Revision by comparison

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Abstract

Since the early 1980s, logical theories of belief revision have offered formal methods for the transformation of knowledge bases or "corpora" of data and beliefs. Early models have dealt with unconditional acceptance and integration of potentially belief-contravening pieces of information into the existing corpus. More recently, models of "non-prioritized" revision were proposed that allow the agent rationally to refuse to accept the new information. This paper introduces a refined method for changing beliefs by specifying constraints on the relative plausibility of propositions. Like the earlier belief revision models, the method proposed is a qualitative one, in the sense that no numbers are needed in order to specify the posterior plausibility of the new information. We use reference beliefs in order to determine the degree of entrenchment of the newly accepted piece of information. We provide two kinds of semantics for this idea, give a logical characterization of the new model, study its relation with other operations of belief revision and contraction, and discuss its intuitive strengths and weaknesses.

Key words: belief revision, theory change, sphere semantics, epistemic entrenchment, AGM approach, iterated revision, non-prioritized revision, severe withdrawal, irrevocable revision, irrefutable revision

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1 Introduction: A new model for plausible reasoning without numbers

Up to now, belief change theories have been facing a dilemma. In *qualitative theories* of the AGM variety (Alchourrón, Gärdenfors and Makinson 1985, Gärdenfors 1988, Gärdenfors and Rott 1995) agents accept new incoming information without further qualification. Usually, there is no deliberation as to whether to accept the input or not. This limitation has been addressed in recent research on "non-prioritized" belief change (Hansson ed. 1997). But *if* an agent accepts a piece of information, no degree of "certainty", "security" or "plausibility" is attached to it – belief is always just *plain* belief. So these models are all rather crude.

In *quantitative theories*, on the other hand, agents do not, or in any case do not have to, accept new information *simpliciter*. In probability theory, plain conditionalisation (which assigns probability 1 to new evidence) can be replaced by Jeffrey conditionalisation with a certain chosen parameter (which specifies the posterior probability of the new evidence at any arbitrary value between 0 and 1, see Jeffrey 1965). In Spohn's (1988) model of ordinal conditional functions (now often called "ranking functions", following a suggestion of Goldszmidt and Pearl's 1992) beliefs are fully accepted, but they can – and must – be accommodated with a chosen value of security (i.e., the level of implausibility of their negations). Obviously, this additional expressive power of quantitative approaches is an advantage. It cannot be denied, however, that it has its price: The meaning of the numbers employed is not clear. While in probability theory numerical values are often thought to be explicable in terms of betting quotients, the precise significance of the numbers in ranking functions remains largely unelucidated so far.[2] But also regarding probabilities, the meaning of numbers can be disputed. Even amongst agents who are ideally "rational," room should be made for characters that are risk-seeking and for those that are risk-averse; there may be effects of regret (Allais' paradox), there may be effects of resource-bounded reasoning capabilities, etc. Real as opposed to ideally rational agents are notoriously bad at reasoning with probabilities.[3] Working with numbers helps us to come to terms with many non-trivial modelling tasks, but it is based on parameters the meaning of which is not fully understood, and it puts very heavy demands on the agents' reasoning capabilities.

The present paper tries to show a way out of the dilemma. On the one hand, we fully agree that there should be ways of specifying the strengths of new beliefs. On the other hand, we are also impressed by the arguments that speak

[2] There are actually a few attempts to provide such an elucidation, see Levi (1996, Chapter 8) and Spohn (1999).

[3] See, for instance, Tversky and Kahneman (1983), Piattelli-Palmarini (1994) and Gigerenzer (2002).

against the use of numbers: their lack of clear meaning and intractability. We want to be able to express more than just injunctions like

$$\text{Accept } \beta$$

à la AGM. At the same time we want to avoid numerical formulations of commands like

$$\text{Accept } \beta \text{ with probability } p.$$

à la Jeffrey,[4] or

$$\text{Accept } \beta \text{ with degree of plausibility } k.$$

à la Spohn.[5] Our idea is that it is a manageable task not only for ideal, but also for real agents to perform *revisions by comparison*. Such revisions do not employ *numbers* as indices for beliefs; it is other *beliefs* that serve as points of reference. The relevant injunction is expressed by sentences like

$$\text{Accept } \beta \text{ with a degree of plausibility that at least equals that of } \alpha.$$

Why should one want to have an operation modelling this kind o belief change? Suppose that a colleague tells us "Graham is negotiating with XXXL Company." Should we accept this piece of information? AGM say yes, Spohn says yes, too, and asks us to fix the parameters; Jeffrey asks us to fix the parameters, but the notion of acceptance does not really fit into his probabilistic framework. In real life, it may be hard to come up with "the right" numbers. What we can do, however, is ask our friend how sure he is of this piece of information. He might say that it is at least as well-confirmed as the claim that Graham has got an offer from Medium Size Company. Another way of obtaining the same sort of comparative information is by juxtaposing our assessments of the reliability of sources. If our colleague is at least as trustworthy and well-informed as another person who has testified to the truth about the offer from Medium Size the other day, we will wish to accept the information about Graham's negotiation at a level of certainty that is at least as high as our degree of belief in the offer from Medium Size. If the latter belief, however, is not firm enough to overcome our doubts about Graham's negotiations with XXXL, we are likely to end up doubting both pieces of information.

The basic idea of this paper is to model degrees of acceptance by what is known in the literature as *epistemic entrenchment* (Gärdenfors 1988, Gärdenfors and Makinson 1988, Rott 2001). In the intended paradigm cases the point of reference will be a sentence α that is not only believed, but is believed with sufficiently high entrenchment – where "sufficiently high" means that α itself will continue to be believed after the belief change has taken place. However,

[4] There is no good concept of plain acceptance in probabilistic contexts. Perhaps a better formulation would therefore be "The probability of β should be set to p."
[5] For general discussions on various measures of plausibility, see Weydert (1994), Schlechta (1997), Freund (1998) and Friedman and Halpern (2001).

we intend to conceive our studies so broadly that this intuitive idea is not a prerequisite for the viability of our formal approach. It will frequently happen that accepting β (to any degree) somehow affects the degree of acceptance of the reference sentence α. This is why we are immediately confronted with difficult problems in the dynamics of epistemic evaluation when performing revision by comparison.[6]

Let us briefly summarize what we have said so far with the help of symbols. First, all the established alternatives to qualitative belief revision draw heavily on numbers. There are different ways of representing (certain aspects of) belief states: Belief sets K (sets of sentences closed under some standard background logic, as in the AGM approach), probability functions P and Spohnian ranking functions κ. While AGM have an operation like $K * \beta$, Jeffrey probabilists have an operation like $P + (\beta, p)$ and Spohn has an operation like $\kappa * (\beta, k)$. Here the letters p and k stand for real numbers in the interval $[0, 1]$ and ordinal numbers respectively. The *expansion* symbol $+$ indicates that both the new information β and its denial are supposed to be "consistent" with P,[7] a requirement that is not necessary in the case of *revision* functions which we denote by the symbol $*$.

In contrast to the numerical approaches, our model avoids reference to poorly interpreted numbers and instead focuses on an operation that can be represented by a formula of the format $K * (\alpha \leq \beta)$. This notation indicates that the revision is actually effected by a statement that compares the degrees of acceptance (more precisely: the entrenchments) of α and β. But there is a crucial asymmetry in the roles of α and β: α is the *reference sentence*, β is the *input sentence*. In order to make this more perspicuous, we will use the notation $K \circ_\alpha \beta$ for the revision-by-comparison operation. The belief state $K \circ_\alpha \beta$ is the one that results after the agent has obeyed the instruction "See to it that the entrenchment of β is at least as firm as the entrenchment of α."

In contrast to previous studies in belief change of the AGM variety, we will study a simple model of implementing revision-by-comparison in terms of a function $K \circ_\alpha \beta$ that takes *two* sentences α and β as arguments. We use the letter K for the notion of a belief set, i.e., a set of sentences closed under logical consequences.[8] As is common in belief revision, the belief set is supposed to come equipped with a belief-revision guiding structure, e.g., with an entrench-

[6] One of the advantages of numbers as points of reference is that they stay stable when other things are moved. Beliefs do not in general share this stately property when other beliefs are changed.

[7] Jeffrey-conditionalization only works if both $P(\beta)$ and $P(\neg\beta)$ are non-zero.

[8] Throughout this paper, we presume without further notice that the background logic is Tarskian, i.e., satisfies Inclusion, Idempotence and Monotony, and that it is compact, supraclassical and satisfies the deduction theorem. In our notation, we switch between the consequence relation $\vdash$ and the consequence operation Cn as we find it convenient.

ment ordering $\leq$. Our notation is not meant to suggest that belief sets are full representations of belief states. Rather, the prior and posterior belief states in a transition effected by a revision can be identified with the belief-revision guiding structures themselves. The function $\circ$ takes a belief state and two sentences as arguments and returns a belief state. Having said this, we believe that our notation which is chosen with a view to maintaining continuity with the AGM tradition will not cause any confusion about the matter.

For the most part of this paper (except in Section 9), we will take the initial belief state (and thus the initial belief set) to be revised as contextually given and fixed. Still, we have to deal with a *dyadic* function taking two sentences as arguments. This makes the axiomatization of the model we propose somewhat more complicated and difficult than AGM-style axiomatizations. Ultimately, however, the dyadic function will turn out to be reducible, through a non-trivial case distinction, to a combination of monadic belief change operations taking only one sentence as an argument.

We have mentioned that the reference sentence is usually supposed to be sufficiently highly entrenched as compared to the (negation of the) input sentence. This is the paradigm case of application of revision-by-comparison, and in this case our operation will behave like an ordinary kind of AGM *revision* by the input sentence. More precisely, the operation we are going to study is an extended kind of AGM revision function that is suitable for repeated ("iterated") use. That iterations are possible is due to the fact that it is not belief sets, but belief-revision guiding structures that are taken to represent doxastic states, and that consequently, structures like entrenchment relations are the primary objects that get changed by belief change functions.[9]

If, however, the reference sentence is too weakly entrenched relative to the (negation of the) input sentence, the attempted revision will fail and end up with a *contraction* of the belief set, more precisely a *severe withdrawal* of the reference sentence (Pagnucco and Rott 1999). This is the first limiting case for which our operation reduces to an operation that is well-known from previous literature. If the agent wants to withdraw (severely) a sentence α from his belief set, this can be achieved by making the falsity, $\perp$, at least as entrenched as α (or equivalently, by making $\neg\alpha$ at least as entrenched as α).

Roughly speaking, the input sentence tends to initiate a revision, the reference sentence is prone to be withdrawn. So the two arguments of the dyadic function are related to the two main operations of AGM-style belief change operations.

A second, quite different limiting case is that of *irrevocable belief change*

[9] Iterations allow a convenient definition of a symmetrical version of revision by comparison. If an agent wants to implement an operation that can be read as the instruction "See to it that the entrenchment of β is *exactly* as firm as the entrenchment of α", she can use the operation $(K \circ_\alpha \beta) \circ_\beta \alpha$. Compare Theorem 14, condition (IT5) below.

(Segerberg 1998). This is the case when the agent makes β maximally entrenched, i.e., as well-entrenched as a tautology. In ordinary language, this happens if the agent comes to consider the truth of the input sentence to be "as sure as fate" or "as clear as day". No doubt is left that β is true, and this is settled once and for all, β is never going to be lost.

Both limiting cases concern operations where the dyadic function is reduced to a monadic function (by taking $\bot$ as the input sentence and $\top$ as the reference sentence respectively). A further interesting operation is the monadic function $* = \circ_\varepsilon$ with some fixed reference sentence ε. Because the agent can never ever recover $\neg\beta$, once such an operation $K * \beta$ has successfully taken place, we call these operations *irrefutable revisions*. Irrefutable belief revision is rather close to belief revision as axiomatized by Alchourrón, Gärdenfors and Makinson (1985). The most striking difference is that it is not necessarily "successful" (it is non-prioritized in the sense of Hansson ed. 1997), but that it invariably respects consistency (even in the case of an inconsistent input sentence). Thus it is even closer to "credibility-limited belief revision" in the sense of Hansson *et al.* (2001).

A detailed study and self-contained axiomatizations of the three special cases of severe withdrawal, irrevocable belief revision and irrefutable belief revision is deferred to a companion paper to the present one (Rott 200∗).

2 Semantics for revision by comparison

Our semantics is motivated by the well-known systems-of-spheres modelling in the style of Grove (1988) (which is in turn inspired by Lewis 1973). Formally, a sphere S is simply a set of possible worlds. A *system of spheres*, or *SOS*, is a non-empty set $\$$ of nested spheres (in the sense of set inclusion) with $\bigcap \$$ as the smallest sphere in $\$$. The belief set $K = K_\$$ associated with a system of spheres $\$$ is the set of sentences that are true throughout the innermost sphere of $\$$. Intuitively, the innermost sphere contains the most plausible worlds, the second innermost sphere (without the innermost one) the second most plausible worlds, and so on.

Let $[\alpha]$ be the set of worlds that satisfy α. We call a sentence α *universal* with respect to the system of spheres $\$$ if $\bigcup \$ \subseteq [\alpha]$. The set W of all possible worlds is not necessarily in $\$$. If it is, we call $\$$ itself universal.

Notice that here and in the following, belief sets are only certain abstractions from belief states, while belief states are represented by some fairly complicated structure (a system of spheres). As is common in all approaches to iterated belief change, belief states, and not just belief sets, are subject to change in the process of belief revision.

The main idea now, expressed in the possible worlds reading, is that the new

Fig. 1. The intended Case

system of spheres generated by the change $\circ_\alpha\beta$ is obtained by shifting the $\neg\beta$-worlds that are closer to the original belief set (according to the prior system of spheres) than the closest $\neg\alpha$-worlds outwards up to the ring where the closest $\neg\alpha$-worlds reside. This yields a new system of spheres that represents the posterior belief state. The intended paradigm case is depicted in Fig. 1. The posterior system of spheres which results from applying the operation $\circ_\alpha\beta$ to \$ will be called \$′.

What exactly does the revision-by-comparison operation $\circ_\alpha\beta$ do? For sentences α that are not universal with respect to \$, we put

(Def \$′ from \$):

$$\boxed{\$' = (\{S \cap [\beta] : S \in \$ \text{ and } S \subseteq [\alpha]\} \cup \{S : S \in \$ \text{ and } S \not\subseteq [\alpha]\}) - \{\emptyset\}}$$

If α is universal, but $\neg\beta$ is not universal w.r.t. \$, we can put $\$' = \{S \cap [\beta] : S \in \$\} - \{\emptyset\}$. The case where both α and $\neg\beta$ are universal w.r.t. \$ is a very special case which we define to yield $\$' = \{\emptyset\}$. We will return to this case later.

Clearly, this gives us another system of spheres. It is important that unless both α and $\neg\beta$ are universal w.r.t. \$, the posterior SOS does not have the empty sphere as the innermost sphere – which means that the belief set $K' = K_{\$'}$ associated with \$′ is a *consistent* theory. This makes the method suitable as a method for revisions by sentences that are incompatible with the prior belief set. Notice that it is not necessary that the sphere determining the "firmness" with which β is accepted is characterized with the help of some sentence α. We could alternatively regard this model as a *sphere-indexed* revision operation, indexed by an arbitrary threshold sphere S^*, but we shall not further follow this line of thought in the present paper.[10]

As a slightly different model, one can use orderings of worlds. Assume that a complete and transitive ordering $\preceq$ over the worlds is given. The correspondence between SOS's and orderings of worlds modelling is as follows. The innermost sphere in \$ consists precisely of the $\preceq$-minimal worlds, and each sphere in \$ is of the form $\{w : w \preceq w' \text{ for some fixed } w'\}$.[11] Conversely, $w \preceq w'$ if and only if each sphere in \$ that contains w' also contains w.

The posterior ordering of worlds that results from applying the operation $\circ_\alpha\beta$ to $\preceq$ will be called $\preceq'$.

Now choose some $\preceq$-minimal world that satisfies $\neg\alpha$, and call it $w_{\neg\alpha}$. We assume that such a world exists provided that α is not logically valid (this corresponds to Lewis's "limit assumption"). If α is logically valid, i.e., if there is no world satisfying $\neg\alpha$, then we fix by convention that $w_{\neg\alpha}$ is chosen as an arbitrary $\preceq$-maximal world (we also assume that there exist $\preceq$-maximal worlds).

Now we can define the posterior ordering $\preceq'$.

(Def $\preceq'$ from $\preceq$):

$$\boxed{w \preceq' w' \text{ iff } \begin{cases} w \preceq w' & \text{and } w_{\neg\alpha} \preceq w' \qquad \textit{or} \\ w \preceq w' & \text{and } w \text{ and } w' \text{ are both } \beta\text{-worlds} \quad \textit{or} \\ w \preceq w_{\neg\alpha} & \text{and } w' \text{ is a } \neg\beta\text{-world} \end{cases}}$$

Observation 1 The ordering $\preceq'$ thus defined is a complete and transitive ordering of worlds.

The world determining the "firmness" with which β is accepted need not be identified with the help of some sentence α. We could alternatively employ this model as a *world-indexed* revision operation, indexed by an arbitrary threshold world w^*, but we refrain from doing so in this paper.

It is easy to check that the relation $\preceq'$ just defined has the following properties:

- For all worlds w, $w \preceq w_{\neg\alpha}$ if and only if $w \preceq' w_{\neg\alpha}$
- For all β-worlds w, $w_{\neg\alpha} \preceq w$ if and only if $w_{\neg\alpha} \preceq' w$
- For all $\neg\beta$-worlds w, $w_{\neg\alpha} \preceq' w$

It will become clear later in this paper that these properties are desirable.

[10] For a similar approach using numbers, see Williams (1995).

[11] Alternatively, one could take this system of spheres minus the set W of all possible worlds. We neglect this ambiguity, although it would lead to different belief sets in certain limiting cases.

3 Revision by comparison in terms of epistemic entrenchment

Systems of spheres are a way of representing how well the agent's beliefs are entrenched. Intuitively, the idea is that a belief β is more entrenched than a belief α, if all the $\neg\beta$-worlds are "farther out" than some of the $\neg\alpha$-worlds. More precisely, an entrenchment relation $\leq$ can be thought of as generated by a system of spheres $\$$ by the following definition:

(Def $\leq$ from $\$$):

$$\boxed{\alpha \leq \beta \quad \text{if and only if} \quad \text{for all } S \in \$,\ \text{if } S \subseteq [\alpha],\ \text{then } S \subseteq [\beta]}$$

Entrenchment relations need not be thought of as deriving from a systems of spheres. They can be studied in their own right, independently from any possible worlds representation. In fact, while we use the possible worlds representation as a motivation for the approach we are advocating, we will turn on entrenchment relations in most of the formal developments to come. The relation between possible worlds and entrenchments was studied by Grove (1988), Gärdenfors (1988), Peppas and Williams (1995) and Pagnucco and Rott (1999), among others.[12]

In order to make entrenchments fit the needs of revision by comparison we slightly generalize the definition of epistemic entrenchment as introduced by Gärdenfors and Makinson (1988).

Let $\mathcal{L}$ be the set of all sentences in a propositional language. An relation $\leq$ over $\mathcal{L}$ is called a relation of *epistemic entrenchment*, if it satisfies the following conditions:

(E1) If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$ (Transitivity)

(E2) If $\alpha \vdash \beta$, then $\alpha \leq \beta$ (Dominance)

(E3) $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$ (Conjunctiveness)

For the following it is important to note that as compared to Gärdenfors and Makinson, we do not include an additional 'Maximality condition' according to which $\beta \leq \alpha$ for every $\beta \in \mathcal{L}$ can hold only if α is a logical truth (this requirement corresponds to the universality of $\$$).[13] For reasons that will

[12] In the converse direction, one can construct a system of spheres from a given entrenchment relation by defining $S \in \$$ if and only if there is an α such that $S = \{w :\ w \text{ satisfies every } \beta \text{ with } \alpha < \beta\}$ (and $S \neq W$). See Pagnucco and Rott (1999, p. 528). Except for some limiting cases, the two constructions dovetail nicely.

[13] We will comment on a missing 'Minimality condition' presently. In later work, Gärdenfors and Makinson (1994) used (E1)–(E3) for defining *expectation orderings* to be applied in nonmonotonic reasoning. Similar relations without Maximality are also prominent in the possibilistic framework, see e.g. Dubois and Prade's (1991) *necessity measures*.

become clear later on, we call sentences that are no less entrenched than $\top$ *irrevocable sentences*. An entrenchment relation $\leq$ is called *trivial* if $\top \leq \bot$; this holds just in case $\leq\, = \mathcal{L} \times \mathcal{L}$.[14]

It follows from (E1) – (E3) that an epistemic entrenchment is a complete pre-order over $\mathcal{L}$. While systems of spheres \$ can be seen as essentially ordering worlds *outside* the set of worlds satisfying all current beliefs, entrenchment orderings essentially order the sentences *within* the set of current beliefs.

We may now define the *belief set* $K = K_{\leq}$ associated with an entrenchment relation $\leq$ as the set of all sentences that are non-minimal under $\leq$:

(Def $K_{\leq}$):

$$\boxed{K_{\leq} \;=\; \begin{cases} \{\alpha : \bot < \alpha\} & \text{if } \bot < \top \\ K_{\bot} & \text{in the trivial case } \top \leq \bot \end{cases}}$$

Here $K_{\bot} = \mathcal{L}$ denotes the inconsistent or "absurd" belief set.

