Basic Entrenchment

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Abstract. In contrast to other prominent models of belief change, models based on epistemic entrenchment have up to now been applicable only in the context of very strong packages of requirements for belief revision. This paper decomposes the axiomatization of entrenchment into independent modules. Among other things it is shown how belief revision satisfying only the 'basic' postulates of Alchourrón, Gärdenfors and Makinson can be represented in terms of entrenchment.

Keywords: Belief revision, logic of theory change, entrenchment

1. Introduction

In the mid-1980s, two by now classical models for belief change were introduced by Alchourrón, Gärdenfors and Makinson (henceforth, 'AGM'). Partial meet and safe contractions and revisions have, in their general axiomatic characterization, six 'basic' properties, they satisfy the so-called *basic AGM postulates* (also known as the *basic Gärdenfors postulates*). If (and only if) special conditions are imposed—transitive relationality in the case of partial meet contraction and continuing up/down and virtual connectedness in the case of safe contraction—we get two *supplementary AGM postulates*. These are the classical results obtained by Alchourrón, Gärdenfors and Makinson (1985) and Alchourrón and Makinson (1985), and this two-level architecture has served as an important frame of reference for many studies in the field up to the present day.

Although partial meet contraction and safe contraction are very different on the face of it, direct connections between the two kinds of constructions were discovered shortly after their introduction (Alchourrón and Makinson 1986). This to some extent explained the similarities in their logical properties.

A further deepening of the understanding of the relation between partial meet contraction and safe contraction was provided by Sven Ove Hansson (1994; 1999, Sections 2.8–2.9, 2.22^{+}–2.23^{+}). He actually studied a generalization of safe contraction that he termed 'kernel contraction.' Using the concept of kernel contraction as a missing link,

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Hansson obtained a number of beautiful results on the relationship between partial meet and safe contraction as applied to *belief bases*, i.e., sets of sentences that need not be logically closed. He showed that even if the class of kernel contractions is restricted to 'smooth' or 'saturated' kernel contractions, it is strictly larger than the class of partial meet contractions. For theories that are logically closed the difference between these two classes of operations vanishes.

Ever since the idea came into being (Grove 1988, Gärdenfors 1988, Gärdenfors and Makinson 1988), entrenchment-based belief changes have been regarded as the third 'classical' AGM-style way of changing beliefs. The situation with respect 'epistemic entrenchment' as a tool for constructing belief changes, however, is quite different from the situation with respect to partial meet and safe contraction.

First, as Hansson (1999, Sect. 2.10) has rightly pointed out, it is very difficult to apply the theory of entrenchment to *belief bases*, i.e., sets of sentences that need not be logically closed. There are finite representations of entrenchment relations, but it is problematic to apply these structures directly to operations of belief change.[1] Philosophically, there is little motivation for insisting that entrenchment satisfy certain logical constraints and at the same time renouncing to place any logical constraints on the set of beliefs.

A second problem, again highlighted by Hansson, is that the standard use of entrenchment orderings in the construction of belief contractions is dependent on these contractions' satisfying the postulate of recovery—and this is arguably the most controversial AGM postulate of all (see Hansson 1999, Sect. 2.3). One way of avoiding this problem is to focus on operations of *belief revision* rather than belief *belief contraction*, and this is what I will do in the following.

My main concern is with a third problem, however. Even in the case of logically closed theories (so-called *belief sets*), entrenchment-based belief change does not come with the neat separation of 'basic' and 'supplementary' conditions that we know from partial meet and safe contraction. The entrenchment-based construction was introduced 1988 only for the rather special case where the full package of all eight ('basic' and 'supplementary') AGM postulates is satisfied or desired.

Later on various liberalizations of the strict logical requirements for entrenchment relations were studied, with corresponding liberalizations of the AGM postulates (Lindström and Rabinowicz 1991, Rott 1992, 2001, Cantwell 2001). The liberalizations are not liberal enough, however. There is no answer in the literature so far whether it is pos-

[1] See the 'E-bases' in Rott (1991) and the 'ensconcements' in Williams (1994), as well as the critical remarks in Rott (2000a).

sible to exactly mirror AGM's distinction between basic and extended conceptions of belief change operations with models using entrenchment relations. It would be desirable (and perhaps even necessary to claim the status of a 'classical' AGM construction) to know whether entrenchment-based belief change operations are applicable in the same wide variety of contexts as partial meet and safe contraction. In this paper, I provide a positive answer to this question and present a theory of *basic entrenchment* that exactly fits basic belief change in the AGM sense, thus showing that the applicability of entrenchment-based constructions is indeed as wide as desired.

Ceteris paribus, a modelling will be considered the more natural, the more flexibile and adaptable it is to different contexts and uses. The present paper should thus help making entrenchment relations recognized as a natural and fully workable tool for belief revision.

A final word by way of introduction. The term 'entrenchment' as it is used here means something like 'comparative retractability' or 'vulnerability'. If ϕ is at most as entrenched as ψ, this means that in a case of doubt when one needs to give up either ϕ or ψ, it is not more difficult or painful to give up ϕ than to give up ψ. Belief revision theorists have never claimed that this idea covers all the connotations that the word 'entrenchment' may possibly carry. First and formost, the word is obviously not used here in the sense made famous in philosophy by Nelson Goodman (1973, Chapter 4); after all, Goodman speaks primarily of the entrenchment of predicates, not of sentences. Second, 'entrenchment' in our sense does not satisfy the expectations of authors like Klee (2000) who argues that a scientific law ought to be more entrenched than any of its instances. While the former may certainly be called more *important* that the latter, it cannot be more *entrenched* in the sense that is used here. It cannot be easier to give up 'If Henry is a raven, he is black' than to give up 'All ravens are black', because as soon as the former is called into serious doubt, the latter gets doubtful *a fortiori*.

2. The AGM postulates for belief revision

In order to make this paper self-contained, I now want review some elements of the classical belief revision theory—the well-known AGM postulates—and isolate three different concepts of coherence that they can be seen as embodying. The presentation in this section follows that of Rott (1999).

A *belief set* is a set of sentences of a given language $\mathcal{L}$, usually conceived as consistent, that is closed under logical consequences. We use $\vdash$

and Cn to indicate the consequence relation and operation governing $\mathcal{L}$, respectively, with the usual understanding that $Cn(H) = \{\phi : H \vdash \phi\}$. Without further indication, we will suppose that the logic is Tarskian (reflexive, idempotent and monotonic), that it includes classical propositional logic, that it is compact, and that it satisfies the deduction theorem. We reserve the letter 'K' for belief sets; '$K_\perp$' will be used to denote the inconsistent set of all $\mathcal{L}$-sentences.

Alchourrón, Gärdenfors and Makinson developed their theories for belief revision functions that model the potential (non-iterated[2]) changes of a given belief set. Such a function $*$ is associated with a belief set K and assigns, for each input sentence ϕ, the revision $K * \phi$ of K that assimilates ϕ. So formally a revision function $*$ associated with K is a function with domain $\mathcal{L}$ and range $\mathbb{K}$ (the set of all belief sets).

A revision function $*$ is usually supposed to satisfy certain conditions. In the belief revision literature these conditions (and those that will follow) are usually called 'rationality postulates.' We use the AGM labels to refer to them.

($*$1) $\quad K * \phi = Cn(K * \phi)$ *(Closure)*

($*$2) $\quad \phi \in K * \phi$ *(Success)*

($*$5) $\quad$ If $\nvdash \neg\phi$, then $K * \phi \neq K_\perp$ *(Consistency)*

($*$6) $\quad$ If $\phi \dashv\vdash \psi$, then $K * \phi = K * \psi$ *(Extensionality)*

In the present paper, I shall treat these postulates as fundamental conditions for belief revision. Roughly speaking, they say that revisions should be made in a way that is *successful* (i.e., the input is actually accepted in the posterior belief state—($*$2)), *inferentially coherent* (i.e., the posterior belief set is logically closed and consistent—($*$1) and ($*$5)) and *content-driven* (i.e., the result does not depend on variations in the surface grammar of the input sentence—($*$6)). Let us call the set consisting of ($*$1), ($*$2), ($*$5) and ($*$6) the set of *basic postulates for belief revision*, and revision functions satisfying them *basic revision functions*.[3]

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[2] In their classical paper, Alchourrón, Gärdenfors and Makinson (1985) sometimes *formally* use revision functions $*$ as binary functions taking various belief sets as their first argument, but this is not in the spirit of what they actually do. I have explained in Rott (1999) why I think that the conception of binary revision functions is not appropriate as a general framework in which to study iterated belief change.

