6$) and ($\star 7$), $\alpha \in K \star (\beta \wedge (\alpha \vee \neg\beta)) \subseteq (K \star \beta) + (\alpha \vee \neg\beta)$, so by propositional logic, $(\alpha \vee \neg\beta) \to \alpha$ is in $K \star \beta$, and thus, by ($\star 1$), β is in $K \star \beta$.

(ii): $K \star \bot \subseteq K \star \alpha$ follows by ($\star 9$), if $K \star \bot = K$; by ($\star 10$), if $K \star \bot \neq K$ and $\alpha \in K \star \alpha$; and by ($\star 2$c), if $\alpha \notin K \star \alpha$. ■

Proof of Theorem 6 (Soundness of Irrefutable Revision)

Let $K \star \alpha$ be defined as $K \circ_\varepsilon \alpha$ for some fixed ε such that $\varepsilon \notin K \circ_\varepsilon \bot$.

($\star$1) follows from (C1).

($\star$2c) Let $K \circ_\varepsilon \alpha \neq K \circ_\varepsilon \bot$. We want to show that $\alpha \in K \circ_\varepsilon \alpha$. By the choice of ε, we have $\varepsilon \notin K \circ_\varepsilon \bot$. Hence by (C1), $\varepsilon \wedge \neg\alpha \notin K \circ_\varepsilon \bot$, and so, by (C3), $K \circ_\varepsilon \bot \subseteq K \circ_{\varepsilon \wedge \neg\alpha} \bot$. Next we show that $\varepsilon \in K \circ_{\varepsilon \wedge \neg\alpha} \bot$. Suppose that this was not the case. Then by (C6), $K \circ_\varepsilon \alpha = K \circ_{\varepsilon\wedge\neg\alpha} \bot$. Thus, since $K \circ_\varepsilon \alpha \neq K \circ_\varepsilon \bot$, we get $K \circ_{\varepsilon\wedge\neg\alpha} \bot \neq K \circ_\varepsilon \bot$. Because $K \circ_\varepsilon \bot \subseteq K \circ_{\varepsilon\wedge\neg\alpha} \bot$, the converse cannot hold. Thus, by (C3), $\varepsilon \in K \circ_{\varepsilon\wedge\neg\alpha} \bot$ which is what we set out to show. We conclude with (C5) that $K \circ_\varepsilon \alpha = (K \circ_{\varepsilon\wedge\neg\alpha} \bot) + \alpha$, and thus $\varepsilon \in K \circ_\varepsilon \alpha$, so by (K∗2), $\alpha \in K \circ_\varepsilon \alpha$, as desired.

(⋆3) follows from (K∗3).

(⋆4) follows from (K∗4).

(⋆5u) Suppose for reductio that $K \circ_\varepsilon \alpha = \mathcal{L}$. Then by (K∗5), $\varepsilon \wedge \neg\alpha \in K \circ_{\varepsilon\wedge\neg\alpha} \bot$. By (C4), $\varepsilon \wedge \neg\alpha \in K \circ_\varepsilon \bot$, so by (C1), $\varepsilon \in K \circ_\varepsilon \bot$, contradicting the choice of ε.

(⋆6) follows from (C2).

(⋆7) follows from (K∗7).

(⋆8) follows from (K∗8).

(⋆9) Let $K \star \bot = K \circ_\varepsilon \bot = K \circ_\bot \bot = K$. By the choice of ε, we know that $\varepsilon \notin K \circ_\varepsilon \bot$. So $\varepsilon \notin K \circ_\bot \bot$. So by Vacuity for $\circ$, $K \star \alpha = K \circ_\varepsilon \alpha = K$ for all α.

(⋆10) First, we note that since by hypothesis $\varepsilon \notin K \circ_\varepsilon \bot$, (Q17) tells us that $K \circ_\varepsilon \bot \subseteq K \circ_\varepsilon \alpha$ for every α. Thus $K \star \bot \subseteq K \star \alpha$ for every α.