Notice that like in the SOS model, belief sets are only abstractions from belief states, while belief states themselves are now represented by the structure of the entrenchment relation. Due to the properties of entrenchment relations, each such belief set $K_{\leq}$ is logically closed. It is easy to check that $K_{\leq} = K_{\$}$ if $\leq$ derives from a system of spheres \$ by (Def $\leq$ from \$).

Definition (Def $K_{\leq}$) validates the Gärdenfors-Makinson axiom of 'Minimality' for entrenchment conditions, according to which for a consistent belief set K, all and only non-elements of K are minimally entrenched with respect to $\leq$. This is the idea on which the following considerations are based. But we also want to draw the reader's attention to the fact that the set of current beliefs might in principle also be defined to be the set of all sentences 'above a certain level' with respect to $\leq$.

In order to make things iterable, we need to revise not only the belief set but also the entrenchment relation from which the belief set is derived. Let $\leq$ be the prior entrenchment relation determining the prior belief set K. Now assume that the agent accepts β at least as certainly as α, and let $\leq'$ be the posterior entrenchment relation determining the posterior belief set $K \circ_{\alpha} \beta$. Then $\leq'$ is defined by

(Def $\leq'$ from $\leq$):

$$\boxed{\gamma \leq' \delta \;\text{ iff }\; \begin{cases} \alpha \wedge (\beta \rightarrow \gamma) \leq (\beta \rightarrow \delta) & \text{, if } \gamma \leq \alpha \\ \gamma \leq \delta & \text{, otherwise} \end{cases}}$$

[14] The trivial entrenchment relation may be regarded as corresponding to the sphere system $\$ = \{\emptyset\}$ and to the constant revision-by-comparison function $K \circ_{\alpha} \beta = K_{\bot}$.

The most distinctive part of this central definition is that the new relation between γ and δ is determined by the relation between the conditionals $\beta \to \gamma$ and $\beta \to \delta$, where the antecedent β is the belief to be accepted. This condition is well-known from irrevocable belief revision (Rott 1991a, Fermé 2000). However, this part of the definition is relevant only as long as certain sentences, namely γ and $\beta \to \delta$, are less entrenched than α.

Definition (Def $\leq'$ from $\leq$) reflects our intended possible worlds reading according to which the system of spheres generated by the change '$\circ_\alpha \beta$' is obtained by shifting the $\neg\beta$-worlds that are more 'central' or 'plausible' than the most central or plausible $\neg\alpha$-worlds out to the ring where the most plausible $\neg\alpha$-worlds reside. We first show that (Def $\leq'$ from $\leq$) indeed captures the semantical recipe to construct revisions-by-comparison in terms of entrenchment.

Theorem 2 *Let \$ be a system of spheres and \$′ be changed through revision by comparison according to (Def \$′ from \$). Let $\leq$ and $\leq'$ be the entrenchment relations derived from \$ and \$′ by (Def $\leq$ from \$) respectively. Then $\leq'$ can be obtained from $\leq$ by (Def $\leq'$ from $\leq$).*

The proof for this theorem, as well as for the observations and theorems to follow, is given in Appendix 1 of this paper.

(Def $\leq'$ from $\leq$) is our official definition on which the following work is based. Like any other definition of posterior entrenchments in terms of prior entrenchments, this definition provides for *iterated* belief change. But we will not address the problem of iteration before Section 9.

Observation 3 *If $\leq$ satisfies transitivity (E1), dominance (E2) and conjunctiveness (E3), then $\leq'$ does so, too.*

Like the prior entrenchment relation $\leq$, the posterior entrenchment relation $\leq'$ is a complete pre-order of all the sentences of the language.

From the revised relation $\leq'$ we can easily define the revised belief set by putting $K \circ_\alpha \beta = K_{\leq'}$. Notice that we cannot take it for granted here that the posterior entrenchment relation $\leq'$ satisfies Non-triviality (the condition $\bot < \top$ used in Rott 1992). In fact, if both $\top \leq \alpha$ and $\top \leq \neg\beta$ (i.e., if $\top \leq \alpha \wedge \neg\beta$), then we find that the posterior entrenchment relation $\leq'$ gives $\gamma \leq' \delta$ for arbitrary γ and δ. This can be interpreted as a kind of *epistemic collapse* and it is most natural to take the corresponding belief set to be the inconsistent theory $K_\bot$.[15] In AGM theory, any inconsistent input leads to an epistemic collapse. In the present model, the epistemic collapse is a very exceptional case that happens only if an agent is told to promote a sentence *the negation of which* is irrevocable to the entrenchment level of an irrevocable sentence.

[15] The set $\{\gamma : \bot <' \gamma\}$ is empty in this case, and thus is not a logically closed set of sentences at all.

After a few simplifying steps, we get the following characterization of the posterior belief set in terms of the prior entrenchment relation:

(Def $\circ$ from $\leq$):

$$\boxed{\gamma \in K \circ_\alpha \beta \;\text{ iff }\; \begin{cases} \neg\beta < \alpha \wedge (\beta \rightarrow \gamma) & \text{or} \\ \alpha < \gamma & \text{or} \\ \top \leq \alpha \wedge \neg\beta & \end{cases}}$$

The most distinctive part of this definition is that γ is in $K \circ_\alpha \beta$ if $\neg\beta < \beta \rightarrow \gamma$. This is typical for AGM-style acceptance of γ in a theory revised by β (Lindström and Rabinowicz 1991, p. 97, and Rott 1991b, p. 144). However, this part of the definition is relevant only as long as certain sentences, namely $\neg\beta$ and γ, are less entrenched than α.

Taking the revision-by-comparison operation $\circ$ as primitive, we can define the prior belief set by suitably choosing reference and input sentences in such a way that the set of original beliefs is not changed at all. Later we shall see that it is expedient to define the belief set $K = K_\circ$ associated with a revision-by-comparison operation $\circ$ as $K \circ_\perp \perp$.[16]

The crucial question now is: How can the definition of the new entrenchment relation $\leq'$ in terms of the old entrenchment relation $\leq$ be converted into an axiomatic characterization of – iterated – revisions by comparison

(a) for belief change with a fixed reference sentence α
(b) for belief change with variable reference sentences

We will tackle the more general problem (b) first and consider the special case (a) under the name "irrefutable belief revision" in the companion paper (Rott 200$*$).

Before doing this, however, we have a closer look at how the operation of revision-by-comparison works.

4 What the operation does

Let again $\leq'$ be the posterior entrenchment relation that determines the revised belief set $K \circ_\alpha \beta$. In order to get a feeling for what the operation $\circ_\alpha\beta$ actually does, we first have a look at the relative posterior entrenchments of α, β, and arbitrary other sentences γ.

[16] This definition is not circular, because, as pointed out above, $\circ$ really operates on belief *states* (like systems of spheres or entrenchment relations), and not on belief *sets*.

case	entrenchment	kind of change
The intended case: AGM-style revision	$\beta < \alpha$ and $\neg\beta < \alpha$	$K \circ_\alpha \beta = K * \beta$
The vacuous case: No change	$\alpha \leq \beta$	$K \circ_\alpha \beta = K$
The unsuccessful case: Withdrawal	$\alpha \leq \neg\beta$ and $\alpha < \top$	$K \circ_\alpha \beta = K \dot{-} \alpha$
The epistemic collapse:	$\top \leq \alpha$ and $\top \leq \neg\beta$	$K \circ_\alpha \beta = K_\perp$

Fig. 2. The four basic cases

Remember that the goal of the operation $\circ_\alpha\beta$ is to make β at least as entrenched as α. It is easy to show that the definition of $\leq'$ indeed guarantees that always

$$\alpha \leq' \beta$$

but that it does not do any extra lifting. β ends up being more entrenched than α only if it was more entrenched than α to begin with:

$$\alpha <' \beta \text{ iff } \alpha < \beta$$

One can in fact show that for an arbitrary sentence γ

$$\alpha <' \gamma \text{ iff } \alpha < \gamma$$

On the other hand, for an arbitrary sentence γ

$$\gamma \leq' \beta \quad \text{iff} \quad \gamma \leq \alpha \text{ or } \gamma \leq \beta$$

Clearly, the entrenchment relation between two sentences that are at least as entrenched as α in the prior belief state will not be affected by the operation $\circ_\alpha\beta$:

$$\text{If } \alpha \leq \gamma \text{ and } \alpha \leq \delta \text{ then: } \gamma \leq' \delta \text{ iff } \gamma \leq \delta$$

In the rest of this section, we distinguish and discuss four basic cases (see Fig. 2). These cases exhaust the space of logical possibilities. They are not disjoint, but where more than one of them applies, they yield identical results.

4.1 The intended paradigm case

The intended application of revision by comparison is the case when the reference sentence α is more entrenched than both the input sentence β and its negation $\neg\beta$, i.e., when $\beta < \alpha$ and $\neg\beta < \alpha$.

In this case, no simplification of (Def $\leq'$ from $\leq$) is possible, but we can telescope the characterization of the posterior belief set (because in this case $\alpha < \gamma$ implies $\neg\beta < \beta \rightarrow \gamma$):

$$K \circ_\alpha \beta = \{\gamma : \neg\beta < \beta \rightarrow \gamma\} = K * \beta$$

where $*$ is the ordinary AGM revision function based on entrenchment, generalized to a context without the Maximality condition for $\leq$. Interpretation: In the ordinary case, where α is selected as a comparatively well-entrenched belief, the new operation coincides with an AGM revision – as far as one-step revisions of belief sets are concerned.

A *special situation* within the paradigm case arises when α is indeed maximally entrenched. If $\top \leq \alpha$, then $\circ_\alpha\beta$ leads to the following change, according to (Def $\leq'$ from $\leq$):

$$\gamma \leq' \delta \ \text{ iff } \ (\beta \rightarrow \gamma) \leq (\beta \rightarrow \delta)$$

Let us briefly consider the special case where $\top \leq \alpha$. Note that in this case, we *always* have $K \circ_\alpha \beta = K * \beta$, where $*$ is entrenchment-based revision in the style of Gärdenfors and Makinson. Correspondingly, we do not have the AGM postulate of Consistency ($K{*}5$). In our account, the condition that the posterior belief set is inconsistent only if $\neg\beta$ is a logical truth is replaced by the condition that the posterior belief set is inconsistent only if $\neg\beta$ is at least as entrenched as $\top$ (that $\neg\beta$ be "irrevocable"). More details about the case $\top \leq \alpha$ are discussed in the companion paper to the present one under the heading *irrevocable belief change* (after Segerberg 1998).

4.2 The vacuous case

The vacuous case is the one in which $\alpha \leq \beta$ holds anyway, so that the goal state of the operation $\circ_\alpha\beta$ – viz., that β be at least as entrenched as α – is present to begin with (for the sphere representation, see Fig. 3). In this case, nothing changes:

Lemma 4 *If $\alpha \leq \beta$, then*

$$\gamma \leq' \delta \ \ \textit{iff} \ \ \gamma \leq \delta$$

And evidently, for the posterior belief set, we get

$$K \circ_\alpha \beta = K$$

Interpretation: If β is already at least as entrenched as α, we don't have to do anything. In effect, this case only differs from the case of section 4.1 if β is

Fig. 3. The vacuous case

Fig. 4. The unsuccessful case

not in the belief set K (and thus, by $\alpha \leq \beta$, α is not in K either). If β is in the belief set, then $K \circ_\alpha \beta = K = K * \beta$, where $*$ is as in AGM, so the case of the previous section subsumes the present one. If β is not in K, however, β is not in $K \circ_\alpha \beta = K$, in contrast to $K * \beta$ which includes β. It turns out that the operation $\circ_\alpha \beta$ can be *unsuccessful* in the sense that β does not get accepted at all. For this reason we sometimes have $K + \beta \not\subseteq K \circ_\alpha \beta$, even if $\neg\beta$ is not in K.

4.3 The unsuccessful case

The unsuccessful case is the one in which $\alpha \leq \neg\beta$ and $\alpha < \top$ (see Fig. 4). Because then $\alpha \wedge (\beta \rightarrow \gamma) \leq \alpha \leq \neg\beta \leq (\beta \rightarrow \delta)$, by (E1) and (E2), we can simplify (Def $\leq'$ from $\leq$) and get that

$$\gamma \leq' \delta \;\text{ iff }\; \gamma \leq \alpha \text{ or } \gamma \leq \delta$$

For the posterior belief set, we obtain

$$K \circ_\alpha \beta = K \ddot{-} \alpha$$

where $\ddot{-}$ is the *severe withdrawal* function based on $\leq$ (Pagnucco and Rott 1999), i.e.,

$$K \ddot{-} \alpha = \{\gamma : \alpha < \gamma\}$$

Interpretation: If $\neg\beta$ is at least as entrenched as α, then the instruction to see to it that the entrenchment of β is at least as firm as the entrenchment of α does not suffice to render β accepted, but rather renders α rejected. The reference sentence just hasn't been strong enough to achieve acceptance of the input sentence. We will presently show that β fails to be accepted in $K \circ_\alpha \beta$ if and only if $\alpha \leq \neg\beta$, and that in this case α is rejected, too. If the agent intends to guarantee that the revision-by-comparison is “unsuccessful” and, thereby, to eliminate the reference sentence, he may choose the ‘input sentence’ $\perp$ – which will not be put into K.

Severe withdrawal is a limiting case of revision by comparison. We have already mentioned another limiting case, that of irrevocable revision. The latter method is basically an extension of AGM revision by an extra condition for iterated revisions. The former method is an alternative to AGM contraction which has not yet been studied as a method for iterated belief change, but its generalization to iterated functions is fairly straightforward. Both limiting cases are studied in Rott (200$*$).

4.4 The epistemic collapse

The epistemic collapse is a consequence of an instruction as absurd as the instruction to accept a contradiction with the strength of a tautology. More precisely, the curious situation arises when $\top \leq \alpha$ and $\top \leq \neg\beta$ (see Fig. 5). The definition (Def $\leq'$ from $\leq$) then reduces to $\gamma \leq' \delta$, for every pair of sentences γ and δ. An equivalent way of expressing the same is just to say that $\top \leq \perp$, from which it follows that $\leq \,=\, \mathcal{L} \times \mathcal{L}$. (Def $K_\leq$) tells us that in this case K' is inconsistent. With revision by comparison alone, we can never recover from a state of epistemic collapse. Once caught in an entrenchment relation with $\top \leq \perp$, no operation of the form ‘$\circ_\alpha\beta$’ will ever lead the agent out of his predicament.

Fig. 5. The epistemic collapse

5 Postulates for one-step revision by comparison

We are now going to present a brief and economical axiomatization, which is unfortunately not very intuitive. Much of it relates a general revision-by-comparison $K \circ_\alpha \beta$ to a revision-by-comparison of the more specific form $K \circ_\gamma \bot$. As already mentioned, such operations are essentially operations of belief contraction (or "belief withdrawal") rather than belief revision. The reader will be able to form a better idea of how $\circ$ works after we have derived a list of properties from the set of axioms.

As is common in the belief revision literature, we use the abbreviation $K + \alpha$ to denote the set $Cn(K \cup \{\alpha\})$. We have six fundamental axioms for one-step revision by comparison.

(C1) $K \circ_\alpha \beta = Cn(K \circ_\alpha \beta)$. (Closure)

(C2) If $Cn(\alpha) = Cn(\gamma)$ and $Cn(\beta) = Cn(\delta)$, then $K \circ_\alpha \beta = K \circ_\gamma \delta$. (Extensionality)

(C3) If $\alpha \notin K \circ_\beta \bot$, then $K \circ_\beta \bot \subseteq K \circ_\alpha \bot$. (Strong Inclusion)

(C4) If $\alpha \in K \circ_\alpha \bot$, then $\alpha \in K \circ_\beta \gamma$. (Irrevocability)

(C5) If $\alpha \in K \circ_{\alpha\wedge\neg\beta} \bot$, then $K \circ_\alpha \beta = (K \circ_{\alpha\wedge\neg\beta} \bot) + \beta$. (Reduction 1)

(C6) If $\alpha \notin K \circ_{\alpha\wedge\neg\beta} \bot$, then $K \circ_\alpha \beta = K \circ_{\alpha\wedge\neg\beta} \bot$. (Reduction 2)

Extensionality and Closure make clear that we are dealing with the "knowledge level" (Nebel 1989), where it is the content that counts, and syntactical variation is irrelevant. Strong Inclusion is a property taken over from an axiomatization of the operation of severe withdrawal (Pagnucco and Rott 1999). Intuitively, it says that if α is not more entrenched than β, then the withdrawal of α results in a superset of the withdrawal of β. According to Irrevocability, if a direct attempt at withdrawing α from K fails, then α will survive any revision by comparison. (C5) and (C6) jointly show that by using a suitable case distinction, the operation of revision by comparison $\circ_\alpha\beta$ with its two arguments (reference sentence α and input sentence β) can essentially be reduced to the operation of $\circ_\alpha\bot$ that takes only one argument. It is important to see that the power of the *dyadic (binary)* function can be simulated by a sophisticated application of a *monadic (unary)* function. It is hard to think of a direct motivation for the reduction postulates (C5) and (C6). What we can say in favour of them is that they make for a compact axiomatization that fits the semantics and has appealing consequences.

From (C5) and (C6), it is easy to derive that $K \circ_\top \top = K \circ_\bot \top = K \circ_\bot \bot$. From the intuitive description and the semantic modellings above, it is clear that none of the operations $\circ_\top\top$, $\circ_\bot\top$ and $\circ_\bot\bot$ introduces any changes to the belief state. So any of the belief sets $K \circ_\top \top$, $K \circ_\bot \top$ and $K \circ_\bot \bot$ can intuitively be taken to represent *the set of current beliefs*, i.e., the belief set $K = K_\circ$ corresponding to a given operation $\circ$. For the sake of simple proofs, we choose $K \circ_\bot \bot$ to take this role, and we say that $\circ$ is a revision-by-comparison operation *on* $K = K \circ_\bot \bot$.

Lemma 5 *Given this definition of K, axioms (C1) – (C6) entail that $\circ$ satisfies*

(Vacuity) *If $\alpha \notin K$, then $K \circ_\alpha \beta = K$.*

Vacuity says that if the reference sentence is not believed itself, then revision by comparison doesn't change the belief set.

6 Derived properties of one-step revision by comparison

We now give a list of properties that have a close connection to the classical AGM (1985) postulates for belief change. First we note that the AGM properties of Closure and Extensionality—(K∗1) and (K∗6) in the usual numbering—have counterparts in (C1) and (C2). But there are more analogies which we reflect by keeping the widely used AGM-numberings, and adding some more descriptive names at the right-hand side.

Observation 6 Let $\circ$ be an operation satisfying (C1) – (C6), and let the set $K = K_\circ$ of current beliefs be defined as $K \circ_\bot \bot$. Then $\circ$ satisfies:

(K∗2)	If $\alpha \in K \circ_\alpha \beta$ then $\beta \in K \circ_\alpha \beta$	(Weak Success)
(K∗3)	$K \circ_\alpha \beta \subseteq K + \beta$	(Inclusion)
(K∗4)	If $\neg\beta \notin K$, then $K \subseteq K \circ_\alpha \beta$	(Preservation)
(K∗5)	$\alpha \wedge \neg\beta \in K \circ_{\alpha\wedge\neg\beta} \bot$ iff $K \circ_\alpha \beta = K_\bot$	(Consistency)
(K∗7)	$K \circ_\alpha (\beta \wedge \gamma) \subseteq (K \circ_\alpha \beta) + \gamma$	(Superexpansion)
(K∗8)	If $\neg\gamma \notin K \circ_\alpha \beta$, then $K \circ_\alpha \beta \subseteq K \circ_\alpha (\beta \wedge \gamma)$	

(Conjunctive Preservation)

(K∗7&8) $K \circ_\alpha (\beta \vee \gamma) = K \circ_\alpha \beta$ or $K \circ_\alpha (\beta \vee \gamma) = K \circ_\alpha \gamma$ or $K \circ_\alpha (\beta \vee \gamma) = K \circ_\alpha \beta \;\cap\; K \circ_\alpha \gamma$ (Disjunctive Factoring)

($K\dot{-}8^+$) If $\alpha \notin K \circ_{\alpha\wedge\beta} \gamma$, then $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\alpha \gamma$

($K\dot{-}$D) $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\alpha \gamma$ or $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\beta \gamma$ (Decomposition)

The Weak Success condition (K∗2) says that if the reference sentence α is retained as a belief, then the revision by the input sentence β is successful in the sense that β actually gets accepted.

Inclusion (K∗3) and Preservation (K∗4) are very close to the AGM postulates with a similar label. AGM, however, use a strengthening of (K∗4), namely,

(K∗4′) If $\neg\beta \notin K$, then $K + \beta \subseteq K \circ_\alpha \beta$. (AGM's 4th condition)

In AGM, (K∗4′) follows immediately from Preservation together with Success ($\alpha \in K * \alpha$) and Closure. In our model, this postulate is not valid, for the same reasons that stand in the way of the unrestricted Success condition $\beta \in K \circ_\alpha \beta$.

Superexpansion (K∗7) and Conjunctive Preservation (K∗8) have analogues in the seventh and eighth AGM postulates for revisions (here: revisions by $\beta \wedge \gamma$). But other than in AGM, (K∗8) does not imply that γ is in $K \circ_\alpha (\beta \wedge \gamma)$ (i.e., it does not imply a certain form of 'success').