[3] Actually I do not think that ($*$5) or its corresponding condition for entrenchment relations (called 'Maximality' below) are core conditions for the notions of belief change and entrenchment. Compare, for the case of belief contractions, Rott (1992) and (2001, especially Sections 6.1.2 and 8.2.5). However, dropping these con-

Postulates $(*1)$ and $(*5)$ taken together embody a notion of *synchronic coherence*. Synchronic notions of coherence are important for belief change, but if they were the only relevant notions of coherence, the theory of belief revision (in the usual sense) would be deprived of its very task. *Theory change then gets reduced to theory choice*: Just the best, most coherent theory will be chosen, regardless of any predecessor theories. Belief change on this account ceases to be grounded on inter-theory relations between prior and posterior belief sets, but is rather driven solely by the structure and properties of the posterior theory. The theory chooser jumps to the theory with the best overall characteristics that fits the data, without any commitment to his earlier theories. Having said this, it may be somewhat ironic to call the collection consisting of $(*1)$, $(*2)$, $(*5)$ and $(*6)$ the set of basic postulates for belief *revision*, but it does not seem to be misleading.

There are two more postulates that AGM also call 'basic', but are somewhat more problematic than those in the first group. They relate the revision function to the set K of currently held beliefs, and express principles of *conservatism* or *minimal change*.

$(*3)$ $\quad K * \phi \subseteq Cn(K \cup \{\phi\})$ *(Expansion)*

$(*4)$ $\quad$ If $\neg\phi \notin K$, then $K \subseteq K * \phi$ *(Preservation)*

These postulates present substantial recommendations of how to perform revisions by input sentences ϕ that are consistent with the prior beliefs in K. Condition $(*3)$ states that the agent should not acquire more beliefs than are necessary on the strength of $(*1)$ and $(*2)$; condition $(*4)$ tells him not to give up more beliefs than are necessary on the strength of $(*5)$.[4] Postulates $(*3)$ and $(*4)$ are vacuously satisfied if the input ϕ is inconsistent with the belief set K (i.e., if $\neg\phi \in K$). They may be regarded as restricted principles of *diachronic coherence*—restricted, that is, to the consistent case. This relational notion of coherence is very different from the synchronic one

ditions makes the technical treatment of limiting cases concerning the inconsistent belief set $K_\perp$ and maximally entrenched sentences (those that are as entrenched as a tautology) a lot more complicated. Since the technical complications in my opinion exceed the philosophical insight gained from relegating $(*5)$ and Maximality to the set of optional condidions, I have decided to treat them as 'basic' in the present paper.

[4] The original AGM conditions actually have as the fourth condition some kind of converse of $(*3)$, viz.,

$(*4')$ $\quad$ If $\neg\phi \notin K$, then $Cn(K \cup \{\phi\}) \subseteq K * \phi$

The additional strength of $(*4')$ over $(*4)$, however, can be gained from conditions $(*1)$ and $(*2)$. In order to avoid redundancies in the axiomatization, we use the more elementary Preservation condition $(*4)$.

codified in $(*1)$ and $(*5)$ which pertains to the properties of a single (posterior) belief state. The intuitive idea of diachronic coherence is that the prior and the posterior belief state (or more generally, the members in a sequence of belief states) somehow 'hang together.' In this sense, conservativity may be interpreted as a strategy aiming at a certain kind of coherence. Let us call revision functions satisfying $(*3)$ and $(*4)$ *c-conservative* (with respect to K, 'c' for 'consistent') or *faithful* (to K).

Although $(*3)$ and $(*4)$ look very straightforward, it is not obvious that they ought to be satisfied. It is characteristic of 'abductive belief revision' as modelled by Pagnucco (1996) and Nayak (2000) that property $(*3)$ does not hold. In the operation of 'belief updates' that are occasioned by changes in the world, $(*4)$ gets violated (Katsuno and Mendelzon 1992). The same is true of the kind of 'foundational belief change' advocated in Rott (2001, Chapter 5), and there are reasons against identifying consistent revisions ('additions') with expansions if the object language contains autoepistemic operators or conditionals (Rott 1991). Further interesting arguments against Preservation are put forward by Rabinowicz (1995) and Levi (1996, Chapters 2 and 3).

Finally, there are two 'supplementary' AGM postulates:

$(*7)$ $\quad K * (\phi \wedge \psi) \subseteq Cn((K * \phi) \cup \{\psi\})$

$(*8)$ $\quad$ If $\neg\psi \notin K * \phi$, then $K * \phi \subseteq K * (\phi \wedge \psi)$

It has frequently been pointed out that $(*7)$ implies $(*3)$ and that $(*8)$ implies $(*4)$—provided that we assume that $K = K * \top$.[5] But saying this tends to obscure the fact that $(*3)$ and $(*4)$ really deal with something completely different from what $(*7)$ and $(*8)$ are about. The former pair compares the prior and the posterior belief set in the case of a revision by an input that is consistent with the prior state. The latter pair compares potential revisions of a belief set by two different, but logically related input sentences, to wit, ϕ and $\phi \wedge \psi$. Seminal results in AGM belief change theory have shown that $(*7)$ and $(*8)$ are equivalent to the existence of a well-behaved, 'rationalizing' structure that can be ascribed to the agent's mental state and is considered to govern his changes of belief. Conditions $(*7)$ and $(*8)$ are about the agent's dispositions to change his beliefs in response to potential inputs. Let us call $(*7)$ and $(*8)$ *dispositional postulates*, and revision functions satisfying $(*7)$ and $(*8)$ *dispositional revision functions*. There are many

[5] If K is consistent, the identity $K = K * \top$ can itself be derived from $(*3)$ and $(*4)$. An alternative approach, taking revision functions as the only primitives of our modelling, would be to interpret the equation $K = K * \top$ as the *definition* of the current belief set. In orthodox AGM theory, however, the equation is satisfied only when K is consistent.

variations on (∗7) and (∗8) in the literature, but only one of them merits special attendance for the purposes of the present paper.

(∗8c) If $\psi \in K * \phi$, then $K * \phi \subseteq K * (\phi \wedge \psi)$

This condition was first discussed in Makinson and Gärdenfors (1991). Given the basic postuates for revisions, it is a weakening of AGM's condition (∗8). Conditions (∗7) and (∗8) have turned out to be particularly strong dispositional postulates,[6] and it is perhaps fair to say that just the dispositional postulates make the AGM theory of belief revision powerful and interesting. However, it is important to see they say nothing about any relation between prior and posterior belief states.

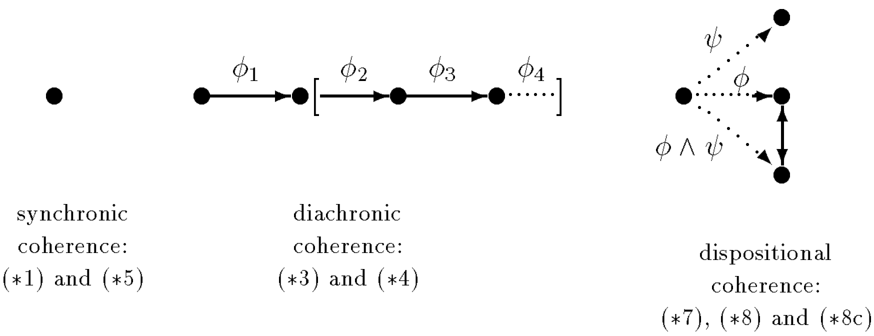

Figure 1. Three types of coherence

Basic, c-conservative and dispositional revision functions, i.e., revision functions satisfying (∗1) through (∗8), are called *AGM revision functions* (compare Figure 1). Notice that Alchourrón, Gärdenfors and Makinson impose no condition whatsoever that encodes a requirement of minimal change for $K * \phi$ in relation to K for the (more interesting) case where ϕ is inconsistent with K. It is a widespread myth that minimal change principles provide the foundation of the existing theories of belief revision, at least as far as the AGM tradition is concerned.[7] This is already evident from the fact that the revision function which sets $K * \phi = Cn(\{\phi\})$ in the inconsistent case (and $K * \phi = Cn(K \cup \{\phi\})$ in the consistent case) perfectly satisfies all the AGM postulates.

[6] See Rott (2001, Chapter 4). As conditions constraining revisions by different inputs, AGM's conditions (∗7) and (∗8) are very powerful indeed. They essentially imply that all beliefs in a belief set are comparable with one another in terms of entrenchment.

[7] Boutilier (1996, p. 264) and Darwiche and Pearl (1997, p. 2) call 'the principle of informational economy' or 'the principle of minimal belief change' the hallmark of the AGM theory. They are echoing familiar prejudices here, promoted by AGM themselves and repeated time and again in the literature. I have criticized the myth of minimal change in belief revision theory in Rott (2000b).

We are now going to unpack and modularize the concept of epistemic entrenchment in a way similar to the modularizing the concept of belief revision.