Secondly, we show that if $K \star \bot \neq K$, then $\bigcap\{K \star \alpha : \alpha \in K \star \alpha\}$ is a *proper* superset of $K \star \bot$ by showing that ε itself is in the former, but not in the latter set. By the choice of ε, we have $\varepsilon \notin K \circ_\varepsilon \bot = K \star \bot$. So it remains to show that ε is in every successful revision, provided that $K \star \bot \neq K$.

Let $K \star \bot \neq K$. Then by Vacuity, $\varepsilon \in K = K \circ_\bot \bot$.

Let $\alpha \in K \star \alpha = K \circ_\varepsilon \alpha$. We want to show that $\varepsilon \in K \star \alpha = K \circ_\varepsilon \alpha$.

By (C5) and (C6) we have

either (case 1) $\varepsilon \in K \circ_{\varepsilon\wedge\neg\alpha} \bot$ and $K \circ_\varepsilon \alpha = (K \circ_{\varepsilon\wedge\neg\alpha} \bot) + \alpha$

or (case 2) $\varepsilon \notin K \circ_{\varepsilon\wedge\neg\alpha} \bot$ and $K \circ_\varepsilon \alpha = K \circ_{\varepsilon\wedge\neg\alpha} \bot$.

If case 1 obtains, we get that $\varepsilon \in (K \circ_{\varepsilon\wedge\neg\alpha} \bot) + \alpha = K \circ_\varepsilon \alpha = K \star \alpha$, as desired.

We now show that because $\alpha \in K \circ_\varepsilon \alpha$, case 2 is actually impossible. From $\varepsilon \notin K \circ_{\varepsilon\wedge\neg\alpha} \bot$, we get by (C1) that $\bot \notin K \circ_{\varepsilon\wedge\neg\alpha} \bot$, so by (C3), $K \circ_{\varepsilon\wedge\neg\alpha} \bot \subseteq K \circ_\bot \bot$. So since $\alpha \in K \circ_\varepsilon \alpha = K \circ_{\varepsilon\wedge\neg\alpha} \bot$, we conclude that $\alpha \in K \circ_\bot \bot$. By (C1), $\varepsilon \notin K \circ_\varepsilon \bot$ implies that $\bot \notin K \circ_\varepsilon \bot$, so $\bot \notin K \circ_\bot \bot$, by (C4). By (C1) again, we get from $\alpha \in K \circ_\bot \bot$ and $\bot \notin K \circ_\bot \bot$ that $\varepsilon \wedge \neg\alpha \notin K \circ_\bot \bot$. So by Vacuity, $K \circ_{\varepsilon\wedge\neg\alpha} \bot = K \circ_\bot \bot$. But in case 2 we have $\varepsilon \notin K \circ_{\varepsilon\wedge\neg\alpha} \bot$. On the other hand, we have already shown from $K \star \bot \neq K$ that $\varepsilon \in K \circ_\bot \bot$. This gives us a contradiction, so we see that case 2 is impossible.

This proves that $\alpha \in K \star \alpha$ entails $\varepsilon \in K \star \alpha$, and hence that $\varepsilon \in \bigcap\{K \star \alpha : \alpha \in K \star \alpha\}$.

Putting everything together, we finally obtain that $K \star \bot \subsetneq \bigcap\{K \star \alpha : \alpha \in K \star \alpha\}$, as desired.

($\star$it) follows from (IT1). ∎

Proof of Theorem 7 (Completeness of Irrefutable Revision)

Let K be a consistent theory, and let $\star$ satisfy ($\star$1), ($\star$2c), ($\star$3), ($\star$4), ($\star$5u) and ($\star$6) – ($\star$10).

In the first step we retrieve an entrenchment relation from $\star$ by means of the following definition, which is similar to, but not identical with, the one used for the completeness proof for irrevocable revision:

$$\alpha \leq \beta \;\text{ iff }\; \alpha \notin K \star \neg(\alpha \wedge \beta) \;\text{ or }\; \beta \in K \star \bot$$

First we show that this relation $\leq$ has all the properties of an entrenchment relation.

For (E1), suppose that $\alpha \leq \beta$ and $\beta \leq \gamma$. This means that

($\alpha \notin K \star \neg(\alpha \wedge \beta)$ or $\beta \in K \star \bot$) and ($\beta \notin K \star \neg(\beta \wedge \gamma)$ or $\gamma \in K \star \bot$).