The seventh and eighth AGM postulates for contractions have counterparts in

($K\dot{-}7$) $K \circ_\alpha \gamma \;\cap\; K \circ_\beta \gamma \subseteq K \circ_{\alpha\wedge\beta} \gamma$ (Conjunctive Overlap)

($K\dot{-}8$) If $\alpha \notin K \circ_{\alpha\wedge\beta} \gamma$, then $K \circ_{\alpha\wedge\beta} \gamma \subseteq K \circ_\alpha \gamma$ (Conjunctive Inclusion)

which follow immediately from ($K\dot{-}$D) and ($K\dot{-}8^+$), respectively.

The following rule also follows immediately from ($K\dot{-}$D):

($K\dot{-}7$&8) $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\alpha \gamma$ or $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\beta \gamma$ or $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\alpha \gamma \;\cap\; K \circ_\beta \gamma$ (Conjunctive Factoring)

Disjunctive Factoring and Conjunctive Factoring correspond to similar AGM properties for revision (by $\beta \vee \gamma$) and contraction (with respect to $\alpha \wedge \beta$) respectively. Decomposition is a strengthening of ($K\dot{-}7$&8). It was mentioned in Alchourrón *et al.* (1985, Observation 6.3(a)) as one of the postulates to characterize maxichoice contraction, and in Pagnucco and Rott (1999, Lemma 2(ii)) as one of the properties of severe withdrawals.

Notice that the typical AGM-style properties ($K\dot{-}7$) and ($K\dot{-}8$) for contractions can be strengthened considerably in the present context.

Observation 6 and its consequences confirm our earlier impression that revision by comparison has characteristics of revision (with respect to the input sentence) and at the same time characteristics of contraction (with respect to the reference sentence). Roughly speaking, one can view $K \circ_\alpha \beta$ as a qualified revision of K by β: The input sentence β is accepted "were it not for the worlds that falsify α". But $K \circ_\alpha \beta$ can at the same time be regarded as a qualified withdrawal of α from K: α is given up in the agent's belief state, "were it not for the worlds that verify β".

Let us finally have a look at a number of further interesting properties of the operation $\circ$, some of which are interesting in themselves, some of them needed to prove other things.

Observation 7 Let $\circ$ be an operation on a belief set K that satisfies (C1) – (C6). Then $\circ$ satisfies:

(Q1) If $\alpha \in K \circ_\beta \perp$ and $\alpha \notin K \circ_\alpha \perp$, then $K \circ_\alpha \perp \subseteq K \circ_\beta \perp$.

(Q2) $\alpha \in K \circ_\alpha \neg\beta$ iff $\alpha \in K \circ_\beta \neg\alpha$.

(Q3) If $\gamma \in K \circ_\alpha \neg\gamma$, then $\gamma \in K \circ_\alpha \beta$.

(Q4) If $\alpha \in K \circ_\alpha \perp$ then $\beta \in K \circ_\alpha \beta$. (Irrevocable Success)

(Q5) $\beta \in K \circ_\top \beta$.

(Q6) $K \circ_{\alpha\wedge\neg\beta} \perp \subseteq K \circ_\alpha \beta$ (Severe withdrawal lower bound)

If $\alpha \notin K \circ_{\neg\beta} \perp$ or $\neg\beta \in K \circ_{\neg\beta} \perp$, then even $K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_\alpha \beta$.

(Q7) $K \circ_\alpha \beta \subseteq K \circ_\top (\alpha \to \beta)$ (AGM revision upper bound)

If $\alpha \in K \circ_{\neg\beta} \perp$ and $\neg\beta \notin K \circ_{\neg\beta} \perp$, then even $K \circ_\alpha \beta = K \circ_\top \alpha \to \beta$.

(Q8) $K \circ_\alpha \perp = K \circ_\alpha \neg\alpha$

(Q9) $K \circ_\alpha \beta = K \circ_\alpha (\alpha \to \beta)$

(Q10) If $\alpha \in K \circ_\alpha \beta$, then $K \circ_\alpha \beta = K \circ_\top \beta$

(Q11) $\beta \in K \circ_\alpha \beta$ iff ($\alpha \in K \circ_\alpha \beta$ or $\beta \in K \circ_\alpha \neg\beta$)

(Q12) $\alpha \in K \circ_\beta \perp$ iff ($\alpha \in K \circ_\beta \neg\alpha$ or $\beta \in K \circ_\beta \perp$)

(Q13) If $\alpha \notin K \circ_\beta \neg\alpha$ and $\beta \notin K \circ_\gamma \neg\beta$ then $\alpha \notin K \circ_\gamma \neg\alpha$

(Q14) If $\alpha \wedge \gamma \notin K \circ_{\alpha\wedge\gamma} \perp$, then $\alpha \notin K \circ_\gamma \neg\alpha$ or $\gamma \notin K \circ_\alpha \neg\gamma$

(Q15) $\alpha \in K \circ_\alpha \perp$ iff $K \circ_\alpha \perp = K_\perp$.

(Q16) If $\alpha \notin K \circ_\alpha \perp$, then $K \circ_\alpha \perp \subseteq K \circ_{\alpha\wedge\beta} \perp$.

(Q17) If $\alpha \notin K \circ_\alpha \perp$, then $K \circ_\alpha \perp \subseteq K \circ_\alpha \beta$.

Properties (Q6) and (Q7) show that our operation $K \circ_\alpha \beta$ is, in a precise sense, "between" an AGM revision by $\alpha \to \beta$ and a severe withdrawal with respect to the negation of this sentence.

7 How to express entrenchments in terms of revisions

A convenient way to express entrenchments in terms of revisions by comparison uses the fixed input sentence $\bot$:

(Def $\le$ from $\circ$)

$$\boxed{\alpha \le \beta \;\text{ iff }\; \alpha \notin K \circ_\beta \bot \;\text{ or }\; \beta \in K \circ_\beta \bot}$$

We first need to make sure that the relation $\le$ so defined is indeed an entrenchment relation in the technical sense introduced in Section 3. Note that $\beta \in K \circ_\beta \bot$ is equivalent to $K \circ_\beta \bot = K_\bot$, by (Q15).

Observation 8 If $\circ$ satisfies (C1) – (C6), then $\le$ as defined by (Def $\le$ from $\circ$) is an entrenchment relation, i.e., it satisfies (E1) – (E3).

Reference is made in (Def $\le$ from $\circ$) to revisions with fixed input sentence $\bot$, equivalent with severe withdrawals.[17] It is actually a unary belief change operation here that defines the entrenchment relation. (Def $\le$ from $\circ$) is the definition that we will use in our proofs, but it is instructive to have a look at a few equivalent versions, some of which use genuine revisions by comparison.

Observation 9 The following conditions are all equivalent:

(a) $\alpha \notin K \circ_\beta \bot$ or $\beta \in K \circ_\beta \bot$

(b) $\alpha \notin K \circ_\beta \neg\alpha$ or $\beta \in K \circ_\beta \bot$

(c) $\alpha \notin K \circ_\alpha \neg\beta$ or $\beta \in K \circ_\beta \bot$

(d) $\alpha \notin K \circ_{\alpha\wedge\beta} \bot$ or $\beta \in K \circ_{\alpha\wedge\beta} \bot$

(e) $\alpha \notin K \circ_\top \neg(\alpha \wedge \beta)$ or $\beta \in K \circ_\top \neg(\alpha \wedge \beta)$

Condition (e) gives perhaps the most interesting variant of (Def $\le$ from $\circ$). Reference is made here to revisions with fixed reference sentence $\top$, equivalent with irrevocable revisions.[18] Again, it is essentially a unary belief change operation that defines the entrenchment relation.

8 Entrenchment-based revision by comparison: Representation theorems for one-step revisions

We show that the class of entrenchment-based functions of revision-by-comparison coincides exactly with the class of functions satisfying (C1) –

[17] Essentially the same condition as in (Def $\le$ from $\circ$) is used in Pagnucco and Rott (1999, pp. 525–526), where it is also shown how to construct systems of spheres from severe withdrawal functions.

[18] See the companion paper (Rott 200$*$). Essentially the same condition is used in Hansson *et al.* (2001, proof of Theorem 13).

(C6).

Theorem 10 (Soundness) Let $\leq$ be an entrenchment ordering that satisfies (E1) – (E3). Furthermore let $\circ$ be the entrenchment-based revision-by-comparison function defined by condition (Def $\circ$ from $\leq$). Then $\circ$ satisfies (C1) – (C6). Moreover, $K_\circ = K_\leq$, and $\leq$ can be retrieved from $\circ$ with the help of (Def $\leq$ from $\circ$).

Theorem 11 (Completeness) Let $\circ$ be a revision-by-comparison function satisfying (C1) – (C6). Then there is an entrenchment relation $\leq$ satisfying (E1) – (E3) such that $\circ$ can be represented as being generated from $\leq$ with the help of (Def $\circ$ from $\leq$), and $K_\leq = K_\circ$.

As may be expected, the entrenchment relation $\leq$ mentioned in the completeness theorem can be derived from $\circ$ with the help of (Def $\leq$ from $\circ$).

Due to Theorem 2, the soundness theorem also applies to sphere-based revision by comparison. Using known results about the very close relationship between systems of spheres and entrenchments (see the references mentioned in Section 3), it is clear that the completeness theorem is valid for sphere-based revision by comparison as well.

9 Entrenchment-based revision by comparison: Representation theorems for iterated revisions

So far we have only looked at one-step revisions. Given an entrenchment relation $\leq$ which represents a belief state, however, we can generate an *iterated* revision-by-comparison operation by repeated applications of (Def $\leq'$ from $\leq$) and (Def $K_\leq$).[19] We would like to know the properties of the repeated operation $\circ$ thus generated. More exactly, we want to know how a two-fold application of $\circ$ relates to single applications of $\circ$. If we have an answer to this question, we can reduce any finite number of revisions by comparison to single applications of $\circ$. There is a price to be paid for this reduction, of course: There will be intricate case distinctions to be made. In addressing the iterated case, then, the general question to ask is under what circumstances we have

$$\phi \ \in \ (K \circ_\alpha \beta) \circ_\gamma \delta$$

However, this case is still very complex. The problem of iteration is a delicate matter even in the case of ordinary AGM revision without any specification of plausibilities for input sentences (see Darwiche and Pearl 1997). Thus clearly,

[19] Remember from Section 3 that (Def $\circ$ from $\leq$) is a consequence of (Def $\leq'$ from $\leq$) and (Def $K_\leq$). The most important part of our construction is the repeated change of the entrenchment relation (or, for that matter: of the system of spheres). As we pointed out in Section 1, our notation leaves implicit the fact that the first argument of $\circ$ is really a belief *state* rather than a belief *set*.

our function that takes three things (a belief state and two sentences) as arguments rather than only two (a belief state and one sentence) must be very involved. What we expect is an effect similar to the effects in the simpler case of iterations of AGM revision, where $(K * \beta) * \delta$ is somehow related to revisions of K by β, δ or $\beta \wedge \delta$. Let us see whether we have analogous structures here.

Fortunately, we can restrict ourselves to a special case of the above, viz. the question what the conditions are for

$$\delta \in \; (K \circ_\alpha \beta) \circ_\gamma \bot$$

By the reduction axioms (C5) and (C6), we know that the answer to this question is all we need to know. This is the crucial condition:

(IT) $\quad (K \circ_\alpha \beta) \circ_\gamma \bot \; =$

$$= \begin{cases} (K \circ_{\beta \to \gamma} \bot) + \beta & \text{if } \; K \circ_\gamma \bot \neq K_\bot \; \text{ and } \; \alpha \in K \circ_\alpha (\beta \wedge \neg\gamma) \\ K \circ_\alpha \bot \; \cap \; K \circ_\gamma \bot & \text{if } \; K \circ_\gamma \bot \neq K_\bot \; \text{ and } \; \alpha \notin K \circ_\alpha (\beta \wedge \neg\gamma) \\ K_\bot & \text{if } \; K \circ_\gamma \bot = K_\bot \end{cases}$$

Unfortunately, (IT) is very complex. Of course it would be nicer if it could be derived from simpler and more elegant, yet valid postulates for iterated belief change, but we do not see how this can be achieved. (IT) will enable us to prove (Def $\leq'$ from $\leq$) from the $\circ$-postulates and (Def $\leq$ from $\circ$). Before doing that, let us extend the soundness result of Theorem 10 to the case of iterated revision-by-comparison:

Theorem 12 Let $\leq$ be an entrenchment relation satisfying (E1) – (E3). Let $\circ$ be the entrenchment-based iterated revision-by-comparison function defined from $\leq$ by condition (Def $\circ$ from $\leq$) and (Def $\leq'$ from $\leq$). Then $\circ$ satisfies (IT).

The following theorem can be seen as an extension of the completeness result Theorem 11:

Theorem 13 Let $\circ$ be an iterated revision-by-comparison function satisfying (C1) – (C6) as well as (IT), and let $\circ'$ be the revision-by-comparison function (associated with $K' = K \circ_\alpha \beta$) which is defined by

$$K' \circ'_\gamma \delta = (K \circ_\alpha \beta) \circ_\gamma \delta$$

If $\leq$ is the entrenchment relation derived from $\circ$ by (Def $\leq$ from $\circ$) and if $\leq'$ is the entrenchment relation derived from $\circ'$ by (Def $\leq$ from $\circ$), then $\leq$ and $\leq'$ satisfy (Def $\leq'$ from $\leq$).

Note that the posterior entrenchment $\leq'$ constructed in Theorem 13 is indeed uniquely determined by the prior entrenchment $\leq$ through (IT), (C5) and (C6).

Condition (IT) is fairly hard to understand. To get a better picture of repeated revisions, we take down the following interesting properties of revision-by-comparison functions that are generated from $\leq$ by means of (Def $\circ$ from $\leq$) and (Def $\leq'$ from $\leq$).

Theorem 14 Let $\circ$ be an iterated revision-by-comparison function for K satisfying (C1) – (C6) as well as (IT). Then $\circ$ satisfies the following conditions:

(IT1) $\quad (K \circ_\alpha \beta) \circ_\alpha \gamma = K \circ_\alpha (\beta \wedge \gamma)$

(IT2) $\quad (K \circ_\alpha \beta) \circ_\gamma \beta = \begin{cases} K \circ_\alpha \beta & \text{, if } \gamma \notin K \circ_\alpha \bot \text{ or } K \circ_\alpha \bot = K_\bot \\ K \circ_\gamma \beta & \text{, otherwise} \end{cases}$

(IT3) $\quad (K \circ_\alpha \beta) \circ_\gamma \beta = (K \circ_\gamma \beta) \circ_\alpha \beta$

(IT4) $\quad (K \circ_\alpha \bot) \circ_\beta \bot = \begin{cases} (K \circ_\alpha \bot) \cap (K \circ_\beta \bot) & \text{, if } K \circ_\alpha \bot, K \circ_\beta \bot \neq K_\bot \\ K_\bot & \text{, otherwise} \end{cases}$

(IT5) $\quad (K \circ_\alpha \beta) \circ_\beta \alpha = \begin{cases} K \circ_\beta \alpha & \text{, if } \alpha \notin K \circ_\beta \bot \text{ or } K \circ_\beta \bot = K_\bot \\ K \circ_\alpha \beta & \text{, otherwise} \end{cases}$

(IT1) says that for identical reference sentences, iterated revisions can be treated as revisions by conjunctions; this entails commutativity of revisions by comparison with a fixed reference sentence. In (IT2), the two cases cannot be conjoined into a set-theoretic combination of $K \circ_\alpha \beta$ and $K \circ_\gamma \beta$, because these sets may be inconsistent, as, e.g., in the case $\alpha < \neg\beta < \gamma$. (IT3) establishes commutativity of revisions by comparison with a fixed input sentence. For the special case of severe withdrawal (the unary operation $\circ_{\ldots}\bot$), (IT4) shows that it is usually possible to take the intersection of $K \circ_\alpha \beta$ and $K \circ_\gamma \beta$. The operation considered in (IT5) is a symmetrical variant of revision-by-comparison that sets α and β to an equal degree of entrenchment.

Now we have found out that iterations of revision-by-comparison functions are fairly well-behaved in some special cases. Other principles that seem plausible at first sight, however, fail. For instance, unrestricted commutativity

$$(K \circ_\alpha \beta) \circ_\gamma \delta = (K \circ_\gamma \delta) \circ_\alpha \beta$$

does not hold in general, as is shown by the following example. Let $\mathcal{L}$ have only four atoms p, q, r and s, and let $K = Cn\{r, p \leftrightarrow s\}$. Then

$$K \circ_p q = K$$

$$(K \circ_p q) \circ_r s = Cn(\{p, r, s\})$$

$$K \circ_r s = Cn(\{p, r, s\})$$

$$(K \circ_r s) \circ_p q = Cn(\{p, q, r, s\})$$

For a simple possible worlds model of the same situation, consider for instance the system of spheres with just one sphere containing four worlds characterized by the literals $\langle p, q, r, s\rangle$, $\langle p, \neg q, r, s\rangle$, $\langle \neg p, q, r, \neg s\rangle$ and $\langle \neg p, \neg q, r, \neg s\rangle$.

10 Discussion

10.1 Related work

In an elegant and dense paper, John Cantwell (1997) presents an idea of *raising* the plausibility of a sentence (more precisely, of a non-belief). Though developed independently, the work presented in this paper turns out to be very close to the way Cantwell's raising mechanism works.

The most significant difference between raising and revision by comparison is that Cantwell does not present his operation as a revision operation, and in fact the new belief set he defines (his "standard of belief", Cantwell 1997, Def. 11) is inconsistent in the belief-contravening case. This problem can be resolved by a second step of consolidation (Cantwell's Def. 14), but consolidation is not an integral part of the idea of raising. It seems fair to say that both in Cantwell's paper and in the present one expansion is viewed as a limiting case which is approached from different directions. For Cantwell, expansion is the limiting case of raising where the raising goes up to the point at which the hypothesis is not just made more plausible, but is actually turned into a new belief. For us, an expansion is the limiting case of a revision by a sentence that is consistent with the original belief set.

When the relative entrenchments of two sentences are shifted, it is hard to say whether the first is raised to the level of the second, or whether the second is lowered to the level of the first. We thus essentially agree with what Cantwell (1997, p. 67) says about the (im-)possibility of comparing degrees of acceptance across purely qualitative structures. We have not thought of revision by comparison as an operation of raising mainly because in our work the unsuccessful case, where revision by comparison reduces to a severe withdrawal, is a very important one. And in this case it seems much more natural to us to say that the input sentence is not raised, but the reference sentence is lowered (to the point at which it is not a belief any more). [20]

[20] We have not investigated here anything like Cantwell's operation $\downarrow_\alpha \beta$ of "lower-

The proposed model is not the only model conceivable that specifies a security level (degree of acceptance) for the input sentence in terms of a reference sentence that typically is already believed. There is an alternative "foundationalist" idea (cf. Bonjour 1999) where one simply *adds* a constraint to an existing set of constraints for possible worlds models or orderings of sentences. Constraints of the form needed are expressible in natural language by conditionals like '*If not both α and β, then (still) β*' which in terms of entrenchment expresses the strict preference $\alpha < \beta$.[21] Negated conditionals of the form '*It is not the case that: if not both α and β, then (still) α*' can be taken to express the weak preference $\alpha \leq \beta$.[22] The task afterwards is to construct one or more distinguished, in some sense minimal or maximal semantic model(s) that satisfy the enlarged set of constraints. In contrast to this idea, the approach that we follow in this paper is a "coherentist" one which incorporates an input like $\alpha \leq \beta$ directly into a given well-balanced relation of epistemic entrenchment, without recourse to the set of sentences or constraints that may have figured as inputs in the epistemic history of the agent. Our operation aims at satisfying the new constraint with minimal perturbation of the prior entrenchment relation and maximal internal coherence.[23] An alternative coherentist account of changing belief states by conditionals is presented in Kern-Isberner (1999).

10.2 Problems

We see two main problems with our model. First, as we have seen, the agent cannot revise by β when $\neg\beta$ is highly entrenched, more precisely, when $\neg\beta$ is at least as entrenched as the reference sentence α. A solution might consist in a strategy of *choosing* as reference sentence some α which is just a little bit more entrenched than $\neg\beta$. This will lead to a posterior belief state in which β is accepted, if only with weak entrenchment. But it is doubtful, at least in some situations or according to some interpretations, whether it makes sense to think of the reference sentence as something that the agent is always free to choose.

A second problematic feature of the model is that it tends to make orderings coarser and coarser along a series of repeated revisions.[24] Semantically speak-

ing" that is supposed to lower the plausibility of β to the plausibility of α.

[21] Compare Rott (1991b, 1997), Lehmann and Magidor (1992), Becher *et al.* (1999). Also see Nayak *et al.* (1996) and Weydert (1998).

[22] A model for dealing with both positive and negative conditional assertions is developed in Booth and Paris (1998).

[23] Coherence is guaranteed by the very fact that the posterior ordering is an entrenchment relation satisfying (E1) – (E3).

[24] If we look at the possible worlds modelling, this description seems intuitively right. Formally, however, the only case of a genuine coarsening (where $\leq'$ is a proper superset of $\leq$) happens if a revision-by-comparison is unsuccessful (i.e., $\beta \notin K \circ_\alpha \beta$).

ing, by throwing the comparatively plausible $\neg\beta$-worlds all together into the first $\neg\alpha$-ring, we lose all plausibility distinctions between them. Similarly, put in terms of entrenchment, we lose all distinctions between those consequences of β which are comparatively poorly entrenched, viz., at most as entrenched as α.