3. Basic entrenchment

What is 'epistemic entrenchment'[8]? Epistemic entrenchment is a binary relation $\leq$ over the sentences in $\mathcal{L}$ that is supposed to constructively govern changes of belief. But it is perhaps expedient to look first at the converse, *re*constructive interpretation of entrenchment. Given a revision function $*$, an entrenchment relation $\leq$ can be *retrieved from* $*$ by means of the following definition:[9]

$$(\leq \text{ from } *) \qquad \phi \leq \psi \quad \text{iff} \quad \phi \notin K * \neg(\phi \wedge \psi) \ \text{ or } \ \vdash \psi$$

This condition expresses essentially what we might call the *meaning of entrenchment*. For the principal case, it says that ϕ is not more entrenched than ψ in an agent's belief state if and only if the revision of the belief state occasioned by the information that not both ϕ and ψ are true leads to a state in which ϕ has been given up.

In order to appreciate the import of this concept, it is necessary to understand that *all basic revision functions* can be represented as revision functions based on an underlying relation of entrenchment.

OBSERVATION 3.1. If $*$ is a basic revision function, i.e., $*$ satisfies $(*1)$, $(*2)$, $(*5)$ and $(*6)$, and if $\leq$ is the entrenchment relation over $\mathcal{L}$ retrieved from $*$ by means of $(\leq$ from $*)$, then $*$ can be reconstructed with the help of $\leq$ in the following way:

$$(* \text{ from } \leq) \qquad \psi \in K * \phi \quad \text{iff} \quad \neg\phi < \phi \supset \psi \ \text{ or } \ \vdash \neg\phi$$

Here $<$ is the asymmetric part of $\leq$. Condition $(*$ from $\leq)$ says that ψ is in $K * \phi$ if the material conditional $\phi \supset \psi$ is strictly more

[8] Actually this name is a misnomer. The relations in question should really be called relations of 'doxastic entrenchment'. Logicians and researchers in AI have no problems in talking about 'knowledge' bases containing 'facts' that may well be false. For philosophers the situation is different. Plato, for one, laboured very hard, especially in the *Meno*, the *Republic* and the *Theaetetus*, to explicate the important differences between *episteme* and *doxa*. I'll continue, however, to use the sloppy way of speaking in order to keep in line with by now widely used terminology.

[9] Essentially the same definition as applied to belief *contraction* is due to Gärdenfors and Makinson (1988). It was transferred to belief *revision*, amongst others, by Lindström and Rabinowicz (1991, p. 97) and Rott (1991, p. 144); also cf. Hansson (1999, Section 3.9).—Constructive 'positive' and 'negative' interpretations of entrenchment are investigated in Rott (2000a).

entrenched than $\neg\phi$, i.e., the negation of the input sentence. Another way of putting things is to say that the material conditional is 'robust' with respect to its antecedent (cf. Jackson 1979, p. 569). If the input sentence is inconsistent, everything may be accepted. When a revision function $*$ is defined from a relation $\leq$ in $\mathcal{L}$ with the help of ($*$ from $\leq$), we say that $*$ is *based* on $\leq$, or that $\leq$ *determines* $*$.

Gärdenfors and Makinson (1988, Theorems 4 and 5) proved a result analogous to Observation 3.1 in the much more restricted context of AGM contraction functions satisfying eight postulates that are analogous to ($*$1) – ($*$8). Rott (1991, Obs. 1) transferred their result to AGM revision functions. The present result shows that only four of the eight AGM postulates are necessary for the applicability of entrenchments in revisions. In fact it is quite surprising to see how weak the properties of a revision function $*$ are that guarantee the existence of a relation $\leq$ that 'rationalizes' $*$. The situation of the approach using ($*$ from $\leq$) thus stands in sharp contrast to so-called partial meet contractions where not only ($*$1) – ($*$6), but also ($*$7) and weakened forms of ($*$8) are necessary in order to secure relationality.[10]

Unfortunately, ($*$ from $\leq$) is not a very transparent condition.[11] And unfortunately, it is not possible to use a more perspicuous representation and let $K * \phi$ be the set of logical consequences of $\{\psi \in K : \neg\phi < \psi\} \cup \{\phi\}$. It is true that $K * \phi$ is included the latter set, but the converse inclusion in general fails. In contrast to the case of AGM revision functions, if $\leq$ is retrieved from a basic revision function $*$, it does not follow from $\neg\phi < \psi$ that $\neg\phi < \phi \supset \psi$. So it does not follow that ϕ is in $K * \psi$.[12]

It is rather surprising how many properties of $\leq$ can be derived if only we know that the revision function $*$ from which it is retrieved is a basic belief revision function.

OBSERVATION 3.2. (a) If $*$ is a basic revision function satisfying ($*$1), ($*$2), ($*$5) and ($*$6), then the entrenchment relation $\leq$ retrieved from $*$ satisfies the following conditions:

(Reflexivity) $\phi \leq \phi$

[10] Cf. Alchourrón, Gärdenfors and Makinson (1985), Rott (1993), and the remarks in Section 4 below.

[11] As the proof of Theorem 3.1 shows, the following condition would equally well serve our purposes:

$$\psi \in K * \phi \quad \text{iff} \quad \phi \supset \neg\psi < \phi \supset \psi \quad \text{or} \quad \vdash \neg\phi$$

Some people find this condition more transparent than ($*$ from $\leq$).

[12] Counterexample: Consider $K = Cn(\{\neg p, q\})$ and let $*$ be a revision function with $K * p = Cn(\{p\})$ and $K * \neg(\neg p \wedge q) = Cn(\{p, q\})$. There is nothing in ($*$1) – ($*$6) that prevents such a function. For $\leq$ retrieved from $*$, we have $\neg p < q$, but neither $\neg p < p \supset q$ nor $q \in K * p$.

(Extensionality) If $\phi \dashv\vdash \psi$ then :
$\phi \leq \chi$ iff $\psi \leq \chi$, and $\chi \leq \phi$ iff $\chi \leq \psi$

(Choice) $\phi \wedge \psi \leq \chi$ iff $\phi \leq \psi \wedge \chi$ or $\psi \leq \phi \wedge \chi$

(Maximality) If $\top \leq \phi$ then $\vdash \phi$

(b) If $*$ in addition satisfies ($*$3), then $\leq$ also satisfies
(K-Minimality) If $\phi \notin K$ then $\phi \leq \psi$

(c) If $*$ in addition satisfies ($*$4), then $\leq$ also satisfies
(K-Representation) If $\phi \in K$ and $\phi \leq \psi$ then $\psi \in K$

(d) If $*$ in addition satisfies ($*$7), then $\leq$ also satisfies
(Continuing up) If $\phi \leq \psi \wedge \chi$ then $\phi \leq \psi$

(e) If $*$ in addition satisfies ($*$8c), then $\leq$ also satisfies
(Continuing down) If $\phi \leq \psi$ then $\phi \wedge \chi \leq \psi$

(f) If $*$ in addition satisfies ($*$7) and ($*$8), then $\leq$ also satisfies
(Transitivity) If $\phi \leq \psi$ and $\psi \leq \chi$ then $\phi \leq \chi$

Relations over $\mathcal{L}$ satisfying Reflexivity, Extensionality, Choice and Maximality will be called *basic entrenchment relations*. Reflexivity and Extensionality need no explanation. The left-hand side and the right-hand side of Choice are both ways of expressing that either ϕ or ψ is given up in a situation in which at least one of ϕ, ψ and χ needs to be given up. Besides being based on ($\leq$ from $*$), this reading derives from the idea that the task of giving up a conjunction $\phi \wedge \psi$ is precisely the same as the task of giving up at least one of ϕ and ψ. The postulate Choice receives its name from its interpretation as a central feature of entrenchments in a framework using 'syntactic' choice functions. It may also be read as saying that the entrenchment of a conjunction is as firm as that of the least entrenched conjunct. Yet another motivation can be drawn from a semantics based on choices between models:

$$\phi \leq \psi \quad \text{iff} \quad [\![\neg\phi]\!] \cap \gamma(\,[\![\neg(\phi \wedge \psi)]\!]\,) \neq \emptyset \quad \text{or} \quad \gamma(\,[\![\neg(\phi \wedge \psi)]\!]\,) = \emptyset$$

Intuitively, this says that ϕ is not more entrenched than ψ if and only if among the most plausible worlds that violate at least one of ϕ and ψ there is a world violating ϕ, provided there are any such worlds. It is now clear that Choice is immediately validated on this semantics. Both the left-hand side and the right-hand side of Choice say that among the most plausible worlds that violate at least one of ϕ, ψ and χ, there is a world violating either ϕ or ψ.[13]

[13] Both syntactic and semantic choice functions are studied in Rott 2001, Chapters 7 and 8.