We want to show that $\alpha \leq \gamma$, i.e., that

$\alpha \notin K \star \neg(\alpha \wedge \gamma)$ or $\gamma \in K \star \bot$.

Suppose that $\gamma \notin K \star \bot$. Then we have

(I) $\beta \notin K \star \neg(\beta \wedge \gamma)$.

By Lemma 5(ii), we get that $\beta \notin K \star \bot$, and hence we also get

(II) $\alpha \notin K \star \neg(\alpha \wedge \beta)$.

We need to show that $\alpha \notin K \star \neg(\alpha \wedge \gamma)$. Suppose for reductio that this is not true. Then $\alpha \in K \star \neg(\alpha \wedge \gamma) = K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma) \wedge (\neg\alpha \vee \neg\gamma)$. Then by ($\star$7) and ($\star$1) $(\neg\alpha \vee \neg\gamma) \rightarrow \alpha \in K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma)$, and by ($\star$1) again, $\alpha \in K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma)$.

Now if $\alpha \wedge \beta \wedge \neg\gamma$ were not in $K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma)$, then by ($\star$8) $K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma) \subseteq K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma) \wedge (\neg\alpha \vee \neg\beta \vee \gamma) = K \star (\neg\alpha \vee \neg\beta)$, and hence $\alpha \in K \star (\neg\alpha \vee \neg\beta)$. But this contradicts (II). Therefore we conclude that $\alpha \wedge \beta \wedge \neg\gamma$ is in $K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma)$.

By ($\star$5u), we get that $\neg\alpha \wedge \beta \wedge \gamma$ is not in $K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma)$. So by ($\star$8), $K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma) \subseteq K \star (\neg\alpha \vee \neg\beta \vee \neg\gamma) \wedge (\alpha \vee \neg\beta \vee \neg\gamma) = K \star (\neg\beta \vee \neg\gamma)$. Hence $\alpha \wedge \beta \wedge \neg\gamma \in K \star (\neg\beta \vee \neg\gamma)$ and also $\beta \in K \star (\neg\beta \vee \neg\gamma)$. But this contradicts (I).

So the supposition that $\alpha \in K \star \neg(\alpha \wedge \gamma)$ has led us to a contradiction. Thus $\alpha \notin K \star \neg(\alpha \wedge \gamma)$ is true, and we have shown that (E1) is satisfied.

For (E2), suppose that $\alpha \vdash \beta$. We need to show that $\alpha \leq \beta$, i.e., that $\alpha \notin K \star \neg(\alpha \wedge \beta)$ or $\beta \in K \star \bot$. By $\alpha \vdash \beta$ and ($\star$6), $K \star \neg(\alpha \wedge \beta) = K \star \neg\alpha$. Suppose that $\alpha \in K \star \neg(\alpha \wedge \beta) = K \star \neg\alpha$. Then by ($\star$5u), $\neg\alpha \notin K \star \neg\alpha$, i.e., $K \star \neg\alpha$ is not successful. But by ($\star$2c), this implies that $K \star \neg\alpha = K \star \bot$. So $\alpha \in K \star \bot$, and by $\alpha \vdash \beta$ and ($\star$1), we get $\beta \in K \star \bot$. Thus (E2) is satisfied.

For (E3), suppose for reductio that neither $\alpha \leq \alpha \wedge \beta$ nor $\beta \leq \alpha \wedge \beta$. This means that both $\alpha \in K \star \neg(\alpha \wedge \alpha \wedge \beta) = K \star \neg(\alpha \wedge \beta)$ and $\beta \in K \star \neg(\beta \wedge \alpha \wedge \beta) = K \star \neg(\alpha \wedge \beta)$ and that $\alpha \wedge \beta \notin K \star \bot$. By ($\star$5u), we get that $\neg(\alpha \wedge \beta) \notin K \star \neg(\alpha \wedge \beta)$, i.e., $K \star \neg(\alpha \wedge \beta)$ is not successful. But by ($\star$2c), this implies that $K \star \neg(\alpha \wedge \beta) = K \star \bot$, so α and β and thus $\alpha \wedge \beta$ are in $K \star \bot$, and we get a contradiction with our hypothesis. Therefore either $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$ is true, and (E3) is satisfied.