A different, but related point is that by using revision-by-comparison alone, we can never build up an informative belief state from the state of complete ignorance.[25] In this state, the only potential reference sentences are the maximally entrenched ones, which *a priori* are only the logical truths. Similarly, as was mentioned before (in Section 4.4), we can never escape the state of epistemic collapse by using revision-by-comparison.

Clearly, this is a disadvantage. Two kinds of remedy are available. First, we can try to find more complicated variations of the proposed model that do not exhibit this kind of weakness. However, it may turn out that the resulting mechanism is so complex as not to allow for a perspicuous logical characterization. Second and perhaps more promising for iterated belief change, we can combine revision-by-comparison with well-known operations of iterated belief change which have the opposite property, i.e., that introduce finer and finer distinctions along a series of revisions. One such model is Boutilier's (1993, 1996) "natural revision" that always moves a certain set of worlds inwards, creating a new sphere in the very center of the system of spheres. Another model for iterated revision that introduces new distinctions has been investigated by Nayak (1994), Nayak, Nelson and Polansky (1996), Konieczny and Pino Pérez (2000), Papini (2001), Nayak, Pagnucco and Peppas (2003) and others. Combining such models (which take only one argument, the input sentence) with revision-by-comparison, there is no danger of ending up with ever more coarse-grained plausibility orderings of worlds or entrenchment orderings of sentences. However, the combination approach burdens us with the task of providing a methodology for deciding when to apply which revision method for what reasons. This is difficult, to be sure, but it may just be the right way to go in a purely qualitative setting. And it is certainly a challenging task for future work.

References

Alchourrón, Carlos, Peter Gärdenfors, and David Makinson: 1985, 'On the logic of theory change: Partial meet contraction and revision functions'. *Journal of Symbolic Logic* **50**, 510–530.

[25] In the system of spheres modelling, the *state of complete ignorance* is represented by $\$ = \{W\}$, in the entrenchment modelling by the relation $\leq \, = \, \{\langle\alpha,\beta\rangle : \nvdash \alpha \text{ or } \vdash \beta\}$.

Becher, Verónica, Eduardo Fermé, Sandra Lazzer, Carlos Oller, Gladys Palau and Ricardo Rodriguez: 1999, 'Some Observations on C. Alchourrón's Theory of Defeasible Conditional', in: *Norms, Logics and Information Systems. New Studies on Deontic Logic and Computer Science (Proceedings DEON'98, Bologna, Italia)*, eds. P. McNamara and H. Prakken, IOS Press, pp. 219–230.

Bonjour, Laurence: 1999, 'The Dialectic of Foundationalism and Coherentism', in J. Greco and E. Sosa (eds.), *The Blackwell Guide to Epistemology*, Oxford, Blackwell, pp. 206–219.

Booth, Richard, and Jeff B. Paris: 1998, 'A Note on the Rational Closure of Knowledge Bases with Both Positive and Negative Knowledge', *Journal of Logic, Language, and Information* **7**, 165–190.

Boutilier, Craig: 1993, 'Revision Sequences and Nested Conditionals', in R. Bajcsy (ed.), *IJCAI-93 – Proceedings of the Thirteenth International Joint Conference on Artificial Intelligence*, 519–525.

Boutilier, Craig: 1996, 'Iterated Revision and Minimal Change of Conditional Beliefs', *Journal of Philosohical Logic* **25**, 263–305.

Cantwell, John: 1997, 'On the logic of small changes in hypertheories', *Theoria* **63**, 54–89.

Darwiche, Adnan, and Judea Pearl: 1997, 'On the Logic of Iterated Belief Revision', *Artificial Intelligence* **89**, 1–29.

Dubois, Didier, and Henri Prade: 1991, 'Epistemic Entrenchment and Possibilistic Logic', *Artificial Intelligence* **50**, 223–239.

Fermé, Eduardo: 2000, 'Irrevocable Belief Revision and Epistemic Entrenchment', *Logic Journal of the IGPL* **8**, 645–652.

Freund, Michael: 1998, 'Preferential Orders and Plausibility Measures', *Journal of Logic and Computation* **8**, 147–158.

Friedman, Nir, and Joseyph Y. Halpern: 2001, 'Plausibility Measures and Default Reasoning', *Journal of the ACM* **48**, 648–685.

Gärdenfors, Peter: 1988, *Knowledge in Flux. Modeling the Dynamics of Epistemic States*, Bradford Books, MIT Press, Cambridge, Mass.

Gärdenfors, Peter, and David Makinson: 1988, 'Revisions of Knowledge Systems Using Epistemic Entrenchment', in Moshe Vardi (ed.), *TARK'88 – Proceedings of the Second Conference on Theoretical Aspects of Reasoning About Knowledge*, Los Altos: Morgan Kaufmann, pp. 83–95.

Gärdenfors, Peter, and David Makinson: 1994, 'Nonmonotonic Inference Based on Expectations', *Artificial Intelligence* **65**, 197–245.

Gärdenfors, Peter, and Hans Rott: 1995, 'Belief revision', in D. M. Gabbay, C. J. Hogger, and J. A. Robinson (eds.), *Handbook of Logic in Artificial Intelligence and Logic Programming Volume IV: Epistemic and Temporal Reasoning*, Oxford University Press, pp. 35–132.

Gigerenzer, Gerd: 2002, *Calculated Risks. How to Know When Numbers Deceive You*, Simon and Schuster, New York.

Goldszmidt, Moises, and Judea Pearl: 1992, 'Rank-based Systems: A Simple Approach to Belief Revision, Belief Update, and Reasoning about Evidence and Actions', *Principles of Knowledge Representation and Reasoning. Proceedings of the Third International Conference (KR'92)*, Morgan Kaufmann, pp. 661–672.

Grove, Adam: 1988, 'Two Modellings for Theory Change', *Journal of Philosophical Logic* **17**, 157–170.

Hansson, Sven Ove (ed.): 1997, *Non-Prioritized Belief Revision*, special issue of *Theoria* **63**, Nos. 1–2.

Hansson, Sven Ove, Eduardo Fermé, John Cantwell and Marcelo Falappa: 2001, 'Credibility Limited Revision', *Journal of Symbolic Logic* **66**, 1581–1596.

Jeffrey, Richard C.: 1965, *The Logic of Decision*, McGraw-Hill, New York, second edition University of Chicago Press, Chicago 1983.

Kern-Isberner, Gabriele: 1999, 'Postulates for Conditional Belief Revision', in Thomas Dean (ed.), *Proceedings of the Sixteenth International Joint Conference on Artificial Intelligence, IJCAI'99*, Morgan Kaufmann, 186–191.

Konieczny, Sébastien, and Ramón Pino Pérez: 2000, 'A Framework for Iterated Revision', *Journal of Applied Non-Classical Logic* **10**, 339–367.

Lehmann, Daniel, and Menachem Magidor: 1992, 'What does a conditional knowledge base entail?', *Artificial Intelligence* **55**, 1–60.

Levi, Isaac: 1996, *For the Sake of Argument*, Cambridge, Cambridge University Press.

Lewis, David: 1973, *Counterfactuals*, Blackwell, Oxford.

Lindström, Sten, and Wlodzimierz Rabinowicz: 1991, 'Epistemic Entrenchment with Incomparabilities and Relational Belief Revision', in André Fuhrmann and Michael Morreau (eds.), *The Logic of Theory Change*, Springer LNAI **465**, Berlin etc., pp. 93–126.

Nayak, Abhaya C.: 1994, 'Iterated Belief Change Based on Epistemic Entrenchment', *Erkenntnis* **41**, 353–390.

Nayak, Abhaya, Maurice Pagnucco, Norman Y. Foo and Pavlos Peppas: 1996, 'Learning from Conditionals: Judy Benjamin's Other Problems', *Proceedings of 12th European Conference on Artificial Intelligence*, Budapest, Wiley, pp. 75–79.

Nayak, Abhaya C., Paul Nelson and Hanan Polansky: 1996, 'Belief Change as Change in Epistemic Entrenchment', *Synthese* **109**, 143–174.

Nayak, Abhaya C., Maurice Pagnucco and Pavlos Peppas: 2003, 'Dynamic Belief Revision Operators', *Artificial Intelligence* **146**, 193–228.

Nebel, Bernhard: 1989, 'A Knowledge Level Analysis of Belief Revision', in Ronald Brachman, Hector Levesque and Ray Reiter (eds.), *Proceedings of the 1st International Conference on Principles of Knowledge Representation and Reasoning*, Morgan Kaufmann, San Mateo, Cal., pp. 301–311.

Pagnucco, Maurice, and Hans Rott: 1999, 'Severe Withdrawal (and Recovery)', *Journal of Philosophical Logic* **28**, 501–547. (Beware of corrupted printing. Correct, complete reprint with original pagination in the February 2000 issue; see publisher's 'Erratum', *Journal of Philosophical Logic* **29**, 2000, 121.)

Papini, Odile: 2001, 'Iterated Revision Operations Stemming from the History of an Agent's Observations', in Mary-Anne Williams and Hans Rott (eds.), *Frontiers of Belief Revision*, Dordrecht: Kluwer, 279–301.

Peppas, Pavlos, and Mary-Anne Williams: 1995, 'Constructive Modelings for Theory Change', *Notre Dame Journal of Formal Logic* **36**, 120–133.

Piattelli-Palmarini, M.: 1994, *Inevitable Illusions: How Mistakes of Reason Rule Our Minds*, New York: John Wiley.

Rott, Hans: 1991, 'Two Methods of Constructing Contractions and Revisions of Knowledge Systems', *Journal of Philosophical Logic* **20**, 149–173.

Rott, Hans: 1991, 'A Non-monotonic Conditional Logic for Belief Revision I', in André Fuhrmann and Michael Morreau (eds.), *The Logic of Theory Change*, Lecture Notes in Computer Science **465**, Berlin: Springer, pp. 135–181.

Rott, Hans: 1992, 'Preferential Belief Change Using Generalized Epistemic Entrenchment', *Journal of Logic, Language and Information* **1**, 45–78.

Rott, Hans: 1997, 'Drawing Inferences from Conditionals', in *Logic, Action and Cognition. Essays in Philosophical Logic*, eds. Eva Ejerhed and Sten Lindström, Serie Trends in Logic, Vol. 2, Dordrecht, Kluwer, pp. 149–179.

Rott, Hans: 2001, *Change, Choice and Inference*, Oxford: Oxford University Press.

Rott, Hans: 200∗, 'Revision by Comparison as a Unifying Framework: Severe Withdrawal, Irrevocable and Irrefutable Revision', Manuscript, University of Regensburg, December 2003.

Schlechta, Karl: 1997, "Filters and Partial Orders", *Journal of the Interest Group in Pure and Applied Logics* **5**, 753–772.

Segerberg, Krister: 1998, 'Irrevocable Belief Revision in Dynamic Doxastic Logic', *Notre Dame Journal of Formal Logic*, **39**, pp. 287–306.

Spohn, Wolfgang: 1988, 'Ordinal Conditional Functions', in William L. Harper and Brian Skyrms (eds.), *Causation in Decision, Belief Change, and Statistics*, Vol. II, Dordrecht: Reidel, pp. 105–134.

Spohn, Wolfgang: 1999, 'Ranking Functions, AGM Style', in *Spinning Ideas, Electronic Essays Dedicated to Peter Gärdenfors on His Fiftieth Birthday*, available at `http://www.lucs.lu.se/spinning/categories/dynamics/Spohn/`.

Tversky, Amos, and Kahneman, Daniel: 1983, 'Extensional Versus Intuitive Reasoning: The Conjunction Fallacy in Probability Judgment', *Psychological Review* **90**, 293–315.

Weydert, Emil: 1994, 'General belief measures', *Proceedings of the Tenth Conference on Uncertainty in Artificial Intelligence (UAI'94)*, R. López de Mantarás and David Poole (eds.), San Francisco, Morgan Kaufmann, 575–582.

Weydert, Emil: 1998, 'System JZ: How to Build a Canonical Ranking Model of a Default Knowledge Base', in A.G. Cohn, L. Schubert and S.C. Shapiro (eds.), *Principles of Knowledge Representation and Reasoning. Proceedings of the Sixth International Conference (KR'98)*, Morgan Kaufmann, pp. 190–201.

Williams, Mary-Anne: 1995, 'Iterated theory base change: A computational model', in *IJCAI-95. Proceedings of the 14th International Joint Conference on Artificial Intelligence*, Vol. 2, San Francisco, Morgan Kaufmann, pp. 1541–1547.

Appendix 1: Proofs

The following preparatory lemma will be helpful in the proofs.

Lemma 0
Let $\circ$ an operation of revision by comparison that satisfies (C1) – (C6). Then:

(a) $K \circ_\alpha \beta \subseteq (K \circ_{\alpha\wedge\neg\beta} \bot) + \beta$.

(b) $K \circ_{\alpha\wedge\neg\beta} \bot \subseteq K \circ_\alpha \beta$.

(c) If $\alpha \in K \circ_\alpha \beta$ then $\alpha \in K \circ_{\alpha\wedge\neg\beta} \bot$.

(d) If $\alpha \in K \circ_\alpha \bot$, then $K \circ_\alpha \bot = K_\bot$.

(e) If $\alpha \in K \circ_\alpha \bot$, then $K \circ_{\alpha\wedge\beta} \bot = K \circ_\beta \bot$.

(f) If $\alpha \notin K \circ_\beta \bot$, then $K \circ_{\alpha\wedge\beta} \bot = K \circ_\alpha \bot$.

(g) If $K \circ_\beta \bot = K \circ_\gamma \bot$, then $K \circ_{\alpha\wedge\beta} \bot = K \circ_{\alpha\wedge\gamma} \bot = K \circ_{\alpha\wedge\beta\wedge\gamma} \bot$.

(h) If $\alpha \notin K \circ_{\alpha\wedge\gamma} \bot$, then $K \circ_{\alpha\wedge\gamma} \bot = K \circ_\alpha \bot$.

(i) If $\alpha \in K \circ_{\alpha\wedge\gamma} \bot$, then $K \circ_{\alpha\wedge\gamma} \bot = K \circ_\gamma \bot$.

(j) If $\alpha \in K \circ_\beta \bot$ and $\beta \notin K \circ_\beta \bot$, then $K \circ_{\alpha\wedge\beta} \bot = K \circ_\beta \bot$.

Proof of Lemma 0.

(a), **(b)** and **(c)** trivially follow from (C5) and (C6).

(d) Let $\alpha \in K \circ_\alpha \bot$. Then it follows by (C2) that $\alpha \in K \circ_{\alpha\wedge\neg\bot} \bot$. Hence by (C5) $K \circ_\alpha \bot = (K \circ_{\alpha\wedge\neg\bot} \bot) + \bot = K_\bot$.

(e) Let $\alpha \in K \circ_\alpha \bot$. It follows by (C4) that $\alpha \in K \circ_{\alpha\wedge\beta} \bot$. If $\beta \in K \circ_{\alpha\wedge\beta} \bot$ it follows by (C1) and (C4) and the previous result (d) that $K \circ_{\alpha\wedge\beta} \bot = K_\bot = K \circ_\beta \bot$. If $\beta \notin K \circ_{\alpha\wedge\beta} \bot$, then by (C4) $\beta \notin K \circ_\beta \bot$, from which it follows by (C1) that $\alpha\wedge\beta \notin K \circ_\beta \bot$. Hence two-fold application of (C3) yields $K \circ_{\alpha\wedge\beta} \bot = K \circ_\beta \bot$.

(f) Let $\alpha \notin K \circ_\beta \bot$. Then by (C3), $K \circ_\beta \bot \subseteq K \circ_\alpha \bot$. It follows by (C1) that $\alpha \wedge \beta \notin K \circ_\beta \bot$. Then by (C3) $K \circ_\beta \bot \subseteq K \circ_{\alpha\wedge\beta} \bot$. It follows by (C4) that $\alpha \notin K \circ_\alpha \bot$, then by (C1) and (C3) $K \circ_\alpha \bot \subseteq K \circ_{\alpha\wedge\beta} \bot$. By (C4) and (C1) either $\alpha \notin K \circ_{\alpha\wedge\beta} \bot$ or $\beta \notin K \circ_{\alpha\wedge\beta} \bot$. Then by (C3) $K \circ_{\alpha\wedge\beta} \bot \subseteq K \circ_\alpha \bot$ or $K \circ_{\alpha\wedge\beta} \bot \subseteq K \circ_\beta \bot$. Hence $K \circ_{\alpha\wedge\beta} \bot = K \circ_\alpha \bot$.

(g) Let $K \circ_\beta \bot = K \circ_\gamma \bot$. We will prove firstly that $K \circ_{\alpha\wedge\beta} \bot = K \circ_{\alpha\wedge\gamma} \bot$. We have the following cases:

(1) $\beta \in K \circ_\beta \bot$. Then by (d) $\gamma \in K \circ_\gamma \bot$. Hence by previous (e) $K \circ_{\alpha\wedge\gamma} \bot = K \circ_\alpha \bot = K \circ_{\alpha\wedge\beta} \bot$.

(2) $\beta \notin K \circ_\beta \bot$. Then by (d) $\gamma \notin K \circ_\gamma \bot$.

(2.1) If $\alpha \notin K \circ_\beta \bot$, then $\alpha \notin K \circ_\gamma \bot$. Hence by previous (f) $K \circ_{\alpha\wedge\beta} \bot = K \circ_\alpha \bot = K \circ_{\alpha\wedge\gamma} \bot$.

(2.2) If $\alpha \in K \circ_\beta \bot$, then $\alpha \in K \circ_\gamma \bot$. Since by (C1) $\alpha \wedge \beta \notin K \circ_\beta \bot$ and $\alpha\wedge\gamma \notin K \circ_\gamma \bot$, (C3) yields that $K \circ_\beta \bot \subseteq K \circ_{\alpha\wedge\beta} \bot$ and $K \circ_\gamma \bot \subseteq K \circ_{\alpha\wedge\gamma} \bot$. So $\alpha \in K \circ_{\alpha\wedge\beta} \bot$ and $\alpha \in K \circ_{\alpha\wedge\gamma} \bot$. Due to (C4) $\beta \notin K \circ_{\alpha\wedge\beta} \bot$ and $\gamma \notin K \circ_{\alpha\wedge\gamma} \bot$. Hence by (C3) $K \circ_{\alpha\wedge\beta} \bot = K \circ_\beta \bot = K \circ_\gamma \bot = K \circ_{\alpha\wedge\gamma} \bot$.
Secondly, applying exactly the same argument as before to $K \circ_{\alpha\wedge\beta} \bot = K \circ_{\alpha\wedge\gamma} \bot$, we get that $K \circ_{\alpha\wedge\beta\wedge\gamma} \bot = K \circ_{\alpha\wedge\gamma\wedge\gamma} \bot = K \circ_{\alpha\wedge\gamma} \bot$, by (C1).

(h) Let $\alpha \notin K \circ_{\alpha\wedge\gamma} \bot$. By (C4) $\alpha \notin K \circ_\alpha \bot$. Then by (C1) $\alpha \wedge \gamma \notin K \circ_\alpha \bot$. Then by two-fold application of (C3), $K \circ_{\alpha\wedge\gamma} \bot = K \circ_\alpha \bot$.

(i) Let $\alpha \in K \circ_{\alpha\wedge\gamma} \bot$. (1) $\gamma \in K \circ_{\alpha\wedge\gamma} \bot$. Then by (C1) and (C4) $\gamma \in K \circ_\gamma \bot$. Hence by (d) $K \circ_{\alpha\wedge\gamma} \bot = K \circ_\gamma \bot = K_\bot$. (2) $\gamma \notin K \circ_{\alpha\wedge\gamma} \bot$. Then by (h) $K \circ_{\alpha\wedge\gamma} \bot = K \circ_\gamma \bot$.

(j) Let $\alpha \in K \circ_\beta \bot$ and $\beta \notin K \circ_\beta \bot$. By (C1) $\alpha \wedge \beta \notin K \circ_\beta \bot$. Then by (C3) $K \circ_\beta \bot \subseteq K \circ_{\alpha\wedge\beta} \bot$, from which it follows by (C1) and (C4) that $\beta \notin K \circ_{\alpha\wedge\beta} \bot$. Hence by (C3), $K \circ_{\alpha\wedge\beta} \bot = K \circ_\beta \bot$. ■

Proof of Observation 1

We refer to the three lines of (Def $\preceq'$ from $\preceq$) as (1), (2) and (3), without further specification.

First we show that $\preceq'$ is complete, i.e., that for any worlds w_1 and w_2, we have that either $w_1 \preceq' w_2$ or $w_2 \preceq' w_1$.

If w_1 and w_2 are both β-worlds, then $w_1 \preceq' w_2$ or $w_2 \preceq' w_1$ by (2) and the completeness of $\preceq$.

So let without loss of generality w_2 be a $\neg\beta$-world. If $w_1 \preceq w_{\neg\alpha}$, then by (3) $w_1 \preceq' w_2$, and we are done.

So suppose that $w_{\neg\alpha} \prec w_1$. If $w_2 \preceq w_1$, then by (1) $w_2 \preceq' w_1$, and we are done.

So suppose finally that $w_1 \prec w_2$. Then $w_{\neg\alpha} \prec w_1 \prec w_2$, so $w_{\neg\alpha} \prec w_2$. Since $w_1 \prec w_2$, we get $w_1 \preceq' w_2$, by (1), and we are done.