The condition of Maximality presented here is slightly stronger than the condition with the same name in Gärdenfors and Makinson (1988) in that it replaces their antecedent 'if $\psi \leq \phi$ for all ψ' by the antecedent 'if $\top \leq \phi$'.[14]

The condition of K-Minimality is only one half of Gärdenfors and Makinson's condition with the same name. The condition of K-Representation was first discussed in Rott (1992). Notice that every entrenchment relation $\leq$ vacuously satisfies both K-Minimality and K-Representation with respect to the inconsistent belief set $K = K_{\perp}$. If there is a consistent belief set K with respect to which $\leq$ satisfies K-Minimality and K-Representation, then K is uniquely determined as $K = \{\phi : \perp < \phi\}$. This solution is immediately derived by substituting $\perp$ for ψ in K-Minimality and K-Representation. It can be shown[15] that $K = \{\phi : \perp < \phi\}$ is logically closed, and since it does not contain $\perp$, it is consistent. Applying ($*$ from $\leq$), we find that it is identical with $K * \top$. — Thus the condition that $\leq$ satisfies K-Minimality and K-Representation with respect to a consistent belief set K can also be expressed by saying that $\leq$ satisfies

($\perp$-Continuing up) If $\phi \leq \perp$ then $\phi \leq \psi$

($\perp$-Transitivity) If $\phi \leq \psi$ and $\psi \leq \perp$ then $\phi \leq \perp$

Standard GM-relations $\leq$ satisfy $\perp$-Continuing up and $\perp$-Transitivity.[16] By convention, we define for each entrenchment relation $\leq$ the associated belief set $K_{\leq} = \{\phi : \perp < \phi\}$. $K_{\leq}$ is a non-empty, logically consistent and closed set of sentences.[17]

Basic relations over $\mathcal{L}$ that also satisfy K-Minimality and K-Representation may be called *faithful* with respect to K. While Reflexivity, Extensionality and Choice are structural properties, Maxi-

[14] It can easily be shown that for relations $\leq$ obtained by ($\leq$ from $*$) from a revision function $*$ satisfying ($*1$) – ($*6$) the following holds:

(a) If $K \neq K_{\perp}$, then $\phi \leq \perp$ iff $\phi \leq \psi$ for all ψ.

(b) $\top \leq \phi$ iff $\psi \leq \phi$ for all ψ.

[15] Using Weak conjunctive splitting and Weak continuing down, see below.

[16] Similar observations can be made with respect to revisions. Every revision function $*$ vacuously satisfies ($*3$) and ($*4$) with respect to the inconsistent belief set $K = \mathcal{L}$. If $*$ satisfies ($*3$) and ($*4$) with respect to a consistent belief set K, then $K = K * \top$. The condition that $*$ satisfies ($*3$) and ($*4$) with respect to a consistent belief set K can also be expressed by saying that $*$ satisfies

($\top$-$*7$) $K * \phi \subseteq (K * \top) + \phi$

($\top$-$*8$) If $\neg\phi \notin K * \top$, then $K * \top \subseteq K * \phi$

which turn out to be special cases of ($*7$) and ($*8$), by $K * \phi = K * (\top \wedge \phi)$. This is a good way of taking either $\leq$ or $*$ as primitive and the belief set K as a derived entity.

[17] See Lemma 3.3.

mality, K-Minimality and K-Representation concern the limiting cases of tautologies and non-beliefs.

The conditions of Continuing up and Continuing down were found to be important for strict relations $<$ in Alchourrón and Makinson (1985) and Rott (1992).[18]

Faithful relations that in addition satisfy Transitivity are called *standard entrenchment relations*. We shall shortly show that they are precisely the entrenchment relations of Gärdenfors (1988) and Gärdenfors and Makinson (1988).[19]

Although the conditions for basic entrenchment are not at all vacuous, they do not guarantee acyclicity.[20] But basic entrenchment relations further satisfy a number of important properties.

LEMMA 3.3. Let $\leq$ satisfy Reflexivity, Extensionality and Choice.
(a) Then it also satisfies the following properties:

(Conjunctiveness)	$\phi \leq \psi$ iff $\phi \leq \phi \wedge \psi$
(Conditionalization)	$\phi \leq \psi$ iff $\phi \leq \phi \supset \psi$
(Connectedness)	$\phi \leq \psi$ or $\psi \leq \phi$
(GM-Dominance)	If $\phi \vdash \psi$ then $\phi \leq \psi$
(GM-Conjunctiveness)	$\phi \leq \phi \wedge \psi$ or $\psi \leq \phi \wedge \psi$
(Weak conjunctive splitting)	If $\phi \wedge \psi \leq \chi$ and $\chi \vdash \phi \wedge \psi$ then $\phi \leq \chi$ or $\psi \leq \chi$
(Weak continuing down)	If $\phi \leq \psi$ and $\chi \vdash \phi$ and

[18] If $\leq$ is the converse complement of $<$, and $<$ is taken to be asymmetric, then Continuing down for $<$ becomes Continuing up for $\leq$, and vice versa. Alchourrón and Makinson (1985, Obs. 4.3 and 5.3) proved that for 'safe contraction', each of Continuing up and Continuing down entails a condition that corresponds to $(*7)$. It is interesting to see that for entrenchment-based revision, the two conditions have quite different effects. — The condition $(*8)$ alone corresponds to the following condition that has little intuitive appeal (cf. Rott 2001, Obs. 31, 32 and 51):

$$\text{If } \phi \leq \psi \text{ and } \phi \wedge \psi \leq \chi \text{ then } \phi < \psi \wedge \chi$$

[19] The term 'standard entrenchment' was suggested in Rott (1992) where it is contrasted with 'generalized entrenchment'. Revision functions based on generalized entrenchments characteristically satisfy some natural weakenings of the dispositional postulates $(*7)$ and $(*8)$. But since generalized entrenchments are not required to satisfy Maximality, they are incomparable in strength with the basic entrenchments which are central for the present paper. Cf. footnote 3 above.

[20] Counterexample: Consider $K = Cn(\{p, q, r\})$ and let $*$ be a revision function with $K * \neg(p \wedge q) = Cn(\{\neg p, q, r\})$, $K * \neg(q \wedge r) = Cn(\{p, \neg q, r\})$ and $K * \neg(p \wedge r) = Cn(\{p, q, \neg r\})$. There is nothing in $(*1)$ – $(*6)$ that excludes such a function. For $\leq$ retrieved from $*$, however, we have $p < q < r < p$.—Transitivity on the other hand immediately entails acyclicity.

	$\phi \wedge \psi \vdash \chi$ then $\chi \leq \psi$
(Weak continuing up)	If $\phi \leq \psi$ and $\psi \vdash \chi$ and
	$\chi \vdash \phi \supset \psi$ then $\phi \leq \chi$
(Closure of $K_{\leq}$)	$K_{\leq} = Cn(K_{\leq})$

(b) If $\leq$ in addition satisfies Maximality, then it also satisfies

(GM-Maximality)	If $\psi \leq \phi$ for all ψ then $\vdash \phi$

(c) If $\leq$ in addition satisfies K-Minimality and K-Representation, then it also satisfies

(GM-Minimality)	If $K \neq K_{\perp}$, then:
	$\phi \leq \psi$ for all ψ iff $\phi \notin K$

(d) If $\leq$ in addition satisfies Transitivity, then it also satisfies Continuing up and Continuing down.

The relations considered in Lemma 3.3(a)-(c) are required to meet far less demanding conditions than the entrenchment relations of Gärdenfors and Makinson (1988) which can only be retrieved from revision functions satisfying the full set of AGM postulates for revision. While basic entrenchment is in general not transitive, standard entrenchment is. In fact Gärdenfors and Makinson characterize their entrenchment relations by the set consisting of GM-Dominance, GM-Conjunctiveness, GM-Maximality, GM-Minimality plus Transitivity. Let us call such relations *GM-entrenchment relations*. We have seen that transitivity is the only GM-feature that basic entrenchments miss. Conversely, we have:

OBSERVATION 3.4. GM-entrenchment relations satisfy all conditions of basic entrenchments.

Next we show that if a basic revision function $*$ is determined by some basic entrenchment relation, then this entrenchment relation is precisely the one which is retrievable from $*$.

OBSERVATION 3.5.[21] Let the entrenchment relation $\leq$ satisfy Reflexivity, Extensionality and Choice, and let $*$ be based on $\leq$. Then $\leq$ can be retrieved from $*$ with the help of ($\leq$ from $*$).

The final result of this section is as it were the converse of Observation 3.2. It shows that the constraints for basic entrenchment

[21] Gärdenfors and Makinson (1988, Theorems 4 and 5) proved an analogous result in the more restrictive context of AGM contraction functions and standard entrenchment relations.

relations make sure that the revisions based on them satisfy the basic postulates for revison functions. Further constraints on entrenchments yield corresponding constraints on revision functions.

OBSERVATION 3.6. (a) If $\leq$ is a basic entrenchment relation satisfying Reflexivity, Extensionality, Choice and Maximality, then the revision function $*$ based on $\leq$ satisfies ($*1$), ($*2$), ($*5$) and ($*6$).

(b) If $\leq$ in addition satisfies K-Minimality, then $*$ satisfies ($*3$).

(c) If $\leq$ in addition satisfies K-Representation, then $*$ satisfies ($*4$).

(d) If $\leq$ in addition satisfies Continuing up, then $*$ satisfies ($*7$).