In the second step, we show that the entrenchment relation thus defined actually generates the irrefutable revision function. We distinguish two cases.

Case 1: $K \star \bot = K$. Then, by ($\star$9), $K \star \alpha = K$ for arbitrary α. The entrenchment relation $\leq$ defined above reduces to

$$\alpha \leq \beta \ \text{ iff } \ \alpha \notin K \ \text{ or } \ \beta \in K$$

As the fixed reference sentence, we choose $\varepsilon = \bot$. Now let us take the retrieved entrenchment relation $\leq$ and define a new revision function $\star'$ by putting $K \star' \delta = K \circ_\varepsilon \delta$ for all δ. This means, by (Def $\circ$ from $\leq$)

$$\gamma \in K \star' \delta \text{ iff } \begin{cases} \neg\delta < \varepsilon \wedge (\delta \to \gamma) & \textit{or} \\ \varepsilon < \gamma & \textit{or} \\ \top \leq \varepsilon \wedge \neg\delta \end{cases}$$

We have to show that $\varepsilon = \bot$ and $\leq$ represent $\star$ in the sense that $K \star \delta = K \star' \delta = K \circ_\varepsilon \delta$ for every δ. The definition of $\leq$, together with the fact that K is consistent, yields that $\bot \leq \beta$ for all β and $\bot < \beta$ for all $\beta \in K$. Thus the above reduces to

$$\gamma \in K \star' \delta \qquad \text{iff} \qquad \bot < \gamma \qquad \text{iff} \qquad \bot \notin K \text{ and } \gamma \in K$$

Using again the fact that K is consistent, this means that $K \star' \delta = K = K \star \delta$ for all δ, as desired.

Case 2: $K \star \bot \neq K$. Then, by ($\star$10), the set $\bigcap\{K \star \alpha : \alpha \in K \star \alpha\} - K \star \bot$ is non-empty. We pick an element ε from this set and use it as the fixed reference sentence for the rest of this proof.

Now we take this ε and the retrieved entrenchment relation $\leq$ and define a new revision function $\star'$ by putting $K \star' \delta = K \circ_\varepsilon \delta$ for all δ. This means, by

(Def $\circ$ from $\leq$)

$$\gamma \in K \star' \delta \text{ iff } \begin{cases} \neg\delta < \varepsilon \wedge (\delta \to \gamma) & \textit{or} \\ \varepsilon < \gamma & \textit{or} \\ \top \leq \varepsilon \wedge \neg\delta \end{cases}$$

We have to show that the chosen ε and $\leq$ represent $\star$ in the sense that $K \star \delta = K \star' \delta = K \circ_\varepsilon \delta$ for every δ.

Let us unravel the above by exploiting the meaning of $\leq$:

$$\gamma \in K \star' \delta \text{ iff } \begin{cases} \varepsilon \wedge (\delta \to \gamma) \in K \star \neg(\neg\delta \wedge (\varepsilon \wedge (\delta \to \gamma))) \text{ and} & \\ \qquad \neg\delta \notin K \star \bot & \textit{or} \\ \gamma \in K \star \neg(\varepsilon \wedge \gamma) \text{ and } \varepsilon \notin K \star \bot & \textit{or} \\ \top \notin K \star \neg(\top \wedge (\varepsilon \wedge \neg\delta)) \text{ or } \varepsilon \wedge \neg\delta \in K \star \bot & \end{cases}$$

But by the choice of ε, we know that $\varepsilon \notin K \star \bot$ is true, and, by ($\star$1), that $\varepsilon \wedge \neg\delta \in K \star \bot$ is false. So the above reduces to

$$\gamma \in K \star' \delta \text{ iff } \begin{cases} \varepsilon \wedge (\delta \to \gamma) \in K \star (\delta \vee \neg\varepsilon) \text{ and } \neg\delta \notin K \star \bot & \textit{or} \\ \gamma \in K \star (\neg\varepsilon \vee \neg\gamma) \end{cases}$$

Assume that $\gamma \in K \star' \delta$. We want to show that $\gamma \in K \star \delta$. We look at the two cases in the reverse order.