Second, we show that $\preceq'$ is transitive. So assume that $w_1 \preceq' w_2$ and $w_2 \preceq' w_3$. By definition (Def $\preceq'$ from $\preceq$), the latter means that one of the following is true:

(a) $w_2 \preceq w_3$ and $w_{\neg\alpha} \preceq w_3$
(b) $w_2 \preceq w_3$ and w_2 and w_3 are both β-worlds
(c) $w_2 \preceq w_{\neg\alpha}$ and w_3 is a $\neg\beta$-world

If (a) is true, then we get from $w_1 \preceq' w_2$, (Def $\preceq'$ from $\preceq$) and the transitivity of $\preceq$ that in any case $w_1 \preceq w_3$ and $w_{\neg\alpha} \preceq w_3$, so by (1), $w_1 \preceq' w_3$.

If (b) is true, then we have to have a closer look at how $w_1 \preceq' w_2$ came about. If $w_1 \preceq w_2$ and $w_{\neg\alpha} \preceq w_2$, then we get from the transitivity of $\preceq$ that $w_1 \preceq w_3$ and $w_{\neg\alpha} \preceq w_3$, so by (1), $w_1 \preceq' w_3$.
If $w_1 \preceq w_2$ and both w_1 and w_2 are β-worlds, then we get from the transitivity of $\preceq$ that $w_1 \preceq w_3$ and both w_1 and w_3 are β-worlds, so by (2), $w_1 \preceq' w_3$.
The case where $w_1 \preceq w_{\neg\alpha}$ and w_2 is a $\neg\beta$-world is not possible, since w_2 has already been assumed to be a β-world.

If (c) is true, then we get from $w_1 \preceq' w_2$, (Def $\preceq'$ from $\preceq$) and the transitivity of $\preceq$ that in any case $w_1 \preceq w_{\neg\alpha}$ and w_3 is a $\neg\beta$-world, so by (3), $w_1 \preceq' w_3$. ■

Proof of Theorem 2

We show that (Def $\leq'$ from $\leq$) captures in terms of entrenchment the semantical recipe to construct revisions-by-comparison. In the following, let $[\alpha]$ denote the set of all worlds that satisfy α.

Let $\$'$ be the system of spheres and $\leq'$ be the entrenchment relation associated with $K \circ_\alpha \beta$ Then we have

$\gamma \leq' \delta$ iff

iff for all $S \in \$'$, if $S \subseteq [\gamma]$, then $S \subseteq [\delta]$

$$\text{iff for all } S \in \$, \begin{cases} \text{If } S \subseteq [\alpha] \text{ and } S \cap [\beta] \subseteq [\gamma], \text{ then } S \cap [\beta] \subseteq [\delta] \text{ and} \\ \text{If } S \not\subseteq [\alpha] \text{ and } S \subseteq [\gamma], \text{ then } S \subseteq [\delta] \end{cases}$$

$$\text{iff for all } S \in \$, \begin{cases} \text{If } S \subseteq [\alpha] \cap [\beta \to \gamma], \text{ then } S \subseteq [\beta \to \delta] \quad \text{(I) and} \\ \text{If } S \not\subseteq [\alpha] \text{ and } S \subseteq [\gamma], \text{ then } S \subseteq [\delta] \qquad \text{(II)} \end{cases}$$

Now let us distinguish cases.

Case 1: $\gamma \leq \alpha$, i.e., for all $S \in \$$, if $S \subseteq [\gamma]$, then $S \subseteq [\alpha]$.

In this case, (II) is vacuously satisfied, and $\gamma \leq' \delta$ reduces to (I), which means by definition that

$\alpha \wedge (\beta \to \gamma) \leq \beta \to \delta$.

Case 2: $\alpha < \gamma$, i.e., there is an $S_0 \in \$$ such that $S_0 \subseteq [\gamma]$ and $S_0 \not\subseteq [\alpha]$.

In this case, $S \subseteq [\alpha]$ implies that $S \subset S_0$, since $\$$ is nested, and that $S \subseteq [\gamma]$.

We show that in this case for every $S \in \$$, the conjunction of (I) and (II) is equivalent with the claim that for all $S \in \$$,

if $S \subseteq [\gamma]$, then $S \subseteq [\delta]$ (II$'$)

First, let $S \in \$$ and suppose that (II$'$) holds. Then clearly (II). For (I), suppose that $S \subseteq [\alpha]$. We have just seen that in this case it follows that $S \subseteq [\gamma]$. Hence by (II$'$), $S \subseteq [\delta] \subseteq [\beta \to \delta]$, as needed for (I).

Conversely, let $S \in \$$ and suppose that (I) and (II) hold. If in addition $S \not\subseteq [\alpha]$ then (II) implies (II′). Moreover, (II) implies that $S_0 \subseteq [\delta]$. So, on the other hand, if in addition $S \subseteq [\alpha]$ then $S \subseteq S_0 \subseteq [\delta]$, as needed for (II′).

(II′) means by definition that $\gamma \leq \delta$.

In sum then, then, we get precisely (Def $\leq'$ from $\leq$). ■

Proof of Observation 3

(E1). Let $\gamma \leq' \delta$ and $\delta \leq' \phi$. Then we have four cases, according to the definition of $\leq'$:

(1) $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \delta)$ and $\alpha \wedge (\beta \to \delta) \leq (\beta \to \phi)$. Due to (E3) we have two subcases:

(1.1) $\alpha \leq \alpha \wedge (\beta \to \delta)$, from which it follows by (E1) that $\alpha \leq (\beta \to \phi)$. By (E2), $\alpha \wedge (\beta \to \gamma) \leq \alpha$, then by (E1), $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \phi)$. Hence, by definition of $\leq'$, $\gamma \leq' \phi$.

(1.2) $(\beta \to \delta) \leq \alpha \wedge (\beta \to \delta)$. Then we have $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \delta) \leq \alpha \wedge (\beta \to \delta) \leq (\beta \to \phi)$, and by (E1) it follows that $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \phi)$. Hence, by definition of $\leq'$, since $\gamma \leq \alpha$, $\gamma \leq' \phi$.

(2) $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \delta)$ and $\delta \leq \phi$. It follows from definition that $\gamma \leq \alpha$ and $\alpha < \delta$. Then by E1, $\alpha < \phi$, and $\gamma \leq \phi$, hence by the definition of $\leq'$, $\gamma \leq' \phi$.

(3) $\gamma \leq \delta$ and $\alpha \wedge (\beta \to \delta) \leq (\beta \to \phi)$. We prove that this is not a valid case. By definition, $\alpha < \gamma$, and $\delta \leq \alpha$, then (E1) yields $\delta < \gamma$, contradiction.

(4) $\gamma \leq \delta$ and $\delta \leq \phi$. Then, by definition, $\delta \leq \alpha$ and $\gamma, \leq \alpha$, and by (E1) we obtain $\gamma \leq \phi$. Hence $\gamma \leq' \phi$.

(E2). Let $\gamma \vdash \delta$. We have two cases:

(1) $\gamma \leq \alpha$. Since $\gamma \vdash \delta$, $(\beta \to \gamma) \vdash (\beta \to \delta)$, then by (E2) $(\beta \to \gamma) \leq (\beta \to \delta)$, and, again by (E2) $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \gamma)$. Then, by (E1), $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \delta)$, hence $\gamma \leq' \delta$.

(2) $\alpha < \gamma$. By (E2), $\gamma \leq \delta$. Hence $\gamma \leq' \delta$.

(E3). We divide the proof in cases:

(1) Let $\alpha < \delta$. We have two subcases:

(1.1) Let $\gamma \leq \alpha$. Then by (E1) and (E2) $\gamma \wedge \delta \leq \alpha$. By (E2), $\delta \leq (\beta \to \delta)$. Then (E1) yields $\alpha \leq (\beta \to \delta)$. By (E2) $\alpha \wedge (\beta \to \gamma) \leq \alpha$, so by (E1) $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \delta)$. By (E2) $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \gamma)$. Hence by (E1) and (E3) $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \gamma) \wedge (\beta \to \delta)$, which is equivalent to $\alpha \wedge (\beta \to \gamma) \leq (\beta \to \gamma \wedge \delta)$, hence $\gamma \leq' \gamma \wedge \delta$ by (Def $\leq'$ from $\leq$).

(1.2) Let $\alpha < \gamma$. Then $\alpha < \gamma \wedge \delta$. By (E3) either $\gamma \leq \gamma \wedge \delta$ or $\delta \leq \gamma \wedge \delta$, hence, by (Def $\leq'$ from $\leq$), either $\gamma \leq' \gamma \wedge \delta$ or $\delta \leq' \gamma \wedge \delta$.

(2) Let $\gamma \leq \alpha$ and $\delta \leq \alpha$, then $\gamma \wedge \delta \leq \alpha$. By (E3), $(\beta \rightarrow \gamma) \leq (\beta \rightarrow \gamma) \wedge (\beta \rightarrow \delta)$ or $(\beta \rightarrow \delta) \leq (\beta \rightarrow \gamma) \wedge (\beta \rightarrow \delta)$, which is equivalent to $(\beta \rightarrow \gamma) \leq (\beta \rightarrow \gamma \wedge \delta)$ or $(\beta \rightarrow \delta) \leq (\beta \rightarrow \gamma \wedge \delta)$. By (E2), $\alpha \wedge (\beta \rightarrow \gamma) \leq (\beta \rightarrow \gamma)$ and $\alpha \wedge (\beta \rightarrow \delta) \leq (\beta \rightarrow \delta)$. Then, by (E1), $\alpha \wedge (\beta \rightarrow \gamma) \leq (\beta \rightarrow \gamma \wedge \delta)$ or $\alpha \wedge (\beta \rightarrow \delta) \leq (\beta \rightarrow \gamma \wedge \delta)$, hence by (Def $\leq'$ from $\leq$), $\gamma \leq' \gamma \wedge \delta$ or $\delta \leq' \gamma \wedge \delta$. ■

Proof of Lemma 4

Let $\alpha \leq \beta$. If $\alpha < \gamma$, the claim is immediate from (Def $\leq'$ from $\leq$). So let $\gamma \leq \alpha$. Then by (Def $\leq'$ from $\leq$), $\gamma \leq' \delta$ means that $\alpha \wedge (\beta \rightarrow \gamma) \leq \beta \rightarrow \delta$.

We first show that this condition implies $\gamma \leq \delta$. On the one hand, we have $\gamma \leq \alpha$ and $\alpha \leq \beta$, so by transitivity we get $\gamma \leq \beta$. On the other hand, we have $\gamma \leq \alpha$ and by Dominance $\gamma \leq (\beta \rightarrow \gamma)$, so by $\alpha \wedge (\beta \rightarrow \gamma) \leq \beta \rightarrow \delta$, we get $\gamma \leq \beta \rightarrow \delta$. By the entrenchment properties, $\gamma \leq \beta$ and $\gamma \leq \beta \rightarrow \delta$ taken together give $\gamma \leq \delta$.

To show the converse, suppose that $\gamma \leq \delta$. This, taken together with the supposition that $\alpha \leq \beta$ and the entrenchment properties, allows us to form the following chain: $\alpha \wedge (\beta \rightarrow \gamma) \leq \beta \wedge (\beta \rightarrow \gamma) \leq \gamma \leq \delta \leq \beta \rightarrow \delta$, so by transitivity, $\alpha \wedge (\beta \rightarrow \gamma) \leq \beta \rightarrow \delta$, and we are done. ■

Proof of Lemma 5

Let $\alpha \notin K \circ_{\perp} \perp$. Then by (C4) and (C1), we know that $\alpha \wedge \neg\beta \notin K \circ_{\alpha\wedge\neg\beta} \perp$, so by (C1), $\perp \notin K \circ_{\alpha\wedge\neg\beta} \perp$. Therefore, by (C3), $K \circ_{\alpha\wedge\neg\beta} \perp \subseteq K \circ_{\perp} \perp$. Thus $\alpha \notin K \circ_{\alpha\wedge\neg\beta} \perp$, and by (C6), we get $K \circ_{\alpha} \beta = K \circ_{\alpha\wedge\neg\beta} \perp$. As a first result, we thus have $K \circ_{\alpha} \beta \subseteq K \circ_{\perp} \perp$. For the converse inclusion, we first note that by (C1), $\alpha \wedge \neg\beta \notin K \circ_{\perp} \perp$. By (C3), then, $K \circ_{\perp} \perp \subseteq K \circ_{\alpha\wedge\neg\beta} \perp$, and since we already showed that $K \circ_{\alpha} \beta = K \circ_{\alpha\wedge\neg\beta} \perp$, we have $K \circ_{\perp} \perp \subseteq K \circ_{\alpha} \beta$. In sum then, $K \circ_{\alpha} \beta = K \circ_{\perp} \perp$, as desired.

Proof of Observation 6

Let $\circ$ satisfy (C1) – (C6), and define $K = K \circ_{\perp} \perp$.

(K∗2). Let $\alpha \in K \circ_{\alpha} \beta$. Then by (Lemma 0(c)) $\alpha \in K \circ_{\alpha\wedge\neg\beta} \perp$. Then by (C5) $K \circ_{\alpha} \beta = (K \circ_{\alpha\wedge\neg\beta} \perp) + \beta$. Hence $\beta \in K \circ_{\alpha} \beta$.

(K∗3). Let $\alpha \notin K$, then by Vacuity, $K \circ_{\alpha} \beta = K \subseteq K + \beta$. If $\neg\beta \in K$, then $K \circ_{\alpha} \beta \subseteq K + \beta = K_{\perp}$. Let $\alpha \in K$ and $\neg\beta \notin K$. Then by (C1) $\alpha\wedge\neg\beta \notin K \circ_{\alpha\wedge\neg\beta} \perp =$ (by Vacuity) $= K$. Then by (C5), due to $\alpha \in K \circ_{\alpha\wedge\neg\beta} \perp$, $K \circ_{\alpha} \beta = (K \circ_{\alpha\wedge\neg\beta} \perp) + \beta = K + \beta$.

(K∗4). Let $\neg\beta \notin K$. If $\alpha \notin K$, by Vacuity $K = K \circ_\alpha \beta$. If $\alpha \in K$, we have proven in (K∗3) that $K \circ_\alpha \beta = K + \beta$. Hence $K \subseteq K \circ_\alpha \beta$.

(K∗5). For one direction let $\alpha \wedge \neg\beta \in K \circ_{\alpha\wedge\neg\beta} \bot$ and the rest follows from Lemma 0(d). For the other direction, let $K \circ_\alpha \beta = K_\bot$. Then $\alpha \in K \circ_\alpha \beta$ and by (Lemma 0(c)) $\alpha \in K \circ_{\alpha\wedge\neg\beta} \bot$ from which it follows by (C5) that $K \circ_\alpha \beta = (K \circ_{\alpha\wedge\neg\beta} \bot) + \beta$. Hence, by (C1), $\neg\beta \in K \circ_{\alpha\wedge\neg\beta} \bot$ and, by (C1) again, $\alpha \wedge \neg\beta \in K \circ_{\alpha\wedge\neg\beta} \bot$.

(K∗7). We split the proof in cases:

(1) $\alpha \wedge \neg(\beta \wedge \gamma) \notin K \circ_{\alpha\wedge\neg(\beta\wedge\gamma)} \bot$. Then by (C1) $\alpha \wedge \neg\beta \notin K \circ_{\alpha\wedge\neg(\beta\wedge\gamma)} \bot$. It follows by (C3) that $K \circ_{\alpha\wedge\neg(\beta\wedge\gamma)} \bot \subseteq K \circ_{\alpha\wedge\neg\beta} \bot$. If $\alpha \notin K \circ_{\alpha\wedge\neg\beta} \bot$, then $K \circ_\alpha (\beta \wedge \gamma)$ =(by (C6)) $K \circ_{\alpha\wedge\neg(\beta\wedge\gamma)} \bot \subseteq K \circ_{\alpha\wedge\neg\beta} \bot = K \circ_\alpha \beta \subseteq (K \circ_\alpha \beta) + \gamma$. If $\alpha \in K \circ_{\alpha\wedge\neg\beta} \bot$, then $K \circ_\alpha (\beta \wedge \gamma) \subseteq (K \circ_{\alpha\wedge\neg(\beta\wedge\gamma)} \bot) + (\beta \wedge \gamma) \subseteq ((K \circ_{\alpha\wedge\neg\beta} \bot) + \beta) + \gamma$ =(by (C6)) $(K \circ_\alpha \beta) + \gamma$.

(2) $\alpha\wedge\neg(\beta\wedge\gamma) \in K\circ_{\alpha\wedge\neg(\beta\wedge\gamma)}\bot$. It follows by (C1) and (C4) that $\alpha \in K\circ_{\alpha\wedge\neg\beta}\bot$, then by (C5) $K\circ_\alpha\beta = (K\circ_{\alpha\wedge\neg\beta}\bot)+\beta$. Then $(K\circ_\alpha\beta)+\gamma = ((K\circ_{\alpha\wedge\neg\beta}\bot)+\beta)+\gamma$. By (C1) and (C4), $\neg(\beta \wedge \gamma) \in K \circ_{\alpha\wedge\neg\beta} \bot$. Hence $(K \circ_\alpha \beta) + \gamma = K_\bot$, from which it follows that $K \circ_\alpha (\beta \wedge \gamma) \subseteq (K \circ_\alpha \beta) + \gamma$.

(K∗8). Let $\neg\gamma \notin K \circ_\alpha \beta$. We split the proof in cases:

(1) $\alpha \in K \circ_{\alpha\wedge\neg\beta} \bot$. Then by (C5) $K \circ_\alpha \beta = (K \circ_{\alpha\wedge\neg\beta} \bot) + \beta$. Then $\neg(\beta \wedge \gamma) \notin K\circ_{\alpha\wedge\neg\beta}\bot$, and $\alpha\wedge\neg(\beta\wedge\gamma) \notin K\circ_{\alpha\wedge\neg\beta}\bot$, from which it follows by (C3) that $\alpha \in K \circ_{\alpha\wedge\neg\beta} \bot \subseteq K \circ_{\alpha\wedge\neg(\beta\wedge\gamma)} \bot$. Then $(K \circ_{\alpha\wedge\neg\beta} \bot) + \beta \subseteq (K \circ_{\alpha\wedge\neg(\beta\wedge\gamma)} \bot) + (\beta \wedge \gamma)$. Hence, by (C5), $K \circ_\alpha \beta \subseteq K \circ_\alpha (\beta \wedge \gamma)$.

(2) $\alpha \notin K \circ_{\alpha\wedge\neg\beta} \bot$. Then by (C6) $K \circ_\alpha \beta = K \circ_{\alpha\wedge\neg\beta} \bot$. By (C1) $\alpha \wedge \neg(\beta \wedge \gamma) \notin K \circ_{\alpha\wedge\neg\beta} \bot$. Hence $K \circ_\alpha \beta = K \circ_{\alpha\wedge\neg\beta} \bot \subseteq$ (by (C3)) $\subseteq K \circ_{\alpha\wedge\neg(\beta\wedge\gamma)} \bot \subseteq$ (by Lemma 0(b)) $\subseteq K \circ_\alpha (\beta \wedge \gamma)$.

(K∗7&8). We split the proof in cases:

(1) $\neg\beta \notin K \circ_{\neg\beta} \bot$ and $\neg\gamma \in K \circ_{\neg\beta} \bot$: It follows by (C1) that $\alpha \wedge \neg\beta \wedge \neg\gamma \notin K\circ_{\neg\beta}\bot$. Then (C3) yields $\neg\gamma \in K\circ_{\alpha\wedge\neg\beta\wedge\neg\gamma}\bot$. By (C1) and (C4) it follows that $\alpha\wedge\neg\beta \notin K\circ_{\alpha\wedge\neg\beta\wedge\neg\gamma}\bot$. Since by (C1) $\alpha\wedge\neg\beta \notin K\circ_{\neg\beta}\bot$ it follows by (C4) that $\alpha\wedge\neg\beta \notin K\circ_{\alpha\wedge\neg\beta}\bot$ and by (C1) that $\alpha\wedge\neg\beta\wedge\neg\gamma \notin K\circ_{\alpha\wedge\neg\beta}\bot$. Then by two-fold application of (C3) $K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \bot = K \circ_{\alpha\wedge\neg\beta} \bot$. If $\alpha \in K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \bot$, it follows that $K \circ_\alpha (\beta \vee \gamma) = (K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \bot) + (\beta \vee \gamma) =$ (due to $\neg\gamma \in K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \bot$) $= (K\circ_{\alpha\wedge\neg\beta\wedge\neg\gamma}\bot)+\beta = (K\circ_{\alpha\wedge\neg\beta}\bot)+\beta = K\circ_\alpha\beta$, by (C5). If $\alpha \notin K\circ_{\alpha\wedge\neg\beta\wedge\neg\gamma}\bot$, it follows by (C6) that $K \circ_\alpha (\beta \vee \gamma) = K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \bot = K \circ_{\alpha\wedge\neg\beta} \bot = K \circ_\alpha \beta$.
(2) $\neg\gamma \notin K \circ_{\neg\gamma} \bot$ and $\neg\beta \in K \circ_{\neg\gamma} \bot$. Due to the symmetry of the case, it follows that $K \circ_\alpha (\beta \vee \gamma) = K \circ_\alpha \gamma$.

(3) $\neg\beta \in K \circ_{\neg\beta} \bot$ or $\neg\gamma \notin K \circ_{\neg\beta} \bot$; and $\neg\gamma \in K \circ_{\neg\gamma} \bot$ or $\neg\beta \notin K \circ_{\neg\gamma} \bot$. Due to (C4), we have two possibilities: (3.1) $\neg\beta \in K \circ_{\neg\beta} \bot$ and $\neg\gamma \in K \circ_{\neg\gamma} \bot$ or (3.2) $\neg\gamma \notin K \circ_{\neg\beta} \bot$ and $\neg\beta \notin K \circ_{\neg\gamma} \bot$.