(e) If $\leq$ in addition satisfies Continuing down, then $*$ satisfies ($*8c$).

(f) If $\leq$ in addition satisfies Transitivity, then $*$ satisfies ($*7$) and ($*8$).

We have now seen a bijective mapping between basic belief revision functions and basic entrenchment relations. The most important and original property characterizing basic entrenchment is a condition we called 'Choice'. Additional constraints can be matched one by one. Two conditions for c-conservative revision functions (also considered 'basic' by AGM) correspond to two conditions for faithful entrenchment relations. Three 'supplementary' conditions for dispositional revision functions correspond to the conditions of Continuing up, Continuing down and Transitivity for entrenchment, respectively. This picture decomposes entrenchment-based belief change in a way that is simlar to the way AGM decomposed partial meet contractions and safe contractions in the 1980s.

In order to secure transitivity for an entrenchment relation retrieved from a revision function $*$, the full power of postulates ($*7$) and ($*8$) is badly needed (Gärdenfors and Makinson 1988). Given the failure of transitivity for basic entrenchment relations, it is remarkable that connectedness follows immediately from ($\leq$ from $*$) if only $*$ satisfies ($*1$) and ($*5$). In the context of non-strict relations of epistemic entrenchment, connectedness is a rather trivial condition, while transitivity is a non-trivial and indeed very strong condition. This surprising finding can be explained by reflecting on the meaning of the relation $\leq$ as given by ($\leq$ from $*$). The point is that $\phi \leq \psi$ does not simply mean that ψ is at least as entrenched as ϕ, but rather that ψ is at least as entrenched *or incomparable* with ϕ (i.e., that ϕ is not more entrenched than ψ). One reason for withdrawing ϕ when forced to give up either ϕ or ψ is that ϕ is less or equally entrenched as ψ, another reason is given when one fails to find a unique standard for the comparison of ϕ and ψ. Once the possibility of incomparabilities is recognized, both the connectedness and the failure of transitivity of $\leq$ are very natural features indeed.[22]

4. The use of relations in belief change operations

Partial meet contractions use choice functions (over sets of maximal non-implying subsets), whereas safe contractions and entrenchment-based contractions use relations (over sentences). It is well-known from the theory of rational choice that some, but not all choice functions can be 'rationalized' by a preference relation, in the sense that it is exactly the 'best elements' according to the preference relation that are in the chosen set. Such are the prescriptions of the method of optimization (maximization or minimization). A common slogan in the classical theory of rational choice has been *rational choice is relational choice*.[23]

In a similar vein, Alchourrón, Gärdenfors and Makinson (1985) studied the impact of relations rationalizing their choice functions, finding that for the rationalizability by a transitive preference relation their two 'supplementary' postulates for belief change are necessary and sufficient. But even rationalizability by a preference relation that is not necessarily transitive puts rather heavy demands on belief change functions.[24]

Interestingly, as we know from Alchourrón and Makinson (1985) and from the above reflections, preference relations can be used even in the context of 'basic' belief change, if we turn to the case of safe contractions and entrenchment-based contractions.

How can this be, given the wide-ranging parallels that can be uncovered between partial meet and entrenchment-based operations?[25] The reason is that in both safe and entrenchment-based belief change operations, the relations are *not* employed in a straightforward minimization

[22] As a general property of orderings, transitivity is of course still to be regarded as more natural than connectedness. This intuition can be complied with if one works with strict entrenchment relations $<$ that are the converse complements of our relations $\leq$. For an explanation of the advantages of this transition, see Rott (1992, p. 50).

[23] Compare for instance Chernoff (1954), Herzberger (1973) and Sen (1997).

[24] For the case of logically finite belief sets, see Rott (1993).

[25] It is actually shown in Rott (2001, Chapters 7–8) that constraints for choice functions on the 'semantic' level (which corresponds to partial meet functions) almost always lead to the same belief-change behaviour as analogous constraints on the 'syntactic' level (which corresponds to entrenchment-based functions, with entrenchments being interpreted as revealed preferences). In the only exceptional case, the sytactic level yields an even *stronger* condition than the semantic level—adding as it were a little to the puzzlement expressed by the above question.

or maximization process, but in more complicated processes. In the partial meet operations of Alchourrón, Gärdenfors and Makinson, they are.[26]

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Appendix: Proofs

Proof of Observation 3.1.

We show that ($*$ from $\leq$) follows from ($\leq$ from $*$) if $*$ satisfies ($*1$), ($*2$), ($*5$) and ($*6$).

First notice that if $\leq$ is retrieved from $*$ with the help of ($\leq$ from $*$), then $\leq$ is connected, by ($*1$), ($*4$) and ($*5$). Now ($\leq$ from $*$) gives us

$\phi \supset \psi \leq \neg\phi$ iff $(\phi \supset \psi \notin K * \neg((\phi \supset \psi) \wedge \neg\phi)$ or $\vdash \neg\phi$

By ($*6$), this is equivalent with

$\phi \supset \psi \leq \neg\phi$ iff $(\phi \supset \psi \notin K * \phi$ or $\vdash \neg\phi)$

By ($*1$) and ($*2$), $\phi \supset \psi \notin K * \phi$ is equivalent with $\psi \notin K * \phi$. Hence

$\phi \supset \psi \leq \neg\phi$ iff $(\psi \notin K * \phi$ or $\vdash \neg\phi)$

By ($*1$) and ($*2$) again, we see that $\vdash \neg\phi$ implies $\psi \in K * \phi$, so we get

$(\phi \supset \psi \not\leq \neg\phi$ or $\vdash \neg\phi)$ iff $\psi \in K * \phi$

But since we already know that $\leq$ is connected, we conclude that

$((\neg\phi \leq \phi \supset \psi$ and $\phi \supset \psi \not\leq \neg\phi)$ or $\vdash \neg\phi)$ iff $\psi \in K * \phi$

which is precisely ($*$ from $\leq$). □

Proof of Observation 3.2.

(a) Let $*$ be a revision function satisfying ($*1$), ($*2$), ($*5$) and ($*6$), and let $\leq$ be retrieved from $*$ with the help of ($\leq$ from $*$).

(Reflexivity). $\phi \leq \phi$ means that either $\phi \notin K * \neg(\phi \wedge \phi)$ or $\vdash \phi \wedge \phi$. Suppose that $\nvdash \phi$. Then, by ($*5$) and ($*6$), $K * \neg(\phi \wedge \phi) = K * \neg\phi \neq K_\perp$. By ($*2$), $\neg\phi \in K * \neg\phi$, so by ($*1$), $\phi \notin K * \neg(\phi \wedge \phi)$, as desired.

(Extensionality) Let $\vdash (\phi \leftrightarrow \psi)$. Let also $\phi \leq \chi$, i.e., $\phi \notin K * \neg(\phi \wedge \chi)$ or $\vdash \chi$. It follows from ($*1$) and ($*6$) that $\psi \notin K * \neg(\psi \wedge \chi)$ or $\vdash \chi$, so $\psi \leq \chi$. Now let $\chi \leq \phi$, i.e., $\chi \notin K * \neg(\phi \wedge \chi)$ or $\vdash \phi$. It follows from ($*6$) that $\chi \notin K * \neg(\psi \wedge \chi)$ or $\vdash \psi$, so $\chi \leq \psi$.

[26] For more details on this, see Rott (2001, Section 8.7).

(Choice) The left-hand side means that $\phi \wedge \psi \notin K * \neg((\phi \wedge \psi) \wedge \chi)$ or $\vdash \chi$. The right-hand side means that $\phi \notin K * \neg(\phi \wedge (\psi \wedge \chi))$ or $\vdash \psi \wedge \chi$ or $\psi \notin K * \neg(\psi \wedge (\phi \wedge \chi))$ or $\vdash \phi \wedge \chi$. It follows immediately from (*1) and (*6) that the right-hand side implies the left-hand side. For the converse direction, let the left-hand side be satisfied. Suppose that $\phi \in K * \neg(\phi \wedge \psi \wedge \chi)$ and $\psi \in K * \neg(\phi \wedge \psi \wedge \chi)$ (this notation is well-defined, thanks to (*6)). Hence by (*1), $\phi \wedge \psi \in K * \neg(\phi \wedge \psi \wedge \chi)$. Therefore $\vdash \chi$. Now suppose for reductio that $\nvdash \psi \wedge \chi$ and $\nvdash \phi \wedge \chi$. It follows that $\nvdash \phi \wedge \psi \wedge \chi$. However, by (*1) we have $\phi \wedge \psi \wedge \chi \in K * \neg(\phi \wedge \psi \wedge \chi)$, contradicting (*2), (*1) and (*5).

(Maximality) That $\top \leq \phi$ means that $\top \notin K * \neg(\top \wedge \phi)$ or $\vdash \phi$. But the former cannot be, by (*1). So $\vdash \phi$.

(b) Now let $*$ in addition satisfy (*3).