Firstly, assume that $\gamma \in K \star (\neg\varepsilon \vee \neg\gamma)$. We can infer that ε is not in $K \star (\neg\varepsilon \vee \neg\gamma)$, since if it were, then $K \star (\neg\varepsilon \vee \neg\gamma)$ could not be successful, by ($\star$5u), hence $K \star (\neg\varepsilon \vee \neg\gamma) = K \star \bot$ by ($\star$2c), and so $\varepsilon \in K \star \bot$, contradicting our choice of ε. Since ε is not in $K \star (\neg\varepsilon \vee \neg\gamma)$, we conclude that $K \star (\neg\varepsilon \vee \neg\gamma)$ is not successful after all, by our choice of ε. By ($\star$2c), then, $K \star (\neg\varepsilon \vee \neg\gamma) = K \star \bot$. But then γ is in $K \star \bot$, and thus it is in $K \star \delta$, too, by Lemma 5(ii).

Secondly, assume that $\varepsilon \wedge (\delta \to \gamma) \in K \star (\delta \vee \neg\varepsilon)$. Since ε is in $K \star (\delta \vee \neg\varepsilon)$, we know that $K \star (\delta \vee \neg\varepsilon)$ is successful. Hence $\neg(\delta \vee \varepsilon) \notin K \star (\delta \vee \neg\varepsilon)$, by ($\star$5u). Therefore, by ($\star$8), $K \star (\delta \vee \neg\varepsilon) \subseteq K \star ((\delta \vee \neg\varepsilon) \wedge (\delta \vee \varepsilon)) = K \star \delta$, by ($\star$6). Now we know that all of $\delta \vee \neg\varepsilon$, ε and $\delta \to \gamma$ are in $K \star (\delta \vee \neg\varepsilon)$. Hence, by ($\star$1), γ is in $K \star (\delta \vee \neg\varepsilon)$, too. And since $K \star (\delta \vee \neg\varepsilon) \subseteq K \star \delta$, we finally get that γ is in $K \star \delta$, as desired.

We have now shown that whenever γ is in $K \star' \delta$, it also holds that γ is in $K \star \delta$.

For the converse, suppose that γ is in $K \star \delta$. Suppose further that $\gamma \notin K \star (\neg\varepsilon \vee \neg\gamma)$. By Lemma 5(ii), we conclude that γ is not in $K \star \bot$. We want to show that $\varepsilon \wedge (\delta \to \gamma) \in K \star (\delta \vee \neg\varepsilon)$ and that $\neg\delta \notin K \star \bot$.

Firstly, we conclude from $\gamma \in K \star \delta$ but $\gamma \notin K \star \bot$, that $K \star \delta \neq K \star \bot$ and $K \star \delta$ is successful (use ($\star$2c)), i.e., $\delta \in K \star \delta$. If $\neg\delta$ were in $K \star \bot$, it would also be in $K \star \delta$, by Lemma 5(ii), so $K \star \delta$ would be inconsistent, contradicting ($\star$5u). So $\neg\delta \notin K \star \bot$.

Secondly, we have that $\gamma \in K \star \delta = K \star ((\delta \vee \neg\varepsilon) \wedge \delta) \subseteq (K \star (\delta \vee \neg\varepsilon)) + \delta$, by ($\star$6) and ($\star$7). So we get from ($\star$1) that $\delta \to \gamma$ is in $K \star (\delta \vee \neg\varepsilon)$. It remains to show that ε is in $K \star (\delta \vee \neg\varepsilon)$, too. Since $K \star \delta$ is successful, we get, Lemma 5(i), that $K \star (\delta \vee \neg\varepsilon)$ is successful, too. But by our choice of ε, this means that ε is in $K \star (\delta \vee \neg\varepsilon)$.

Thus $K \star' \delta = K \star \delta$ for all δ, as desired.

Finally, regarding iterations, the case is similar to irrevocable revisions: A comparative inspection of ($\star$it) and (IT1) shows that $(K \star \alpha) \star \beta = (K \circ_\varepsilon \alpha) \circ_\varepsilon \beta$ for all α and β. ■