(3.1) By (C4) $\neg\beta$ and $\neg\gamma \in K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \perp$. Then it follows by Lemma 0(i) that $K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \perp = K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_{\alpha\wedge\neg\gamma} \perp$. If $\alpha \notin K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \perp$, it follows by (C6) that $K \circ_\alpha (\beta \vee \gamma) = K \circ_\alpha \beta = K \circ_\alpha \gamma$. Now let $\alpha \in K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \perp$. We know that $K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \perp + (\beta \vee \gamma) = K \circ_{\alpha\wedge\neg\beta} \perp + \beta = K_\perp = K \circ_{\alpha\wedge\neg\gamma} \perp + \gamma$. Hence, by (C5), $K \circ_\alpha (\beta \vee \gamma) = K \circ_\alpha \beta = K \circ_\alpha \gamma$.
(3.2) It follows by (C3) that $K \circ_{\neg\beta} \perp = K \circ_{\neg\gamma} \perp$. Then by Lemma 0(g) $K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \perp = K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_{\alpha\wedge\neg\gamma} \perp$. Then $(K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \perp) + (\beta \vee \gamma) = (K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \perp) + \beta \cap (K \circ_{\alpha\wedge\neg\beta\wedge\neg\gamma} \perp) + \gamma = (K \circ_{\alpha\wedge\neg\beta} \perp) + \beta \cap (K \circ_{\alpha\wedge\neg\gamma} \perp) + \gamma$. Hence, by (C5) and (C6), $K \circ_\alpha (\beta \vee \gamma) = K \circ_\alpha \beta \cap K \circ_\alpha \gamma$.

(K$\dot{-}$8$^+$). Let $\alpha \notin K \circ_{\alpha\wedge\beta} \gamma$, By Lemma 0(b) $\alpha \notin K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp$, and by (C1), $\alpha \wedge \beta \notin K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp$. So by (C6), $K \circ_{\alpha\wedge\beta} \gamma = K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp$. By (C1), $\alpha \wedge \neg\gamma \notin K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp$. Then by Lemma 0(h), $K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp = K \circ_{\alpha\wedge\neg\gamma} \perp$. α is not in this set. Hence by (C6), $K \circ_\alpha \gamma = K \circ_{\alpha\wedge\neg\gamma} \perp$. Hence $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\alpha \gamma$.

(K$\dot{-}$D). In the proof of (K$\dot{-}$8$^+$) we have proved that $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\alpha \gamma$ if $\alpha \notin K \circ_{\alpha\wedge\beta} \gamma$. Similarly, if $\beta \notin K \circ_{\alpha\wedge\beta} \gamma$, then $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\beta \gamma$. Let $\alpha \wedge \beta \in K \circ_{\alpha\wedge\beta} \gamma$. Then it follows by Lemma 0(c) that $\alpha \wedge \beta \in K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp$ and by (C5) $K \circ_{\alpha\wedge\beta} \gamma = (K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp) + \gamma$.

(1) If $\neg\gamma \in K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp$, it follows by (C1), (C4) and Lemma 0(d) that $K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp = K \circ_{\alpha\wedge\neg\gamma} \perp = K \circ_{\beta\wedge\neg\gamma} \perp = K_\perp$, from which it follows by (C5) that $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\alpha \gamma = K \circ_\beta \gamma$.

(2) If $\neg\gamma \notin K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp$, it follows by (C1) that $\alpha \wedge \neg\gamma \notin K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp$ and $\beta \wedge \neg\gamma \notin K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp$. Then by Lemma 0(h) $K \circ_{\alpha\wedge\beta\wedge\neg\gamma} \perp = K \circ_{\alpha\wedge\neg\gamma} \perp = K \circ_{\beta\wedge\neg\gamma} \perp$. Hence by (C5), $K \circ_{\alpha\wedge\beta} \gamma = K \circ_\alpha \gamma = K \circ_\beta \gamma$.

■

Proof of Observation 7

(Q1). Let $\alpha \in K \circ_\beta \perp$ and $\alpha \notin K \circ_\alpha \perp$. Then, by (C1), $\alpha \wedge \beta \notin K \circ_\alpha \perp$. From (C3) it follows that $K \circ_\alpha \perp \subseteq K \circ_{\alpha\wedge\beta} \perp$.

(1) If $\beta \in K \circ_\beta \perp$, then by Lemma 0 (d), $K \circ_\alpha \perp \subseteq K \circ_\beta \perp = K_\perp$.

(2) If $\beta \notin K \circ_\beta \perp$, then by (C1) and (C3) $K \circ_\beta \perp \subseteq K \circ_{\alpha\wedge\beta} \perp$, so $\alpha \in K \circ_{\alpha\wedge\beta} \perp$, so by (C4) $\beta \notin K \circ_{\alpha\wedge\beta} \perp$, from which it follows that $K \circ_{\alpha\wedge\beta} \perp = K \circ_\beta \perp$. Hence $K \circ_\alpha \perp \subseteq K \circ_\beta \perp$.

(Q2). We first show that $\alpha \in K \circ_\alpha \neg\beta$ iff $\alpha \in K \circ_{\alpha\wedge\beta} \perp$. Let $\alpha \in K \circ_\alpha \neg\beta$. The assumption that $\alpha \notin K \circ_{\alpha\wedge\beta} \perp$ implies by (C6) that $K \circ_\alpha \neg\beta = K \circ_{\alpha\wedge\beta} \perp$ and we have a contradiction. Therefore $\alpha \in K \circ_{\alpha\wedge\beta} \perp$. For the converse, let $\alpha \in K \circ_{\alpha\wedge\beta} \perp$. Then, by Lemma 0(b), $\alpha \in (K \circ_{\alpha\wedge\beta} \perp) + \neg\beta = K \circ_\alpha \neg\beta$. Now we show that $\alpha \in K \circ_\beta \neg\alpha$ iff $\alpha \in K \circ_{\alpha\wedge\beta} \perp$. Let $\alpha \in K \circ_\beta \neg\alpha$, then by Lemma 0(a) $\alpha \in (K \circ_{\alpha\wedge\beta} \perp) + \neg\alpha$, hence $\alpha \in K \circ_{\alpha\wedge\beta} \perp$. For the converse, let $\alpha \in K \circ_{\alpha\wedge\beta} \perp$, by Lemma 0(b), $\alpha \in K \circ_\beta \neg\alpha$.

(Q3). Let $\gamma \in K \circ_\alpha \neg\gamma$. Then, by Lemma 0(a) $\gamma \in (K \circ_{\alpha\wedge\gamma} \perp) + \neg\gamma$, then $\gamma \in K \circ_{\alpha\wedge\gamma} \perp$. By Lemma 0(i), $K \circ_{\alpha\wedge\gamma} \perp = K \circ_\alpha \perp$. If $\alpha \in K \circ_\alpha \perp$, by (C1), $\alpha \wedge \gamma \in K \circ_{\alpha\wedge\gamma} \perp$, then by (C4) and (C1) $\gamma \in K \circ_\alpha \beta$. Now let $\alpha \notin K \circ_\alpha \perp$, then $\alpha \wedge \neg\beta \notin K \circ_\alpha \perp$, then by (C3) $K \circ_\alpha \perp \subseteq K \circ_{\alpha\wedge\neg\beta} \perp$. Hence by Lemma 0 (b) $\gamma \in K \circ_\alpha \perp \subseteq K \circ_\alpha \beta$.

(Q4). Let $\alpha \in K \circ_\alpha \perp$. By (C4), $\alpha \in K \circ_\alpha \beta$, hence (K$*$2) yields $\beta \in K \circ_\alpha \beta$.

(Q5). Follows trivially from (C1) and (K$*$2)

(Q6). See Lemma 0(b).

For the second part, we have two cases: (1) Let $\alpha \notin K \circ_{\neg\beta} \perp$: Then it follows by Lemma 0(f) that $K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_\alpha \perp$ and by (C4) that $\alpha \notin K \circ_\alpha \perp$. Hence by (C6) $K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_\alpha \beta$. (2) Let $\neg\beta \in K \circ_{\neg\beta} \perp$: Then it follows by Lemma 0(e) that $K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_\alpha \perp$. If $\alpha \notin K \circ_\alpha \perp$ then (C6) $K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_\alpha \beta$. If $\alpha \in K \circ_\alpha \perp$ then (C1) and (C4) yield $\alpha \wedge \neg\beta \in K \circ_{\alpha\wedge\neg\beta} \perp$, from which it follows by Lemma 0(d) that $K \circ_{\alpha\wedge\neg\beta} \perp = K_\perp$. Hence by Lemma 0(b) $K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_\alpha \beta$.

(Q7). By (C5), $K \circ_\top (\alpha \to \beta) = (K \circ_{\alpha\wedge\neg\beta} \perp) + (\alpha \to \beta)$. If $\alpha \in K \circ_{\alpha\wedge\neg\beta} \perp$ it follows by (C5) that $K \circ_\alpha \beta = (K \circ_{\alpha\wedge\neg\beta} \perp) + \beta = (K \circ_{\alpha\wedge\neg\beta} \perp) + (\alpha \to \beta)$. If $\alpha \notin K \circ_{\alpha\wedge\neg\beta} \perp$ it follows by (C6) that $K \circ_\alpha \beta = K \circ_{\alpha\wedge\neg\beta} \perp \subseteq (K \circ_{\alpha\wedge\neg\beta} \perp) + (\alpha \to \beta)$.

For the second part, let $\alpha \in K \circ_{\neg\beta} \perp$ and $\neg\beta \notin K \circ_{\neg\beta} \perp$. Then by Lemma 0(j), $K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_{\neg\beta} \perp$. Then by (C5), $K \circ_\alpha \beta = (K \circ_{\alpha\wedge\neg\beta} \perp) + \beta$. It follows by (C5) that $K \circ_\top (\alpha \to \beta) = (K \circ_{\top\wedge\neg(\alpha\to\beta)} \perp) + (\alpha \to \beta) =$ (by C2) $(K \circ_{\alpha\wedge\neg\beta} \perp) + (\alpha \to \beta) = (K \circ_{\alpha\wedge\neg\beta} \perp) + \beta$. Hence $K \circ_\alpha \beta = K \circ_\top (\alpha \to \beta)$.

(Q8). If $\alpha \notin K \circ_\alpha \perp$, then by (C2) and (C6) $K \circ_\alpha \neg\alpha = K \circ_\alpha \perp$. If $\alpha \in K \circ_\alpha \perp$, then by Lemma 0(d), $K \circ_\alpha \perp = K_\perp$ and by (C5) $K \circ_\alpha \neg\alpha = (K \circ_\alpha \perp) + \neg\alpha = K_\perp$.

(Q9). It follows by (C2) that $K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_{\alpha\wedge\neg(\alpha\to\beta)} \perp$. If $\alpha \in K \circ_{\alpha\wedge\neg(\alpha\to\beta)} \perp$, then $(K \circ_{\alpha\wedge\neg(\alpha\to\beta)} \perp) + (\alpha \to \beta) = (K \circ_{\alpha\wedge\neg\beta} \perp) + \beta$. Applying (C5) and (C6) to both $K \circ_\alpha \beta$ and $K \circ_\alpha (\alpha \to \beta)$, then, we see that they are identical.

(Q10). Let $\alpha \in K \circ_\alpha \beta$, then by Lemma 0(c) $\alpha \in \alpha \in K \circ_{\alpha\wedge\neg\beta} \perp$. By Lemma 0(h) it follows that $K \circ_{\alpha\wedge\neg\beta} \perp = K \circ_{\neg\beta} \perp =$ (C2) $K \circ_{\top\wedge\neg\beta} \perp$. Hence by (C5) $K \circ_\alpha \beta = (K \circ_{\alpha\wedge\neg\beta} \perp) + \beta = (K \circ_{\neg\beta} \perp) + \beta = K \circ_\top \beta$.

(Q11). ($\Leftarrow$) If $\alpha \in K \circ_\alpha \beta$, follows by (K*2). If $\beta \in K \circ_\alpha \neg\beta$ follows by (Q3). ($\Rightarrow$) Let $\beta \in K \circ_\alpha \beta$ and $\alpha \notin K \circ_\alpha \beta$. Then by Lemma 0 (b), $\alpha \notin K \circ_{\alpha\wedge\neg\beta} \perp$, $\alpha \wedge \beta \notin K \circ_{\alpha\wedge\neg\beta} \perp$. By (C3), $K \circ_{\alpha\wedge\neg\beta} \perp \subseteq K \circ_{\alpha\wedge\beta} \perp$. By (C6) $\beta \in K \circ_{\alpha\wedge\neg\beta} \perp \subseteq K \circ_{\alpha\wedge\beta} \perp$. Thus by Lemma 0(b), $\beta \in K \circ_\alpha \neg\beta$.

(Q12). ($\Rightarrow$) Let $\alpha \in K \circ_\beta \perp$ and $\beta \notin K \circ_\beta \perp$. Then by Lemma 0(j) $K \circ_\beta \perp = K \circ_{\beta\wedge\alpha} \perp$. Hence by Lemma 0(b) $\alpha \in K \circ_\beta \neg\alpha$.

($\Leftarrow$) If $\beta \in K \circ_\beta \perp$, the left-hand side follows by Lemma 0(d). If $\alpha \in K \circ_\beta \neg\alpha$, it follows by (Q3).

(Q13). Let $\alpha \notin K \circ_\beta \neg\alpha$ and $\beta \notin K \circ_\gamma \neg\beta$. (C4) yields $\beta \notin K \circ_\beta \bot$. (1) $\gamma \in K \circ_\gamma \bot$. Then by (C4), $\gamma \in K \circ_{\alpha\wedge\gamma} \bot$, from which it follows by (C4) that $\alpha \notin K \circ_{\alpha\wedge\gamma} \bot$, then $\alpha \notin (K \circ_{\alpha\wedge\gamma} \bot) + \neg\alpha$, hence by Lemma 0 (a), $\alpha \notin K \circ_\gamma \neg\alpha$. (2) $\gamma \notin K \circ_\gamma \bot$. Then by (Q12) $\alpha \notin K \circ_\beta \bot$ and $\beta \notin K \circ_\gamma \bot$. It follows from (C3) that $K \circ_\gamma \bot \subseteq K \circ_\beta \bot \subseteq K \circ_\alpha \bot$, then $\alpha \notin K \circ_\gamma \bot$, hence, by (Q3) $\alpha \notin K \circ_\gamma \neg\alpha$.

(Q14). Let $\alpha \wedge \gamma \notin K \circ_{\alpha\wedge\gamma} \bot$. Then either $\alpha \notin K \circ_{\alpha\wedge\gamma} \bot$ or $\gamma \notin K \circ_{\alpha\wedge\gamma} \bot$. Hence by (C2) and (C6) $\alpha \notin K \circ_\gamma \neg\alpha$ or $\gamma \notin K \circ_\alpha \neg\gamma$.

(Q15). Follows trivially from (K$*$5), replacing β by $\bot$.

(Q16). Let $\alpha \notin K \circ_\alpha \bot$, then by (C1) $\alpha \wedge \beta \notin K \circ_\alpha \bot$, from which it follows by (C3) that $K \circ_\alpha \bot \subseteq K \circ_{\alpha\wedge\beta} \bot$.

(Q17) follows from (Q6) and (Q16). ■

Proof of Observation 8

We show that $\leq$ indeed satisfies (E1) – (E3).

(E1) Let $\alpha \leq \beta$ and $\beta \leq \gamma$, that is, $\alpha \notin K \circ_\beta \bot$ or $\beta \in K \circ_\beta \bot$, and also $\beta \notin K \circ_\gamma \bot$ or $\gamma \in K \circ_\gamma \bot$, by (Def $\leq$ from $\circ$).

We need to show that $\alpha \leq \gamma$, i.e. $\alpha \notin K \circ_\gamma \bot$ or $\gamma \in K \circ_\gamma \bot$. Assume that $\gamma \notin K \circ_\gamma \bot$. Then $\beta \notin K \circ_\gamma \bot$, and hence, by (C3), $K \circ_\gamma \bot \subseteq K \circ_\beta \bot$. By (C4), we know that $\beta \notin K \circ_\beta \bot$. Hence $\alpha \notin K \circ_\beta \bot$, and since $K \circ_\gamma \bot$ is a subset of $K \circ_\beta \bot$, we also have $\alpha \notin K \circ_\gamma \bot$, as desired.

(E2) Let $\alpha \vdash \beta$. In order to see that $\alpha \leq \beta$, we need to show that $\alpha \notin K \circ_\beta \bot$ or $\beta \in K \circ_\beta \bot$. But this is immediate from (C1).

(E3) In order to see that either $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$, we need to show that either $\alpha \notin K \circ_{\alpha\wedge\beta} \bot$ or $\alpha \wedge \beta \in K \circ_{\alpha\wedge\beta} \bot$, or else $\beta \notin K \circ_{\alpha\wedge\beta} \bot$ or $\alpha \wedge \beta \in K \circ_{\alpha\wedge\beta} \bot$. But this is immediate from (C1). ■

Proof of Observation 9

That (a) and (b) are equivalent follows from (Q12).

That (b) and (c) are equivalent follows from (Q2).

(c) $\rightarrow$ (d): (1) $\alpha \notin K \circ_\alpha \neg\beta$. Then by Lemma 0(b), $\alpha \notin K \circ_{\alpha\wedge\beta} \bot$. (2) $\beta \in K \circ_\beta \bot$. Then by (C4) $\beta \in K \circ_{\alpha\wedge\beta} \bot$.

(d) $\rightarrow$ (e): By (C5), since $\top \in K \circ_{\top\wedge(\alpha\wedge\beta)} \bot$ we have $K \circ_\top \neg(\alpha\wedge\beta) = (K \circ_{\top\wedge(\alpha\wedge\beta)} \bot) + \neg(\alpha \wedge \beta)$. So if $\alpha \notin K \circ_{\alpha\wedge\beta} \bot$, by (C1) $\neg(\alpha \wedge \beta) \rightarrow \alpha \notin K \circ_{\alpha\wedge\beta} \bot$, so $\alpha \notin (K \circ_{\top\wedge(\alpha\wedge\beta)} \bot) + \neg(\alpha\wedge\beta) = K \circ_\top \neg(\alpha\wedge\beta)$. If $\beta \in K \circ_{\alpha\wedge\beta} \bot = K \circ_{\top\wedge(\alpha\wedge\beta)} \bot \subseteq (K \circ_{\top\wedge(\alpha\wedge\beta)} \bot) + \neg(\alpha \wedge \beta)$ follows immediately that $\beta \in K \circ_\top \neg(\alpha \wedge \beta)$.

(e) $\rightarrow$ (a): Let $\alpha \in K \circ_\beta \bot$ and $\beta \notin K \circ_\beta \bot$. Then $\alpha \wedge \beta \notin K \circ_\beta \bot$. By (C3), $K \circ_\beta \bot \subseteq K \circ_{\alpha\wedge\beta} \bot$, then $\alpha \in K \circ_{\alpha\wedge\beta} \bot = K \circ_{\top\wedge(\alpha\wedge\beta)} \bot \subseteq K \circ_\top \neg(\alpha \wedge \beta)$. Since

$\beta \notin K \circ_\beta \bot$ we have $\alpha \wedge \beta \notin K \circ_{\alpha\wedge\beta} \bot$, by (C4) and (C1), since $\alpha \in K \circ_{\alpha\wedge\beta} \bot$, we have $\beta \notin K \circ_{\alpha\wedge\beta} \bot$. By (C1) and (C2) then $\neg(\alpha \wedge \beta) \to \beta \notin K \circ_{\top\wedge(\alpha\wedge\beta)} \bot$ from which it follows by Lemma 0(a) that $\beta \notin K \circ_\top \neg(\alpha \wedge \beta)$.

■

Proof of Theorem 10 (Soundness)

(C1). Let $\gamma \in Cn(K \circ_\alpha \beta)$. Then, by the compactness of the underlying logic, there is a finite subset $\{\delta_1, \ldots, \delta_n\} \subseteq K \circ_\alpha \beta$, such that $\{\delta_1, \ldots, \delta_n\} \vdash \gamma$. We have to prove that $\gamma \in K \circ_\alpha \beta$. If $\top \vdash \alpha \wedge \neg\beta$, then by definition, $K \circ_\alpha \beta = \mathcal{L}$, hence trivially $\gamma \in K \circ_\alpha \beta$. So let $\top \nvdash \alpha \wedge \neg\beta$.

We first show that $\delta_1 \wedge \ldots \wedge \delta_n \in K \circ_\alpha \beta$. For this purpose we are going to prove that if $\delta_1 \in K \circ_\alpha \beta$ and $\delta_2 \in K \circ_\alpha \beta$ then $\delta_1 \wedge \delta_2 \in K \circ_\alpha \beta$. The rest follows by iteration of the same procedure. So suppose $\delta_1 \in K \circ_\alpha \beta$ and $\delta_2 \in K \circ_\alpha \beta$.

By (E3), we have that either $\delta_1 \leq \delta_1 \wedge \delta_2$ or $\delta_2 \leq \delta_1 \wedge \delta_2$. We can assume without loss of generality that $\delta_1 \leq \delta_1 \wedge \delta_2$ from which it follows that $\delta_1 \leq \delta_2$.