(K-Minimality) Assume that $\phi \notin K$. By (*3), $K * \neg(\phi \wedge \psi) \subseteq Cn(K \cup \{\neg(\phi \wedge \psi)\})$. But since $\phi \notin K$, we get from the fact that K is a belief set that $(\neg(\phi \wedge \psi)) \supset \phi \notin K$ either. So $\phi \notin Cn(K \cup \{\neg(\phi \wedge \psi)\})$ and $\phi \notin K * \neg(\phi \wedge \psi)$. Thus $\phi \leq \psi$.

(c) Now let $*$ in addition satisfy (*4).

(K-Representation) Let $\phi \in K$ and $\phi \leq \psi$. The latter means that $\phi \notin K * \neg(\phi \wedge \psi)$ or $\vdash \psi$. Suppose for reductio that $\psi \notin K$. Then, since K is a belief set, $\phi \wedge \psi \notin K$, so by (*4) $K \subseteq K * \neg(\phi \wedge \psi)$. So since $\phi \in K$, we also have $\phi \in K * \neg(\phi \wedge \psi)$. Thus it must be the case that $\vdash \psi$. Hence $\psi \in K$, by K's being a belief set, and we have a contradiction again.

(d) Now let $*$ in addition satisfy (*7).

(Continuing up) Let $\phi \leq \psi \wedge \chi$, which means that $\phi \notin K * \neg(\phi \wedge \psi \wedge \chi)$ or $\vdash \psi \wedge \chi$. We have to show that $\phi \leq \psi$, that is, $\phi \notin K * \neg(\phi \wedge \psi)$ or $\vdash \psi$. If $\vdash \psi \wedge \chi$, then $\vdash \psi$, so this case is trivial. Let now $\phi \notin K * \neg(\phi \wedge \psi \wedge \chi)$. By (*1) and (*6), this is equivalent with $(\neg\phi \vee \neg\psi \vee \chi) \supset \phi \notin K * (\neg\phi \vee \neg\psi \vee \neg\chi)$. By (*1) again, this is equivalent with $\phi \notin Cn((K * (\neg\phi \vee \neg\psi \vee \neg\chi)) \cup \{\neg\phi \vee \neg\psi \vee \chi\})$. By (*7), it follows that $\phi \notin K * ((\neg\phi \vee \neg\psi \vee \neg\chi) \wedge (\neg\phi \vee \neg\psi \vee \chi))$, which means, by (*6), that $\phi \notin K * \neg(\phi \wedge \psi)$, which is what we wanted to show.

(e) Now let $*$ in addition satisfy (*8c).

(Continuing down) Let $\phi \leq \psi$, which means that $\phi \notin K * \neg(\phi \wedge \psi)$ or $\vdash \psi$. We have to show that $\phi \wedge \chi \leq \psi$, that is, $\phi \wedge \chi \notin K * \neg(\phi \wedge \psi \wedge \chi)$ or $\vdash \psi$. The case $\vdash \psi$ is trivial. Let now $\nvdash \psi$ and $\phi \notin K * \neg(\phi \wedge \psi)$, which, by (*6), is equivalent with $\phi \notin K * ((\neg\phi \vee \neg\psi \vee \neg\chi) \wedge (\neg\phi \vee \neg\psi \vee \chi))$. Suppose for reductio that $\phi \wedge \chi \in K * \neg(\phi \wedge \psi \wedge \chi) = K * (\neg\phi \vee \neg\psi \vee \neg\chi)$. By (*1), then $\neg\phi \vee \neg\psi \vee \chi \in K * (\neg\phi \vee \neg\psi \vee \neg\chi)$. By (*8c), then, $K * (\neg\phi \vee \neg\psi \vee \neg\chi) \subseteq K * ((\neg\phi \vee \neg\psi \vee \neg\chi) \wedge (\neg\phi \vee \neg\psi \vee \chi))$. Therefore, we have $\phi \notin K * (\neg\phi \vee \neg\psi \vee \neg\chi)$. Using $(*1)$, however, we see that this contradicts our supposition that $\phi \wedge \chi \in K * (\neg\phi \vee \neg\psi \vee \neg\chi)$, so we are done.

(f) Finally, let $*$ in addition satisfy $(*7)$ and $(*8)$. For this case (or rather, the analogous case of AGM *contraction* functions, cf. Alchourrón, Gärdenfors and Makinson, 1985, and Gärdenfors, 1988) it is proved in Gärdenfors and Makinson (1988, Theorem 5) that $\leq$ satisfies Transitivity. □

Proof of Lemma 3.3.

(a) Conjunctiveness. By Choice, $\phi \leq \phi \wedge \psi$ is equivalent with $\phi \wedge \phi \leq \psi$, which is, by Extensionality, equivalent with $\phi \leq \psi$.

Conditionalization follows from Conjunctiveness and Extensionality.

Connectedness. By Reflexivity, we have $\phi \wedge \psi \leq \phi \wedge \psi$. Thus, by Choice, either $\phi \leq \psi \wedge \phi \wedge \psi$ or $\psi \leq \phi \wedge \phi \wedge \psi$. By Extensionality, this means that either $\phi \leq \phi \wedge \psi$ or $\psi \leq \psi \wedge \phi$, so by Conjunctiveness either $\phi \leq \psi$ or $\psi \leq \phi$.

GM-Dominance. Let $\phi \vdash \psi$. Since $\phi \leq \phi$ by Reflexivity, it follows from Extensionality that $\phi \leq \phi \wedge \psi$. So by Choice $\phi \wedge \phi \leq \psi$, so by Extensionality again $\phi \leq \psi$.

GM-Conjunctiveness follows from Connectedness and Conjunctiveness.

Weak conjunctive splitting. Let $\phi \wedge \psi \leq \chi$ and $\chi \vdash \phi \wedge \psi$. Choice gives us either $\phi \leq \psi \wedge \chi$ or $\psi \leq \phi \wedge \chi$. Hence, by, Extensionality, either $\phi \leq \chi$ or $\psi \leq \chi$.

Weak continuing down. Let $\phi \leq \psi$, $\phi \wedge \psi \vdash \chi$ and $\chi \vdash \phi$. By Conjunctiveness, we have $\phi \leq \phi \wedge \psi$, so by Extensionality $\phi \leq (\phi \wedge \chi) \wedge \psi$. Hence by Choice $\phi \wedge (\phi \wedge \chi) \leq \psi$, so by Extensionality $\chi \leq \psi$.

Weak continuing up. Let $\phi \leq \psi$, $\psi \vdash \chi$ and $\chi \vdash \phi \supset \psi$. Then by Conjunctiveness, $\phi \leq \phi \wedge \psi$, so by Extensionality, $\phi \leq \phi \wedge \chi$, thus by Conjunctiveness again, $\phi \leq \chi$.

Closure of $K_{\leq}$. Suppose that $\{\phi_1, \ldots, \phi_n\} \vdash \psi$ and $\psi \notin K_{\leq} = \{\phi : \bot < \phi\}$. By Connectedness, $\psi \leq \bot$. Then by Extensionality, $\psi \leq (\phi_1 \wedge \ldots \wedge \phi_n) \wedge \bot$. Then by Choice, $\psi \wedge (\phi_1 \wedge \ldots \wedge \phi_n) \leq \bot$. Then by Extensionality again, $\phi_1 \wedge \ldots \wedge \phi_n \leq \bot$. Hence, by repeated application of Choice, $\phi_i \leq \bot$, i.e., $\phi_i \notin K_{\leq}$ for some i.

(b) GM-Maximality follows immediately from Maximality.

(c) GM-Minimality. One half follows immediately from K-Minimality. For the other half, let $K \neq K_{\bot}$ and $\phi \leq \psi$ for all ψ. Assume for reductio that $\phi \in K$. Choose some ψ that is not in K. Such a ψ exists since $K \neq K_{\bot}$. By K-Representation, we get that $\phi \not\leq \psi$, contradicting one of our assumptions.

(d) Both Continuing up and Continuing down follow immediately from GM-Dominance and Transitivity. □

Proof of Observation 3.4.

Reflexivity follows from GM-dominance. Extensionality follows from GM-dominance and Transitivity. K-Minimality follows from GM-Minimality. Since $\psi \vdash \top$, Maximality follows from GM-Maximality, GM-Dominance and Transitivity. K-Representation follows from GM-Minimality and Transitivity. The proof for Choice is the only one that is not immediate; we need to show that

$$\phi \wedge \psi \leq \chi \text{ iff } \phi \leq \psi \wedge \chi \text{ or } \psi \leq \phi \wedge \chi$$

From left to right. Suppose that $\phi \wedge \psi \leq \chi$. By GM-Conjunctiveness, either $\phi \leq \phi \wedge \psi$ or $\psi \leq \phi \wedge \psi$. Suppose the former (the other case is analogous). By GM-Dominance $\phi \wedge \psi \leq \psi$, so by Transitivity $\phi \leq \psi$. Since $\phi \wedge \psi \leq \chi$, it also holds, by Transitivity, that $\phi \leq \chi$. But either $\psi \leq \psi \wedge \chi$ or $\chi \leq \psi \wedge \chi$, by GM-Conjunctiveness. So in any case, by Transitivity again, $\phi \leq \psi \wedge \chi$.
From right to left. Suppose that either $\phi \leq \psi \wedge \chi$ or $\psi \leq \phi \wedge \chi$. Suppose the former (the other case is analogous). By GM-Dominance, it holds that $\phi \wedge \psi \leq \phi$ and $\psi \wedge \chi \leq \chi$. In sum, then, $\phi \wedge \psi \leq \phi \leq \psi \wedge \chi \leq \chi$, giving us $\phi \wedge \psi \leq \chi$ by twofold application of Transitivity. □

Proof of Observation 3.5.

We show that ($\leq$ from $*$) follows from ($*$ from $\leq$) if $\leq$ satisfies Reflexivity, Extensionality and Choice. We know from Lemma 3.3 that $\leq$ also satisfies GM-Dominance and Conjunctiveness.
($*$ from $\leq$) gives us

$\phi \in K * \neg(\phi \wedge \psi)$ iff $((\neg\neg(\phi \wedge \psi) \leq \neg\neg(\phi \wedge \psi) \vee \phi$ and $\neg\neg(\phi \wedge \psi) \vee \phi \not\leq \neg\neg(\phi \wedge \psi))$ or $\vdash \neg\neg(\phi \wedge \psi))$

Hence, by Extensionality,

$\phi \in K * \neg(\phi \wedge \psi)$ iff $((\phi \wedge \psi \leq \phi$ and $\phi \not\leq \phi \wedge \psi)$ or $\vdash \phi \wedge \psi)$

or equivalently

$\phi \notin K * \neg(\phi \wedge \psi)$ iff $((\phi \wedge \psi \not\leq \phi$ or $\phi \leq \phi \wedge \psi)$ and $\nvdash \phi \wedge \psi)$

But $\phi \wedge \psi \not\leq \phi$ is impossible, by GM-Dominance, and $\phi \leq \phi \wedge \psi$ is equivalent with $\phi \leq \psi$, by Conjunctiveness. A simplified formulation thus is

$\phi \notin K * \neg(\phi \wedge \psi)$ iff $\phi \leq \psi$ and $\nvdash \phi \wedge \psi$

Moreover, $\vdash \phi \wedge \psi$ implies $\vdash \psi$ which in turn implies $\phi \leq \psi$, by GM-Dominance. Hence we have:

$(\phi \notin K * \neg(\phi \wedge \psi)$ or $\vdash \phi \wedge \psi)$ iff $\phi \leq \psi$

This, however, is just ($\leq$ from $*$). □

Proof of Observation 3.6.

(a) Let $\leq$ satisfy Reflexivity, Extensionality, Choice and Maximality, and let $*$ be based on $\leq$. We know from Lemma 3.3 that it follows that $\leq$ satisfies Connectedness, Weak conjunctive splitting and Weak continuing down. If $\vdash \neg\phi$, then $K * \phi = K_\bot$ by ($*$ from $\leq$), and conditions ($*1$) – ($*6$) are all satisfied. So let $\not\vdash \neg\phi$ in the following proofs of (a), (b) and (c).

($*1$). We prove that $K * \phi = Cn(K * \phi)$ by first showing that $K * \phi$ is closed under conjunctions and secondly showing that $K * \phi$ is closed under singleton entailments. The claim then follows from the compactness of Cn. We first show that

$$\text{If } \psi \in K * \phi \text{ and } \chi \in K * \phi, \text{ then } \psi \wedge \chi \in K * \phi$$

Since we have assumed that $\not\vdash \neg\phi$, the antecedent means that $\neg\phi < \phi \supset \psi$ and $\neg\phi < \phi \supset \chi$, and the consequent means that $\neg\phi < \phi \supset (\psi \wedge \chi)$. Since $\leq$ is connected, what we need to prove is that $\phi \supset (\psi \wedge \chi) \leq \neg\phi$ implies that either $\phi \supset \psi \leq \neg\phi$ or $\phi \supset \chi \leq \neg\phi$ is true. But this follows directly from Extensionality and Weak conjunctive splitting. Next we show that

$$\text{If } \psi \in K * \phi \text{ and } \psi \vdash \chi, \text{ then } \chi \in K * \phi$$

Since $\not\vdash \neg\phi$, the antecedent means that $\neg\phi < \phi \supset \psi$ and $\psi \vdash \chi$, and the consequent means that $\neg\phi < \phi \supset \chi$. By Connectedness, we need to prove that $\psi \vdash \chi$ and $\phi \supset \chi \leq \neg\phi$ taken together imply $\phi \supset \psi \leq \neg\phi$. But since $\psi \vdash \chi$ implies $\psi \wedge \chi \vdash \phi \supset \psi \vdash \phi \supset \chi$, this follows directly from Weak continuing down.

($*2$). For $\phi \in K * \phi$, we need to show that $\neg\phi < \phi \supset \phi$ is true. But by Connectedness and Extensionality, this means that $\top \not\leq \neg\phi$. Since we are assuming that $\not\vdash \neg\phi$, this follows from Maximality.

($*5$). We still assume that $\not\vdash \neg\phi$. We want to show that $K * \phi \neq K_\bot$. Thanks to ($*1$) which we have already verified, it suffices to show that $\bot \notin K * \phi$. This means we need to show that $\not\vdash \neg\phi$ and $\neg\phi \not< \phi \supset \bot$. The former is true by hypothesis. The latter means, by Extensionality, that $\neg\phi \not< \neg\phi$. But this follows from Reflexivity.

($*6$). Let $\phi \dashv\vdash \psi$. We need to show that $\neg\phi < \phi \supset \chi$ if and only if $\neg\psi < \psi \supset \chi$. Or equivalently, that $\phi \supset \chi \leq \neg\phi$ if and only if $\psi \supset \chi \leq \neg\psi$. This follows from Extensionality.

(b) Now let $\leq$ in addition satisfy K-Minimality.

($*3$). In order to show that $K * \phi \subseteq Cn(K \cup \{\phi\})$, assume that $\psi \in K * \phi$, that is, that either $\vdash \neg\phi$ or $\neg\phi < \phi \supset \psi$. We need to show that $\psi \in Cn(K \cup \{\phi\})$, i.e., since K is a belief set, that $\phi \supset \psi \in K$. Since we have assumed that $\not\vdash \neg\phi$, we know that $\phi \supset \psi \not\leq \neg\phi$. Hence, by K-Minimality, $\phi \supset \psi \in K$, as desired.

(c) Now let $\leq$ in addition satisfy K-Representation.

$(*4)$. Let $\neg\phi \notin K$. In order to show that $K \subseteq K * \phi$, assume that $\psi \in K$. We need to show that either $\vdash \neg\phi$ or $\neg\phi < \phi \supset \psi$. Since we are assuming $\nvdash \neg\phi$, we show the latter. Suppose for reductio that it is false, that is, by Connectedness, that $\phi \supset \psi \leq \neg\phi$. Since $\psi \in K$, we know from the fact that K is a belief set that $\phi \supset \psi \in K$. Hence, by K-Representation, $\neg\phi \in K$ as well, contradicting our hypothesis.

(d) Now let $\leq$ in addition satisfy Continuing up.

$(*7)$. Let $\chi \in K * (\phi \wedge \psi)$, which means that $\neg(\phi \wedge \psi) < (\phi \wedge \psi) \supset \chi$ or $\vdash \neg(\phi \wedge \psi)$. We want to show that $(K * \phi) \cup \{\psi\} \vdash \chi$, i.e., by the deduction theorem for $\vdash$ and $(*1)$, $\psi \supset \chi \in K * \phi$, which again means, by ($*$ from $\leq$), $\neg\phi < \phi \supset (\psi \supset \chi)$ or $\vdash \neg\phi$. If $\vdash \neg(\phi \wedge \psi)$, then $(K * \phi) \cup \{\psi\}$ is inconsistent, by $(*1)$ and $(*2)$, so the claim is trivial. Let therefore $\nvdash \neg(\phi \wedge \psi)$ and $\neg(\phi \wedge \psi) < (\phi \wedge \psi) \supset \chi$. Suppose for reductio that $\nvdash \neg\phi$ and $\phi \supset (\psi \supset \chi) \leq \neg\phi$, or equivalently, by Extensionality, $(\phi \wedge \psi) \supset \chi \leq (\neg\phi \vee \neg\psi) \wedge (\neg\phi \vee \psi)$, By Continuing up and again Extensionality, this implies that $(\phi \wedge \psi) \supset \chi \leq \neg(\phi \wedge \psi)$, and we have found a contradiction.

(e) Now let $\leq$ in addition satisfy Continuing down.