If $\alpha < \delta_1$, then by (E1) $\alpha < \delta_1 \wedge \delta_2$, hence by (Def $\circ$ from $\leq$), $\delta_1 \wedge \delta_2 \in K \circ_\alpha \beta$.

If on the other hand $\delta_1 \leq \alpha$, it follows by (Def $\circ$ from $\leq$) that $\neg\beta < \alpha \wedge (\beta \to \delta_1)$. If $\alpha < \delta_2$, by (E1) – (E3) we obtain that $\neg\beta < \alpha \wedge (\beta \to \delta_1) \leq \alpha < \delta_2 \leq \beta \to \delta_2$. If $\delta_2 \leq \alpha$, then by (Def $\circ$ from $\leq$) $\neg\beta < \alpha \wedge (\beta \to \delta_2)$. Hence in any case $\neg\beta < \alpha \wedge (\beta \to \delta_2)$. From $\neg\beta < \alpha \wedge (\beta \to \delta_1)$ and $\neg\beta < \alpha \wedge (\beta \to \delta_2)$ it follows by (E3) and (E1) that $\neg\beta < \alpha \wedge (\beta \to (\delta_1 \wedge \delta_2))$. Hence $\delta_1 \wedge \delta_2 \in K \circ_\alpha \beta$.

By repeated use of this argument, we get that $\delta_1 \wedge \ldots \wedge \delta_n \in K \circ_\alpha \beta$. We have $\delta_1 \wedge \ldots \wedge \delta_n \vdash \gamma$, from which it follows by (E2) that $\delta_1 \wedge \ldots \wedge \delta_n \leq \gamma$. If $\alpha < \delta_1 \wedge \ldots \wedge \delta_n$, then by (E1) $\alpha < \gamma$, hence $\gamma \in K \circ_\alpha \beta$. If on the other hand $\delta_1 \wedge \ldots \wedge \delta_n \leq \alpha$, then by (Def $\circ$ from $\leq$) $\neg\beta < \alpha \wedge (\beta \to (\delta_1 \wedge \ldots \wedge \delta_n))$. By (E2) $\alpha \wedge (\beta \to (\delta_1 \wedge \ldots \wedge \delta_n)) \leq \alpha \wedge (\beta \to \gamma)$, so by (E1) $\neg\beta < \alpha \wedge (\beta \to \gamma)$. Hence $\gamma \in K \circ_\alpha \beta$.

(C2) Trivial.

(C3). Let $\alpha \notin K \circ_\beta \bot$. Then it follows by definition of $\circ$ that $\alpha \leq \beta$ and $\beta < \top$. Let $\gamma \in K \circ_\beta \bot$. Then $\beta < \gamma$; (E1) yields that $\alpha < \gamma$, then $\gamma \in K \circ_\alpha \bot$, hence $K \circ_\beta \bot \subseteq K \circ_\alpha \bot$.

(C4). Let $\alpha \in K \circ_\alpha \bot$. Then it follows by definition of $\circ$ that $\top \leq \alpha$. (1) $\beta < \alpha$. Then by definition of $\circ$, $\alpha \in K \circ_\beta \gamma$. (2) $\alpha \leq \beta$. Then by (E1) – (E3) $\top \leq \beta \wedge \alpha \leq \beta \wedge (\gamma \to \alpha)$. (2.1) $\neg\gamma < \beta \wedge (\gamma \to \alpha)$. Then by definition of $\circ$, $\alpha \in K \circ_\beta \gamma$. (2.2) $\beta \wedge (\gamma \to \alpha) \leq \neg\gamma$. Then by (E1) - (E3) $\top \leq \beta \wedge \neg\gamma$. Hence by definition of $\circ$, $\alpha \in K \circ_\beta \gamma$.

(C5). Let $\alpha \in K \circ_{\alpha\wedge\neg\beta} \bot$. Then by definition of $\circ$ we have two cases: (1) $\top \leq \alpha \wedge \neg\beta$. Then by definition of $\circ$ it follows that $K \circ_\alpha \beta = (K \circ_{\alpha\wedge\neg\beta} \bot) + \beta = K_\bot$. (2) $\alpha \wedge \neg\beta < \alpha$. From (1) we can assume that $\alpha \wedge \neg\beta < \top$. By (E1) – (E3) $\alpha \wedge \neg\beta < \alpha \wedge (\beta \to \gamma)$ iff $\alpha \wedge \neg\beta < \beta \to \gamma$ and since by (E3) $\neg\beta \leq \alpha \wedge \neg\beta$, (E1) yields $\neg\beta < \alpha \wedge (\beta \to \gamma)$ iff $\alpha \wedge \neg\beta < \beta \to \gamma$. Hence by definition of $\circ$, $\gamma \in K \circ_\alpha \beta$ iff $\beta \to \gamma \in K \circ_{\alpha\wedge\neg\beta} \bot$.

(C6). Let $\alpha \notin K \circ_{\alpha\wedge\neg\beta} \bot$. Then by definition of $\circ$ $\alpha\wedge\neg\beta < \top$ and $\alpha \leq \alpha\wedge\neg\beta$. By (E1) - (E3) for all γ it follows that $\alpha \wedge (\beta \to \gamma) \leq \alpha \leq \alpha \wedge \neg\beta \leq \neg\beta$. Then for all γ, $\neg\beta \not< \alpha \wedge (\beta \to \gamma)$. Hence by definition of $\circ$, $\gamma \in K \circ_\alpha \beta$ iff $\alpha < \gamma$ iff $\alpha \wedge \neg\beta < \gamma$ iff $\gamma \in K \circ_{\alpha\wedge\neg\beta} \bot$.

($K_\circ = K_\leq$). $K_\circ$ is defined to be $K \circ_\bot \bot$. But by (Def $\circ$ from $\leq$), $\alpha \in K \circ_\bot \bot$ if and only if

$$\begin{cases} \neg\bot < \bot \wedge (\bot \to \alpha) & \text{or} \\ \bot < \alpha & \text{or} \\ \top \leq \bot \wedge \neg\bot & \end{cases}$$

which reduces (since $\top < \bot$ is impossible) to the disjunction $\bot < \alpha$ or $\top \leq \bot$. But this is exactly what the definition of $K_\leq$ comes down to.

($\leq$ from $\circ$). Finally, we show that $\leq$ can be retrieved from $\circ$ with the help of ($\leq$ from $\circ$).

($\Rightarrow$). Let $\alpha \leq \beta$ and $\beta \notin K \circ_\beta \bot$. By definition of $\circ$, $\beta < \top$. Then $\beta \not< \alpha$ and $\top \not\leq \beta$, hence by definition of $\circ$, $\alpha \notin K \circ_\beta \bot$.

($\Leftarrow$) Let $\beta \in K \circ_\beta \bot$. By definition of $\circ$, $\top \leq \beta$. By (E2) $\alpha \leq \top$, then by (E1) $\alpha \leq \beta$. Let $\alpha \notin K \circ_\beta \bot$. Hence by definition of $\circ$, $\alpha \leq \beta$. ∎

Proof of Theorem 11 (Completeness)

We define $\leq$ from $\circ$ with the help of (Def $\leq$ from $\circ$). Due to Observation 8 $\leq$ is an entrenchment relation that satisfies (E1)-(E3). Then we use $\leq$ to construct a revision-by-comparison operation $\circ'$ with the help of (Def $\circ$ from $\leq$). We need to show that $\circ' = \circ$.

Let $\gamma \in K \circ'_\alpha \beta$.

By (Def $\circ$ from $\leq$), this means that

either $\neg\beta < \alpha \wedge (\beta \to \gamma)$

or $\alpha < \gamma$

or $\top \leq \alpha \wedge \neg\beta$

By (Def $\leq$ from $\circ$), this means that

either (I) $\alpha \wedge (\beta \to \gamma) \in K \circ_{\neg\beta} \bot$ and $\neg\beta \notin K \circ_{\neg\beta} \bot$

or (II) $\gamma \in K \circ_\alpha \bot$ and $\alpha \notin K \circ_\alpha \bot$

or (III) $\alpha \wedge \neg\beta \in K \circ_{\alpha\wedge\neg\beta} \bot$

First we show that each of (I), (II) and (III) implies $\gamma \in K \circ_\alpha \beta$.

Suppose that (I) is true. By (C1) $\alpha \in K \circ_{\neg\beta} \bot$. Then by Lemma 0(j) $K \circ_{\neg\beta} \bot = K \circ_{\alpha\wedge\neg\beta} \bot$ from which it follows by (C1) that $(\beta \to \gamma) \in K \circ_{\alpha\wedge\neg\beta} \bot$. Hence by (C5) $\gamma \in K \circ_{\alpha} \beta$.

Next suppose that (II) is true. Then by (Q12) $\gamma \in K \circ_{\alpha} \neg\gamma$, from which we get by (Q3) that $\gamma \in K \circ_{\alpha} \beta$.

Now suppose that (III) is true. Then by (K∗5) $K \circ_{\alpha} \beta = K_{\bot}$, hence $\gamma \in K \circ_{\alpha} \beta$.

Now we show that conversely $\gamma \in K \circ_{\alpha} \beta$ implies one of (I), (II) and (III).

Let $\gamma \in K \circ_{\alpha} \beta$. Then, according to (C5) and (C6) we have the following subcases:

(1) $\beta \to \gamma \in K \circ_{\alpha\wedge\neg\beta} \bot$ and $\alpha \in K \circ_{\alpha\wedge\neg\beta} \bot$. If $\neg\beta \in K \circ_{\alpha\wedge\neg\beta} \bot$, it follows by (C1) that (III) is satisfied. If $\neg\beta \notin K \circ_{\alpha\wedge\neg\beta} \bot$, it follows by (C4) that $\neg\beta \notin K \circ_{\neg\beta} \bot$ and by Lemma 0(h) that $K \circ_{\alpha\wedge\neg\beta} \bot = K \circ_{\neg\beta} \bot$. Then by (C1) $\alpha \wedge (\beta \to \gamma) \in K \circ_{\neg\beta} \bot$, hence (I) is satisfied.

(2) $\gamma \in K \circ_{\alpha\wedge\neg\beta} \bot$ and $\alpha \notin K \circ_{\alpha\wedge\neg\beta} \bot$. By (C4) $\alpha \notin K \circ_{\alpha} \bot$ and by Lemma 0(h) $K \circ_{\alpha\wedge\neg\beta} \bot = K \circ_{\alpha} \bot$. Hence $\gamma \in K \circ_{\alpha} \bot$ and so (II) is satisfied.

Finally, we show that $K_{\leq} = K_{\circ}$. By the definition of $K_{\leq}$, we know that $\alpha \in K_{\leq}$ iff either $\bot < \alpha$ or $\top \leq \bot$. But by (Def $\leq$ from $\circ$), this means that $\alpha \in K_{\leq}$ if and only if

$$\begin{cases} \text{either} \quad (\alpha \in K \circ_{\bot} \bot \ \text{ and } \ \bot \notin K \circ_{\bot} \bot) \\ \text{or} \quad (\top \notin K \circ_{\bot} \bot \ \text{ or } \ \bot \in K \circ_{\bot} \bot) \end{cases}$$

which reduces, thanks to (C1), to the disjunction $\alpha \in K \circ_{\bot} \bot$ or $\bot \in K \circ_{\bot} \bot$. This in turn reduces, again thanks to (C1), to $\alpha \in K \circ_{\bot} \bot$. Thus we have shown that $K_{\leq}$ is identical with $K \circ_{\bot} \bot = K_{\circ}$. ■

Proof of Theorem 12

For this proof, we rely on the "translation" of facts about one-step revisions by comparison into the language of entrenchments, i.e., on (Def $\circ$ from $\leq$), and on the properties of entrenchments.

We begin with the left-hand side of (IT).

$\delta \in \ (K \circ_{\alpha} \beta) \circ_{\gamma} \bot$

iff (by (Def $\circ$ from $\leq$) and a few simplifications)

either $\gamma <' \delta$ *or* $\top \leq' \gamma$

iff (by (Def $\leq'$ from $\leq$))

either $\beta \to \gamma < \alpha \wedge (\beta \to \delta)$ and $\delta \leq \alpha$ or

$\gamma < \delta$ and $\alpha < \delta$

or $\alpha \wedge (\beta \rightarrow \top) \leq (\beta \rightarrow \gamma)$ and $\top \leq \alpha$ or

$\top \leq \gamma$ and $\alpha < \top$

Since $\alpha \wedge (\beta \rightarrow \top)$ is equivalent with α and $\leq$ is transitive, the left-hand side of (IT) is satisfied iff one of the following holds:

$$\begin{cases} \text{(a)} \quad \beta \rightarrow \gamma < \alpha \wedge (\beta \rightarrow \delta) \text{ and } \delta \leq \alpha \\ \text{(b)} \quad \gamma < \delta \text{ and } \alpha < \delta \\ \text{(c)} \quad \top \leq \beta \rightarrow \gamma \text{ and } \top \leq \alpha \\ \text{(d)} \quad \top \leq \gamma \text{ and } \alpha < \top. \end{cases}$$

Now we turn to the right-hand side of the condition (IT). What will it mean, in terms of entrenchment, that $\delta \in (K \circ_\alpha \beta) \circ_\gamma \bot$ if we employ the right-hand side of (IT)? The conditions spelt out in the following lines are to be read disjunctively. (Def $\circ$ from $\leq$) gives us that

$$\begin{cases} ((\beta \rightarrow \gamma < \beta \rightarrow \delta) \text{ or } \top \leq \beta \rightarrow \gamma) \text{ and } \gamma < \top \text{ and } (\beta \rightarrow \gamma < \alpha \text{ or } \top \leq \alpha) \\ \text{or} \\ (\alpha < \delta \text{ or } \top \leq \alpha) \text{ and } (\gamma < \delta \text{ or } \top \leq \gamma) \text{ and } \gamma < \top \text{ and } \alpha \leq \beta \rightarrow \gamma \text{ and } \alpha < \top \\ \text{or} \\ \top \leq \gamma \end{cases}$$

Now let us split up the first two lines into four disjuncts each

$$\begin{cases} (1) & \beta \to \gamma < \beta \to \delta \text{ and } \gamma < \top \text{ and } \beta \to \gamma < \alpha \\ (2) & \beta \to \gamma < \beta \to \delta \text{ and } \gamma < \top \text{ and } \top \leq \alpha \\ (3) & \top \leq \beta \to \gamma \text{ and } \gamma < \top \text{ and } \beta \to \gamma < \alpha \\ (4) & \top \leq \beta \to \gamma \text{ and } \gamma < \top \text{ and } \top \leq \alpha \\ \text{or} & \\ (5) & \alpha < \delta \text{ and } \gamma < \delta \text{ and } \gamma < \top \text{ and } \alpha \leq \beta \to \gamma \text{ and } \alpha < \top \\ (6) & \alpha < \delta \text{ and } \top \leq \gamma \text{ and } \gamma < \top \text{ and } \alpha \leq \beta \to \gamma \text{ and } \alpha < \top \\ (7) & \top \leq \alpha \text{ and } \gamma < \delta \text{ and } \gamma < \top \text{ and } \alpha \leq \beta \to \gamma \text{ and } \alpha < \top \\ (8) & \top \leq \alpha \text{ and } \top \leq \gamma \text{ and } \gamma < \top \text{ and } \alpha \leq \beta \to \gamma \text{ and } \alpha < \top \\ \text{or} & \\ (9) & \top \leq \gamma \end{cases}$$

Of these conditions, (6), (7) and (8) are plainly inconsistent, while (3) is inconsistent with entrenchment properties. (1), (2) and (5) can easily be reduced by the application of entrenchment properties. Condition (2) in fact implies (1) and can thus be dropped. In (4), the condition $\gamma < \top$ can be dropped due to the presence of the disjunct (9). In sum, we get the following:

$$\begin{cases} (1) & \beta \to \gamma < \beta \to \delta \text{ and } \beta \to \gamma < \alpha \\ (4) & \top \leq \beta \to \gamma \text{ and } \top \leq \alpha \\ (5) & \alpha < \delta \text{ and } \gamma < \delta \text{ and } \alpha \leq \beta \to \gamma \\ (9) & \top \leq \gamma \end{cases}$$

In order to prove that condition (IT) is valid, it remains to prove that the disjunction of (1), (4), (5) and (9) is equivalent to the disjunction of (a), (b), (c) and (d).

First, we check that (1)-or-(4)-or-(5)-or-(9) implies (a)-or-(b)-or-(c)-or-(d).

(1) implies that either (a) is the case or

$\beta \to \gamma < \beta \to \delta$ and $\beta \to \gamma < \alpha$ and $\alpha < \delta$

are all true. But this implies (b) (since $\beta \to \gamma < \delta$ implies $\gamma < \delta$).

(4) implies (c). (5) implies (b). (9) implies that either (d) is the case or

$\top \leq \gamma$ and $\top \leq \alpha$

are both true. But this implies (c) (since $\top \leq \gamma$ implies $\top \leq \beta \to \gamma$, by (E2) and (E1)).

Second, we check that (a)-or-(b)-or-(c)-or-(d) implies (1)-or-(4)-or-(5)-or-(9).

(a) implies (1). (b) implies that either (5) is the case or

$\gamma < \delta$ and $\alpha \leq \delta$ and $\beta \to \gamma < \alpha$

are all true. But this implies (1) (since $\beta \to \gamma < \alpha \leq \delta \leq \beta \to \delta$).

(c) implies (4). (d) implies (9).

This finally proves that (IT) is valid. ∎

Proof of Theorem 13

Let the revision-by-comparison function $\circ$ satisfy (C1) – (C6) as well as (IT). Let $\leq$ be derived from $\circ$ and let $\leq'$ be derived from $\circ'$ with the help of (Def $\leq$ from $\circ$), where $\circ'$ for $K' = K \circ_\alpha \beta$ is defined by

$$K' \circ'_\gamma \delta = (K \circ_\alpha \beta) \circ_\gamma \delta$$

We want to show that $\leq$ and $\leq'$ satisfy (Def $\leq'$ from $\leq$).

Let $\gamma \leq' \delta$.

By (Def $\leq$ from $\circ$), this means that

$\gamma \notin (K \circ_\alpha \beta) \circ_\delta \bot$ or $\delta \in (K \circ_\alpha \beta) \circ_\delta \bot$.

Using (IT), we can give the following list of all ways of making this condition true (read the list as a long disjunction). (1)–(4) correspond to $\gamma \notin (K \circ_\alpha \beta) \circ_\delta \bot$, (5)–(7) correspond to $\delta \in (K \circ_\alpha \beta) \circ_\delta \bot$.

(1) $\beta \to \gamma \notin K \circ_{\beta\to\delta} \bot$ and $K \circ_\delta \bot \neq K_\bot$ and $\alpha \in K \circ_\alpha (\beta \wedge \neg\delta)$

(2) $\gamma \notin K \circ_\alpha \bot$ and $K \circ_\delta \bot \neq K_\bot$ and $\alpha \notin K \circ_\alpha (\beta \wedge \neg\delta)$

(3) $\gamma \notin K \circ_\delta \bot$ and $K \circ_\delta \bot \neq K_\bot$ and $\alpha \notin K \circ_\alpha (\beta \wedge \neg\delta)$

(4) $\gamma \notin K_\bot$ and $K \circ_\delta \bot = K_\bot$

(5) $\beta \to \delta \in K \circ_{\beta\to\delta} \bot$ and $K \circ_\delta \bot \neq K_\bot$ and $\alpha \in K \circ_\alpha (\beta \wedge \neg\delta)$

(6) $\delta \in K \circ_\alpha \bot \cap K \circ_\delta \bot$ and $K \circ_\delta \bot \neq K_\bot$ and $\alpha \notin K \circ_\alpha (\beta \wedge \neg\delta)$

(7) $\delta \in K_\bot$ and $K \circ_\delta \bot = K_\bot$

Now (4) is plainly impossible, and (6) is inconsistent with (K$*$5): $\delta \in K \circ_\delta \bot$ implies that $K \circ_\delta \bot = K_\bot$. The second conjunct of (3) and the first conjunct of (7) can obviously be dropped. So we end up with a disjunction of

(1) $\beta \to \gamma \notin K \circ_{\beta\to\delta} \bot$ and $K \circ_\delta \bot \neq K_\bot$ and $\alpha \in K \circ_\alpha (\beta \wedge \neg\delta)$

(2) $\gamma \notin K \circ_\alpha \perp$ and $K \circ_\delta \perp \neq K_\perp$ and $\alpha \notin K \circ_\alpha (\beta \wedge \neg\delta)$

(3) $\gamma \notin K \circ_\delta \perp$ and $\alpha \notin K \circ_\alpha (\beta \wedge \neg\delta)$

(5) $\beta \to \delta \in K \circ_{\beta\to\delta} \perp$ and $K \circ_\delta \perp \neq K_\perp$ and $\alpha \in K \circ_\alpha (\beta \wedge \neg\delta)$

(7) $K \circ_\delta \perp = K_\perp$

Now we know from the proof of Theorem 11 that $\circ$ can be generated from $\leq$ with the help of (Def $\circ$ from $\leq$). For the special case where β equals $\perp$, (Def $\circ$ from $\leq$) simplifies to $\gamma \in K \circ_\alpha \perp$ iff either $\alpha < \gamma$ or $\top \leq \alpha$, and consequently, $K \circ_\alpha \perp = K_\perp$ iff $\top \leq \alpha$. Using these facts, we can "translate" the remaining five cases into conditions about entrenchments (to be read disjunctively):

(A1) $\beta \to \gamma \leq \beta \to \delta$ and $\beta \to \delta < \top$ and $\delta < \top$ and ($\beta \to \delta < \alpha$ or $\top \leq \alpha$)

(A2) $\gamma \leq \alpha$ and $\alpha < \top$ and $\delta < \top$ and ($\alpha \leq \beta \to \delta$ and $\alpha < \top$)

(A3) $\gamma \leq \delta$ and $\delta < \top$ and ($\alpha \leq \beta \to \delta$ and $\alpha < \top$)

(A5) $\top \leq \beta \to \delta$ and $\delta < \top$ and ($\beta \to \delta < \alpha$ or $\top \leq \alpha$)

(A7) $\top \leq \delta$

Let us now simplify (A1). We can drop $\delta < \top$, since it is implied by $\beta \to \delta < \top$ (using E1–E3). We can also drop $\top \leq \alpha$, since together with $\beta \to \delta < \top$, it implies $\beta \to \delta < \alpha$ (using E1–E3). Having the latter as a conjunct, we can finally drop what is implied by it, namely $\beta \to \delta < \top$. In (A5), note that $\top \leq \beta \to \delta$ rules out $\beta \to \delta < \alpha$. Because of (A7), we can finally drop all conjuncts of the form $\delta < \top$ in (A2), (A3) and (A5). So we have the following simplified array of conditions (to be read disjunctively):

(A1) $\beta \to \gamma \leq \beta \to \delta$ and $\beta \to \delta < \alpha$

(A2) $\gamma \leq \alpha$ and $\alpha \leq \beta \to \delta$ and $\alpha < \top$

(A3) $\gamma \leq \delta$ and $\alpha \leq \beta \to \delta$ and $\alpha < \top$

(A5) $\top \leq \beta \to \delta$ and $\top \leq \alpha$

(A7) $\top \leq \delta$

On the other hand, the two conditions of (Def $\leq'$ from $\leq$) can be split up into three conditions (likewise to be read disjunctively):

(I) $\alpha \leq \beta \to \delta$ and $\gamma \leq \alpha$

(II) $\beta \to \gamma \leq \beta \to \delta$ and $\gamma \leq \alpha$

(III) $\gamma \leq \delta$ and $\alpha < \gamma$

It remains to be shown that each of the conditions in the more complicated list implies a disjunction of the conditions in the last list, and vice versa. This is what we will check now, using the entrenchment conditions (E1) – (E3) without further explicit mentioning.