$(*8c)$. Let $\psi \in K * \phi$ and $\chi \in K * \phi$, which means that $\neg\phi < \phi \supset \psi$ or $\vdash \neg\phi$, as well as $\neg\phi < \phi \supset \chi$ or $\vdash \neg\phi$. We want to show that $\chi \in K*(\phi\wedge\psi)$, which means that $\neg(\phi\wedge\psi) < (\phi\wedge\psi) \supset \chi$ or $\vdash \neg(\phi\wedge\psi)$. If $\vdash \neg\phi$, then $\vdash \neg(\phi\wedge\psi)$, so this case is trivial. Let now $\nvdash \neg\phi$, $\neg\phi < \phi \supset \psi$ and $\neg\phi < \phi \supset \chi$.

Suppose for reductio that $(\phi \wedge \psi) \supset \chi \leq \neg(\phi \wedge \psi)$ and $\nvdash \neg(\phi \wedge \psi)$. By Extensionality, the former is the same as $\neg\phi \vee \neg\psi \vee \chi \leq \neg\phi \vee \neg\psi$. By Continuing down, we can conclude that $(\neg\phi \vee \psi) \wedge (\neg\phi \vee \neg\psi \vee \chi) \leq \neg\phi\vee\neg\psi$, and thus, by Extensionality, $(\neg\phi\vee\psi)\wedge(\neg\phi\vee\chi) \leq \neg\phi\vee\neg\psi$. By Conjunctiveness, we get $(\neg\phi\vee\psi)\wedge(\neg\phi\vee\chi) \leq (\neg\phi\vee\neg\psi)\wedge((\neg\phi\vee\psi)\wedge(\neg\phi\vee\chi))$, and, by Extensionality again, $(\neg\phi\vee\psi)\wedge(\neg\phi\vee\chi) \leq \neg\phi$. By Choice, it follows that either $(\neg\phi\vee\psi) \leq (\neg\phi\vee\chi)\wedge\neg\phi$ or $(\neg\phi\vee\chi) \leq (\neg\phi\vee\psi)\wedge\neg\phi$. By Extensionality, this means that either $\phi \supset \psi \leq \neg\phi$ or $\phi \supset \chi \leq \neg\phi$, which contradicts the above assumption that both $\neg\phi < \phi \supset \psi$ and $\neg\phi < \phi \supset \chi$, so we are done.

(f) $(*7)$ and $(*8)$. Finally let $\leq$ in addition satisfy Transitivity. Hence, by Lemma 3.3, $\leq$ satisfies all the conditions of Gärdenfors and Makinson (1988). In Theorem 4 of this paper it is shown that $*$ satisfies $(*7)$ and $(*8)$ (or rather, that the corresponding contraction function satisfies corresponding postulates $(\dot{-}7)$ and $(\dot{-}8)$, cf. Alchourrón, Gärdenfors and Makinson, 1985, and Gärdenfors, 1988). □

References

Alchourrón, Carlos, Peter Gärdenfors and David Makinson: 1985, 'On the Logic of Theory Change: Partial Meet Contraction Functions and Their Associated Revision Functions', *Journal of Symbolic Logic* **50**, 510–530.

Alchourrón, Carlos, and David Makinson: 1985, 'On the Logic of Theory Change: Safe Contraction', *Studia Logica* **44**, 405–422.

Alchourrón, Carlos, and David Makinson: 1986, 'Maps Between Some Different Kinds of Contraction Function: The Finite Case', *Studia Logica* **45**, 187–198.

Boutilier, Craig: 1996, 'Iterated Revision and Minimal Change of Conditional Beliefs', *Journal of Philosophical Logic* **25**, 263–305.

Cantwell, John: 2001, 'Two Notions of Epistemic Entrenchment', in Mary-Anne Williams and Hans Rott (eds.), *Frontiers of Belief Revision*, Dordrecht: Kluwer, pp. 221–245.

Chernoff, Herman: 1954, 'Rational Selection of Decision Functions', *Econometrica* **22**, 423–443.

Darwiche, Adnan, and Judea Pearl: 1997, 'On the Logic of Iterated Belief Revision', *Artificial Intelligence* **89**, 1–29.

Gärdenfors, Peter: 1988, *Knowledge in Flux. Modeling the Dynamics of Epistemic States*, Bradford Books, MIT Press, Cambridge, Mass.

Gärdenfors, Peter, and David Makinson: 1988, 'Revisions of Knowledge Systems Using Epistemic Entrenchment', in Moshe Vardi (ed.), *TARK'88 – Proceedings of the Second Conference on Theoretical Aspects of Reasoning About Knowledge*, Los Altos: Morgan Kaufmann, pp. 83–95.

Goodman, Nelson: 1973, *Fact, Fiction, and Forecast*, Bobbs-Merrill, Indianapolis and New York, 3rd edition (first edition 1955).

Grove, Adam: 1988, 'Two Modellings for Theory Change', *Journal of Philosophical Logic* **17**, 157–170.

Hansson, Sven O.: 1994, 'Kernel Contraction', *Journal of Symbolic Logic* **59**, 845–859.

Hansson, Sven O.: 1999, *A Textbook of Belief Dynamics. Theory Change and Database Updating*, Kluwer, Dordrecht.

Herzberger, Hans G.: 1973, 'Ordinal Preference and Rational Choice', *Econometrica* **41**, 187–237.

Jackson, Frank: 1979, 'Assertion and Conditionals', *Philosohpical Review* **88**, 565–589.

Katsuno, Hirofumi, and Alberto O. Mendelzon: 1992, 'On the Difference between Updating a Knowledge Base and Revising it', in Peter Gärdenfors (ed.), *Belief Revision*, Cambridge University Press, Cambridge, pp. 183–203.

Klee, Robert: 2000, 'Problems with Formal Models of Epistemic Entrenchment as Applied to Scientific Theories', *Synthese* **122**, 313–320.

Levi, Isaac: 1996, *For the Sake of Argument*, Cambridge University Press, Cambridge.

Lindström, Sten, and Wlodzimierz Rabinowicz: 1991, 'Epistemic Entrenchment with Incomparabilities and Relational Belief Revision', in André Fuhrmann and Michael Morreau (eds.), *The Logic of Theory Change*, Springer LNAI **465**, Berlin etc., pp. 93–126.

Makinson, David, and Peter Gärdenfors: 1991, 'Relations Between the Logic of Theory Change and Nonmonotonic Logic', in André Fuhrmann and Michael Morreau (eds.), *The Logic of Theory Change*, Springer LNAI **465**, Berlin etc., pp. 185–205.

Nayak, Abhaya: 2000, 'Acceptance Without Minimality', in Manuel Ojeda-Aciego et al. (eds.), *Logics in Artificial Intelligence. European Workshop, JELIA 2000*, Springer LNAI **1919**, Berlin etc., pp. 163–178.

Pagnucco, Maurice: 1996, *The Role of Abductive Reasoning Within the Process of Belief Revision*, PhD thesis, Department of Computer Science, University of Sydney, February 1996.

Rabinowicz, Wlodzimierz: 1995, 'Stable Revision, or Is Preservation Worth Preserving?', in André Fuhrmann and Hans Rott (eds.), *Logic, Action and Information: Essays on Logic in Philosophy and Artificial Intelligence*, de Gruyter, Berlin, pp. 101–128.

Rott, Hans: 1991, 'A Non-monotonic Conditional Logic for Belief Revision I', in André Fuhrmann and Michael Morreau (eds.), *The Logic of Theory Change*, Springer LNAI **465**, Berlin etc., pp. 135–181.

Rott, Hans: 1992, 'Preferential Belief Change Using Generalized Epistemic Entrenchment', *Journal of Logic, Language and Information* **1**, 45–78.

Rott, Hans: 1993, 'Belief Contraction in the Context of the General Theory of Rational Choice', *Journal of Symbolic Logic* **58**, pp. 1426–1450.

Rott, Hans: 1999, 'Coherence and Conservatism in the Dynamics of Belief. Part I: Finding the Right Framework', *Erkenntnis* **50**, 387–412.

Rott, Hans: 2000a, ' "Just Because": Taking Belief Bases Seriously', in Samuel R. Buss, Petr Hájek and Pavel Pudlák (eds.), *Logic Colloquium '98 – Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic held in Prague*, Lecture Notes in Logic, Vol. 13, Urbana, Ill.: Association for Symbolic Logic, pp. 387–408.

Rott, Hans: 2000b, 'Two Dogmas of Belief Revision', *Journal of Philosophy* **97**, 503–522.

Rott, Hans: 2001, *Change, Choice and Inference*, Oxford University Press, Oxford.

Sen, Amartya K.: 1997, 'Maximization and the Act of Choice', *Econometrica* **65**, 745–779.

Williams, Mary-Anne: 1994, 'On the Logic of Theory Base Change', in Craig MacNish, David Pearce and Luis M. Pereira (eds.), *Logics in Artificial Intelligence*, Springer LNAI **838**, Berlin etc., pp. 86–105.