(A1) implies (II), since it entails $\gamma < \alpha$.

(A2) implies (I).

(A3) implies (I), if $\gamma \leq \alpha$; it implies (III), if $\alpha < \gamma$.

(A5) implies (I) [and also (II)].

(A7) implies the disjunction of (I) and (III) [and also the disjunction of (II) and (III)].

Now for the converse direction.

(I) implies (A2), if $\alpha < \top$; it implies (A5), if $\top \leq \alpha$.

(II) implies (A1), if $\beta \rightarrow \delta < \alpha$. It implies (A2), if both $\alpha \leq \beta \rightarrow \delta$ and $\alpha < \top$. It implies (A5), if both $\alpha \leq \beta \rightarrow \delta$ and $\top \leq \alpha$.

(III) implies (A3).

In sum, we have shown that (Def $\leq'$ from $\leq$) is satisfied by $\leq$ and $\leq'$. ∎

Proof of Theorem 14

Let $\circ$ be an iterated revision-by-comparison function for K satisfying (C1) – (C6) as well as (IT).

The following proofs for the iteration conditions (IT1), (IT2), (IT3) and (IT5) will not be made by direct use of (IT), but via entrenchment representations of one-step revisions by comparison. Let $\leq$ be the entrenchment relation derived from $\circ$ by (Def $\leq$ from $\circ$) and let $\leq'$ be the entrenchment relation derived from $\circ'$ by (Def $\leq$ from $\circ$) as explained in Theorem 13. Then we know from Theorems 11 and 13 that the function $\circ$ can be generated from $\leq$ and $\leq'$ with the help of (Def $\circ$ from $\leq$) and that $\leq$ and $\leq'$ are related by (Def $\leq'$ from $\leq$). We will use these results freely in the following. (The proof of (IT4) will proceed by a direct derivation from (IT).)

(IT1). $\delta \in (K \circ_\alpha \beta) \circ_\alpha \gamma$ is true iff (by Def $\circ$ from $\leq$)

either (case 1) $\neg\gamma <' \alpha \wedge (\gamma \rightarrow \delta)$

or (case 2) $\alpha <' \delta$.

or (case 3) $\top \leq' \alpha \wedge \neg\gamma$.

In case 1 we have (by Def $\leq'$ from $\leq$)

either $\alpha \wedge (\gamma \to \delta) \leq \alpha$ and $(\beta \to \neg\gamma) < \alpha \wedge (\beta \to (\alpha \wedge (\gamma \to \delta)))$

or $(\alpha < \alpha \wedge (\gamma \to \delta))$ and $\neg\gamma < \alpha \wedge (\gamma \to \delta)$.

But the latter case is impossible: $\alpha < \alpha \wedge (\gamma \to \delta)$ contradicts the entrenchment postulates (E1)–(E3).

So we are left with the first case in which $(\beta \to \neg\gamma) < \alpha \wedge (\beta \to (\alpha \wedge (\gamma \to \delta)))$ is equivalent with $\neg(\beta \wedge \gamma) < \alpha \wedge ((\beta \wedge \gamma) \to \delta))$. Moreover, $\alpha \wedge (\gamma \to \delta) \leq \alpha$ is guaranteed by the entrenchment properties.

So in sum, case 1 reduces to $\neg(\beta \wedge \gamma) < \alpha \wedge ((\beta \wedge \gamma) \to \delta))$.

In case 2 we have

either $\delta \leq \alpha$ and $(\beta \to \alpha) < \alpha \wedge (\beta \to \delta)$

or $\alpha < \delta$ and $\alpha < \delta$.

But the former case is impossible, since $(\beta \to \alpha) < \alpha$ is impossible, and hence $(\beta \to \alpha) < \alpha \wedge (\beta \to \delta)$ is impossible, by the entrenchment postulates.

Thus we remain with the latter case, i.e., $\alpha < \delta$.

So in sum, case 2 reduces to $\alpha < \delta$.

In case 3 we have

either $\top \leq \alpha$ and $\alpha \wedge (\beta \to \top) \leq \beta \to (\alpha \wedge \neg\gamma)$

or $\alpha < \top$ and $\top \leq \alpha \wedge \neg\gamma$.

By the entrenchment properties, the lower line is impossible, and this reduces to

$\top \leq \alpha$ and $\top \leq \beta \to \neg\gamma$

or, what is the same, to $\top \leq \alpha \wedge \neg(\beta \wedge \gamma)$.

Since either case 1 or case 2 or case 3 obtains, we remain with the condition

$\neg(\beta \wedge \gamma) < \alpha \wedge ((\beta \wedge \gamma) \to \delta)$ *or*

$\alpha < \delta$ *or*

$\top \leq \alpha \wedge \neg(\beta \wedge \gamma)$

which defines the fact that $\delta \in K \circ_\alpha (\beta \wedge \gamma)$, by (Def $\circ$ from $\leq$).

(IT2). By (Def $\circ$ from $\leq$), $\delta \in (K \circ_\alpha \beta) \circ_\gamma \beta$ iff

either $\neg\beta <' \gamma \wedge (\beta \to \delta)$

or $\gamma <' \delta$

or $\top \leq' \gamma \wedge \neg\beta$

By (Def $\leq'$ from $\leq$), this means

$(*)$: *either* $(\beta \to \neg\beta) < \alpha \wedge (\beta \to (\gamma \wedge (\beta \to \delta)))$ and $\gamma \wedge (\beta \to \delta) \leq \alpha$

or $\neg\beta < \gamma \wedge (\beta \to \delta)$ and $\alpha < \gamma \wedge (\beta \to \delta)$

or $(\beta \to \gamma) < \alpha \wedge (\beta \to \delta)$ and $\delta \leq \alpha$

or $\gamma < \delta$ and $\alpha < \delta$

or $\alpha \wedge (\beta \to \top) \leq \beta \to (\gamma \wedge \neg\beta)$ and $\top \leq \alpha$

or $\top \leq \gamma \wedge \neg\beta$ and $\alpha < \top$

By the entrenchment properties, $\gamma \wedge (\beta \to \delta) \leq \alpha$ holds iff either $\gamma \leq \alpha$ or $\beta \to \delta \leq \alpha$. So $(*)$ can be simplified to

$(**)$: *either* $\neg\beta < \alpha \wedge (\beta \to \gamma) \wedge (\beta \to \delta)$ and $\gamma \leq \alpha$

or $\neg\beta < \alpha \wedge (\beta \to (\gamma \wedge \delta))$ and $\beta \to \delta \leq \alpha$

or $\neg\beta < \gamma \wedge (\beta \to \delta)$ and $\alpha < \gamma \wedge (\beta \to \delta)$

or $(\beta \to \gamma) < \alpha \wedge (\beta \to \delta)$ and $\delta \leq \alpha$

or $\gamma < \delta$ and $\alpha < \delta$

or $\top \leq \alpha \wedge \neg\beta$

or $\top \leq \gamma \wedge \neg\beta$ and $\alpha < \top$

Now we distinguish two cases.

Let in *Case 1* be $\gamma \leq \alpha$. Lines 3 and 7 of $(**)$ are impossible in this case, and the rest reduces to

either $\neg\beta < \alpha \wedge (\beta \to (\gamma \wedge \delta))$

or $\neg\beta < \alpha \wedge (\beta \to (\gamma \wedge \delta))$ and $\beta \to \delta \leq \alpha$

or $(\beta \to \gamma) < \alpha \wedge (\beta \to \delta)$ and $\delta \leq \alpha$

or $\alpha < \delta$

or $\top \leq \alpha \wedge \neg\beta$

Here the second line can be dropped since it implies the first. The last term of the third line is superfluous because of the fourth line. We end with the complex

either $\neg\beta < \alpha \wedge (\beta \to (\gamma \wedge \delta))$

or $(\beta \to \gamma) < \alpha \wedge (\beta \to \delta)$

or $\alpha < \delta$

or $\top \leq \alpha \wedge \neg\beta$

It is easy to check that the first two lines of this complex are equivalent with $\neg\beta < \alpha \wedge (\beta \to \delta)$.

Thus, by (Def $\circ$ from $\leq$), in Case 1 we have $\delta \in (K \circ_\alpha \beta) \circ_\gamma \beta$ iff $\delta \in K \circ_\alpha \beta$.

Consider now *Case 2* where $\alpha < \gamma$. Lines 1, 4 and 6 of $(**)$ are impossible in this case, and the rest reduces to

either $\neg\beta < \alpha \wedge (\beta \rightarrow (\gamma \wedge \delta))$ and $\beta \rightarrow \delta \leq \alpha$

or $\neg\beta < \gamma \wedge (\beta \rightarrow \delta)$ and $\alpha < \beta \rightarrow \delta$

or $\gamma < \delta$

or $\top \leq \gamma \wedge \neg\beta$

Since $\beta \rightarrow \delta \leq \alpha < \gamma$ implies $\beta \rightarrow \delta \leq \alpha \wedge (\beta \rightarrow \gamma)$, we can simplify the first line of this complex to $\neg\beta < \beta \rightarrow \delta$ and $\beta \rightarrow \delta \leq \alpha$, Since $\alpha < \gamma$, we can equivalently say that $\neg\beta < \gamma \wedge (\beta \rightarrow \delta)$ and $\beta \rightarrow \delta \leq \alpha$.

So the first two lines of this complex are equivalent with $\neg\beta < \gamma \wedge (\beta \rightarrow \delta)$.

Thus, by (Def $\circ$ from $\leq$), in Case 2 we have $\delta \in (K \circ_\alpha \beta) \circ_\gamma \beta$ iff $\delta \in K \circ_\gamma \beta$.

Since $\gamma \leq \alpha$ encodes $\gamma \notin K \circ_\alpha \bot$ or $K \circ_\alpha \bot = K_\bot$ (by Q15), our case distinction matches exactly the one formulated in (IT2).

(IT3). From the proof of (IT2), we know that $\delta \in (K \circ_\alpha \beta) \circ_\gamma \beta$ can be decided by a case distinction. If $\gamma \leq \alpha$, then the condition is

either $\neg\beta < \alpha \wedge (\beta \rightarrow \delta)$

or $\alpha < \delta$

or $\top \leq \alpha \wedge \neg\beta$

If, on the other hand, $\alpha < \gamma$, then the condition is

either $\neg\beta < \gamma \wedge (\beta \rightarrow \delta)$

or $\gamma < \delta$

or $\top \leq \gamma \wedge \neg\beta$

Using a perfectly symmetrical argument, we decide $\delta \in (K \circ_\gamma \beta) \circ_\alpha \beta$ by a case distinction. If $\alpha \leq \gamma$, then the condition is

either $\neg\beta < \gamma \wedge (\beta \rightarrow \delta)$

or $\gamma < \delta$

or $\top \leq \gamma \wedge \neg\beta$

If, on the other hand, $\gamma < \alpha$, then the condition is

either $\neg\beta < \alpha \wedge (\beta \rightarrow \delta)$

or $\alpha < \delta$

or $\top \leq \alpha \wedge \neg\beta$

In order to prove that $(K \circ_\alpha \beta) \circ_\gamma \beta = (K \circ_\gamma \beta) \circ_\alpha \beta$, we check the cases. Obviously, the conditions for the inclusion of δ are identical if $\alpha < \gamma$ or if

$\gamma < \alpha$.

It remains to check the case where both $\alpha \leq \gamma$ and $\gamma \leq \alpha$. By the conditions for entrenchment relations, equal entrenchment of α and γ entails that

$$\begin{aligned} \neg\beta < \alpha \wedge (\beta \to \delta) \quad &\text{iff} \quad \neg\beta < \gamma \wedge (\beta \to \delta) \\ \alpha < \delta \quad &\text{iff} \quad \gamma < \delta \\ \top \leq \alpha \wedge \neg\beta \quad &\text{iff} \quad \top \leq \gamma \wedge \neg\beta \end{aligned}$$

So in this last case, the conditions agree as well, and we are done.

(IT4). $(K \circ_\alpha \bot) \circ_\beta \bot$

$$= (\text{by IT}) \begin{cases} (K \circ_{\bot \to \beta} \bot) + \bot & \text{, if } K \circ_\beta \bot \neq K_\bot \text{ and } \alpha \in K \circ_\alpha (\bot \wedge \neg\beta) \\ (K \circ_\alpha \bot) \cap (K \circ_\beta \bot) & \text{, if } K \circ_\beta \bot \neq K_\bot \text{ and } \alpha \notin K \circ_\alpha (\bot \wedge \neg\beta) \\ K_\bot & \text{, if } K \circ_\beta \bot = K_\bot \end{cases}$$

$$= \begin{cases} K_\bot & \text{, if } K \circ_\beta \bot \neq K_\bot \text{ and } \alpha \in K \circ_\alpha \bot \\ (K \circ_\alpha \bot) \cap (K \circ_\beta \bot) & \text{, if } K \circ_\beta \bot \neq K_\bot \text{ and } \alpha \notin K \circ_\alpha \bot \\ K_\bot & \text{, if } K \circ_\beta \bot = K_\bot \end{cases}$$

$$= (\text{by Q15}) \begin{cases} (K \circ_\alpha \bot) \cap (K \circ_\beta \bot) & \text{, if } K \circ_\alpha \bot, K \circ_\beta \bot \neq K_\bot \\ K_\bot & \text{, otherwise} \end{cases}$$

(IT5). By (Def $\circ$ from $\leq$), we have that

$\gamma \in (K \circ_\alpha \beta) \circ_\beta \alpha$

holds if and only if

either $\neg\alpha <' \beta \wedge (\alpha \to \gamma)$

or $\beta <' \gamma$

or $\top \leq' \beta \wedge \neg\alpha$

By (Def $\leq'$ from $\leq$), this means that

either $\beta \to \neg\alpha < \alpha \wedge (\beta \to (\beta \wedge (\alpha \to \gamma)))$ and $\beta \wedge (\alpha \to \gamma) \leq \alpha$

or $\neg\alpha < \beta \wedge (\alpha \to \gamma)$ and $\alpha < \beta \wedge (\alpha \to \gamma)$

or $\beta \to \beta < \alpha \wedge (\beta \to \gamma)$ and $\gamma \leq \alpha$

or $\beta < \gamma$ and $\alpha < \gamma$

or $\alpha \wedge (\beta \to \top) \leq (\beta \to (\beta \wedge \neg\alpha))$ and $\top \leq \alpha$

or $\top \leq \beta \wedge \neg\alpha$ and $\alpha < \top$

Given the properties of entrenchments, this collection of conditions reduces to the following complex of conditions:

$(*)$: *either* $\neg\alpha \vee \neg\beta < \alpha \wedge (\beta \rightarrow \gamma)$ and $\beta \wedge (\alpha \rightarrow \gamma) \leq \alpha$

or $\neg\alpha < \beta \wedge (\alpha \rightarrow \gamma)$ and $\alpha < \beta \wedge (\alpha \rightarrow \gamma)$

or $\beta < \gamma$ and $\alpha < \gamma$

or $\top \leq \alpha \wedge \neg\beta$

or $\top \leq \beta \wedge \neg\alpha$ and $\alpha < \top$

Now we make a case distinction. *Case 1* is $\alpha \leq \beta$. In this case (*) reduces to

either $\neg\alpha \vee \neg\beta < \alpha$ and $\neg\alpha \vee \neg\beta < \beta \rightarrow \gamma$ and $\beta \wedge (\alpha \rightarrow \gamma) \leq \alpha$

or $\alpha, \neg\alpha < \beta$ and $\alpha, \neg\alpha < \alpha \rightarrow \gamma$

or $\beta < \gamma$

or $\top \leq \bot$

or $\top \leq \beta \wedge \neg\alpha$

Taken together with $\alpha \leq \beta$, the first two lines of this complex imply $\neg\alpha < \beta \wedge (\alpha \rightarrow \gamma)$. (Note that the first line taken together with $\alpha \leq \beta$ implies $\neg\alpha \vee \neg\beta < \beta$ and $\neg\alpha \vee \neg\beta < \gamma$.)

Conversely, $\neg\alpha < \beta \wedge (\alpha \rightarrow \gamma)$ taken together with $\alpha \leq \beta$ implies that one of the first two lines of the complex is satisfied. (Notice that $\neg\alpha < \beta \wedge (\alpha \rightarrow \gamma)$ implies that both $\neg\alpha < \beta$ and $\neg\alpha < \alpha \rightarrow \gamma$, by the entrenchment conditions. Assuming then that the second line is not satisfied implies that either $\beta \leq \alpha$ or $\alpha \rightarrow \gamma \leq \alpha$, hence $\beta \wedge (\alpha \rightarrow \gamma) \leq \alpha$. Also, $\neg\alpha < \beta \wedge (\alpha \rightarrow \gamma)$ implies $((\beta \wedge (\alpha \rightarrow \gamma)) \rightarrow \neg\alpha) < \beta \wedge (\alpha \rightarrow \gamma)$, i.e., $\neg\alpha \vee \neg\beta \vee \neg\gamma < \beta \wedge (\alpha \rightarrow \gamma)$, and this yields $\neg\alpha \vee \neg\beta \vee \neg\gamma < \gamma$ as well as $\neg\alpha \vee \neg\beta \vee \neg\gamma < \alpha$.)

In sum, we find in Case 1 that $\gamma \in (K \circ_\alpha \beta) \circ_\beta \alpha$ iff

$\neg\alpha < \beta \wedge (\alpha \rightarrow \gamma)$ or

$\beta < \gamma$ or

$\top \leq \beta \wedge \neg\alpha$

i.e., by (Def $\circ$ from $\leq$), iff $\gamma \in K \circ_\beta \alpha$.

Continuing with the case distinction, we now turn to *Case 2* where $\beta < \alpha$. Lines 2 and 5 of $(*)$ are incompatible with this case, and the rest reduces to

either $\neg\alpha \vee \neg\beta < \alpha \wedge (\beta \rightarrow \gamma)$

or $\alpha < \gamma$

or $\top \leq \alpha \wedge \neg\beta$

The first line of this implies $\neg\beta < \alpha \wedge (\beta \rightarrow \gamma)$.

Conversely, $\neg\beta < \alpha \wedge (\beta \to \gamma)$ taken together with $\beta < \alpha$ implies the first line of the complex just written down. (Notice that $\neg\beta < \alpha \wedge (\beta \to \gamma)$ implies $((\alpha \wedge (\beta \to \gamma)) \to \neg\beta) < \alpha \wedge (\beta \to \gamma)$, i.e., $(\neg\alpha \vee \neg\beta \vee \neg\gamma) < \alpha \wedge (\beta \to \gamma)$, by the entrenchment conditions.)

In sum, we find in Case 2 that $\gamma \in (K \circ_\alpha \beta) \circ_\beta \alpha$ iff

$$\begin{array}{ll} \neg\beta < \alpha \wedge (\beta \to \gamma) & \text{or} \\ \alpha < \gamma & \text{or} \\ \top \leq \alpha \wedge \neg\beta & \end{array}$$

i.e., by (Def $\circ$ from $\leq$), iff $\gamma \in K \circ_\alpha \beta$.

Since $\alpha \leq \beta$ encodes $\alpha \notin K \circ_\beta \bot$ or $K \circ_\beta \bot = K_\bot$ (by Q15), our case distinction is exactly the one formulated in (IT5). ■