Logic and Choice

A Perspective on Belief Revision and Nonmonotonic Reasoning

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1. Introduction

The purpose of this paper is to link the logical theories of nonmonotonic reasoning and belief change to the general theory of rational choice that derives from micro-economic theorizing.

Choice or selection functions are a classical device in the semantic study of conditionals, belief revision and nonmonotonic reasoning (see for instance Lewis 1973, Alchourrón, Gärdenfors and Makinson 1985, Delgrande 1987). It has been pointed out repeatedly that particular properties of choice functions are needed to yield some particular properties of the desired logics (Lewis 1973, p. 58; Nute 1980, p. 22, 1994, p. 370; Delgrande 1987, p. 114; Lamarre 1991, p. 362; Schlechta 1992, p. 682, and 1996; Rott 1993 and 1994; Lindström 1994). What is missing, however, is a unified treatment of nonmonotonic reasoning and belief revision in the general framework of rational choice that makes use of the vast body of literature available there. There are a lot of concepts, insights and techniques in rational choice that wait to be utitized by researchers working in logic, artificial intelligence and computer science.

The present paper intends to give an indication of the potential that lies hidden in this connection. We cannot give the intuitive motivation for properties and constructions mentioned below, but have to refer the reader to the original work in the respective fields. It is the bridge between hitherto hardly related areas that is important. A detailed discussion with an embedding in a broader epistemological context is given in Rott (1996).

2. A General Concept of Practical Rationality: Postulates for Coherent Choice

In the type of situation we consider, an agent is presented with a set S of options or alternatives (for instance, commodities, political parties, ways of acting, or beliefs). This set among the elements of which the agent is to choose is called the *menu* or the *issue*. If the choice function is applied to S, it returns the *choice set* $\sigma(S)$ of all those elements of S that are considered as "best choices" in S.

Let X be a set and $\mathcal{X}$ be a non-empty set of subsets of X. Then formally, a *choice function* (or *selection function*) over $\mathcal{X}$ is a function $\sigma : \mathcal{X} \rightarrow Pow(X)$ such that the choice set $\sigma(S)$ is a subset of S, for every $S \in \mathcal{X}$.

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ed. Itzhak Gilboa, San Francisco: Morgan Kaufmann 1988, pp. 235–248.

The *classical theory of rational choice* was developed during the last 50 years mainly by economists, with important contributions by Samuelson, Houthakker, Arrow, Chernoff, Uzawa, Richter, Sen and Herzberger. Excellent systematic surveys of the field are given by Aizerman and Malishevski (1981), Sen (1982), Suzumura (1983), Aizerman (1985), and Moulin (1985).

The following conditions impose *coherence constraints* (also called *consistency constraints*) on choices across varying menus. They do not specify conditions for choices pertaining to a single menu. Rationality is explicated in this context by an appeal to coherence considerations constraining the choices in a certain menu by choices made in related menus. Here are four central conditions, together with some weaker and stronger variants:

(I) If $S \subseteq S'$, then $S \cap \sigma(S') \subseteq \sigma(S)$

(I^-) If $S \subseteq S'$ and $\sigma(S') \subseteq S$, then $\sigma(S') \subseteq \sigma(S)$

(II) $\sigma(S) \cap \sigma(S') \subseteq \sigma(S \cup S')$

(II^+) If $x \in \sigma(S)$ and $y \in \sigma(S')$, then $x \in \sigma(S \cup S')$ or $y \in \sigma(S \cup S')$

(III) If $S \subseteq S'$ and $\sigma(S') \subseteq S$, then $\sigma(S) \subseteq \sigma(S')$

(IV) If $S \subseteq S'$ and $\sigma(S') \cap S \neq \emptyset$, then $\sigma(S) \subseteq \sigma(S')$

(IV^+) If $S \subseteq S'$, then $\sigma(S) \subseteq \sigma(S')$

Now for some conditions concerning limiting cases. There are two situations in which an agent who is supposed to choose, does not have a genuine need or a genuine possibility to choose. This happens when either there are absolutely satisfactory options which the agent is ready to take no matter what else is at issue, or there are only absolutely unacceptable options which the agent would never consider worth taking. Consider the following postulates:

(Faith1) $\forall S \in \mathcal{X}:$ if $S \cap \sigma(X) \neq \emptyset$ then $\sigma(S) \subseteq \sigma(X)$

(Faith2) $\forall S \in \mathcal{X}: S \cap \sigma(X) \subseteq \sigma(S)$

(Success) If $S \neq \emptyset$, then $\sigma(S) \neq \emptyset$

($\emptyset$1) If $S \subseteq S'$ and $\sigma(S') = \emptyset$, then $\sigma(S) = \emptyset$

($\emptyset$2) If $S \subseteq S'$ and $\sigma(S) = \emptyset$, then $\sigma(S') \cap S = \emptyset$

We shall say that σ satisfies Faith with respect to a set R if it satisfies (Faith1) and (Faith2), and $\sigma(X) = R$. According to (Success) there are no unacceptable options, but ($\emptyset$1) and ($\emptyset$2) are plausible conditions for the case when (Success) fails.

The classical theory of choice and preference is characterized by the idea that *rational choice is relational choice*. Choice sets are taken to be sets of *best elements* under some context-independent preference relation. One formalization of this idea of rationalizability is based on an asymmetric relation $<$, and puts, for all $S \in \mathcal{X}$ such that $\sigma(S)$ is non-empty,

$$\sigma(S) \;=\; \min_{<}(S) \;=\; \{x \in S : y < x \text{ for no } y \in S\}$$

The following observation gives a rough indication as to why conditions (I), (II), (III) and (IV) play a central role in rational choice theory (cf. Moulin 1985):

Observation 1. Let σ be a choice function which satisfies ($\emptyset$1) and can take as arguments all and only the finite subsets of a given domain X.
(a) σ is rationalizable iff it is rationalizable by the preference relation $<$ defined by

$$x < y \quad \text{iff} \quad x \in \sigma(\{x,y\}) \text{ and } y \notin \sigma(\{x,y\})$$

(b) σ is rationalizable iff it satisfies (I) and (II).
(c) σ is rationalizable by a transitive relation $<$ iff it satisfies (I), (II), and (III).
(d) σ is rationalizable by a modular relation $<$ iff it satisfies (I) and (IV).

3. Concepts of Theoretical Rationality: Postulates for Belief Change and Nonmonotonic Reasoning

3.1. Postulates for contractions of belief sets

Belief contraction is the operation where some belief ϕ is removed from the set K of current beliefs. We will always presume that K is logically closed. Removing ϕ from a belief set is non-trivial because the contracted set should be closed again.

Most of the following postulates have gained prominence in the literature on belief change as studied in the research program of Alchourrón, Gärdenfors and Makinson ("AGM") (Alchourrón, Gärdenfors and Makinson 1985; Gärdenfors 1988, Chapter 3, Gärdenfors and Rott 1995). We regroup and rename the postulates while attaching some more commonly used labels at the right-hand sides:

(BI)	$K\dot{-}\phi \cap K\dot{-}\psi \subseteq K\dot{-}(\phi\wedge\psi)$	($\dot{-}$7)
(BI$^-$)	If $\psi \in K\dot{-}(\phi\wedge\psi)$, then $K\dot{-}\phi \subseteq K\dot{-}(\phi\wedge\psi)$	($\dot{-}$7c)
(BII$^\gamma$)	$K\dot{-}(\phi\wedge\psi) \subseteq Cn(K\dot{-}\phi \cup K\dot{-}\psi)$	($\dot{-}$8vwd)
(BII$^\delta$)	$K\dot{-}(\phi\wedge\psi) \subseteq Cn(K\dot{-}\phi \cup \{\neg\psi\}) \cup Cn(K\dot{-}\psi \cup \{\neg\phi\})$	($\dot{-}$8wd)
(BII$^+$)	$K\dot{-}(\phi\wedge\psi) \subseteq K\dot{-}\phi \cup K\dot{-}\psi$	($\dot{-}$8d)
(BIII)	If $\psi \in K\dot{-}(\phi\wedge\psi)$, then $K\dot{-}(\phi\wedge\psi) \subseteq K\dot{-}\phi$	($\dot{-}$8c)
(BIV)	If $\phi \notin K\dot{-}(\phi\wedge\psi)$, then $K\dot{-}(\phi\wedge\psi) \subseteq K\dot{-}\phi$	($\dot{-}$8)
(BIV$^+$)	$K\dot{-}(\phi\wedge\psi) \subseteq K\dot{-}\phi$	($\dot{-}$8m)
(BFaith1)	If $\phi \notin K$, then $K \subseteq K\dot{-}\phi$	($\dot{-}$3)
(BFaith2)	$K\dot{-}\phi \subseteq K$	($\dot{-}$2)
(BSuccess)	If $\phi \in K\dot{-}\phi$, then $\phi \in Cn(\emptyset)$	($\dot{-}$4)

(B∅1) If $\phi \wedge \psi \in K\dot{-}(\phi \wedge \psi)$, then $\phi \in K\dot{-}\phi$

(B∅2) If $\phi \in K\dot{-}\phi$, then $\phi \in K\dot{-}\phi \wedge \psi$

There are three more "basic" AGM postulates that will not be related to choice-theoretic considerations:

(Closure) $K\dot{-}\phi = Cn(K\dot{-}\phi)$ ($\dot{-}1$)

(Recovery) $K \subseteq Cn((K\dot{-}\phi) \cup \{\phi\})$ ($\dot{-}5$)

(Extensionality) If $Cn(\phi) = Cn(\psi)$, then $K\dot{-}\phi = K\dot{-}\psi$ ($\dot{-}6$)

3.2. Postulates for nonmonotonic inference relations

We distinguish between *inference operations Inf* which are important for various kinds of common-sense reasoning and Tarskian *consequence operations Cn* which are truth-preserving. Here is a list of important properties that an inference operation may or should have, even if it violates the condition of Monotony.

(NI) $Inf(\phi) \cap Inf(\psi) \subseteq Inf(\phi \vee \psi)$ (Or)

(NI^-) If $\psi \in Inf(\phi)$, then $Inf(\phi \wedge \psi) \subseteq Inf(\phi)$ (Cut)

(NII^γ) $Inf(\phi \vee \psi) \subseteq Cn(Inf(\phi) \cup Inf(\psi))$ (Very Weak Disjunctive Rationality)

(NII^δ) $Inf(\phi \vee \psi) \subseteq Cn(Inf(\phi) \cup \{\psi\}) \cup Cn(Inf(\psi) \cup \{\phi\})$ (Weak Disjunctive Rationality)

(NII^+) $Inf(\phi \vee \psi) \subseteq Inf(\phi) \cup Inf(\psi)$ (Disjunctive Rationality)

(NIII) If $\psi \in Inf(\phi)$, then $Inf(\phi) \subseteq Inf(\phi \wedge \psi)$ (Cumulative Monotony)

(NIV) If $\neg\psi \notin Inf(\phi)$, then $Inf(\phi) \subseteq Inf(\phi \wedge \psi)$ (Rational Monotony)

(NIV^+) $Inf(\phi) \subseteq Inf(\phi \wedge \psi)$ (Monotony)

(NFaith1) If $\neg\phi \notin Inf(\top)$, then $Inf(\top) \subseteq Inf(\phi)$ (Weak Rational Monotony)

(NFaith2) $Inf(\phi) \subseteq Cn(Inf(\top) \cup \{\phi\})$ (Weak Conditionalization)

(NSuccess) If $Cn(\phi) \neq L$, then $Inf(\phi) \neq L$ (Consistency Preservation)

(N∅1) If $\bot \in Inf(\phi)$, then $\bot \in Inf(\phi \wedge \psi)$

(N∅2) If $\neg\phi \in Inf(\phi)$, then $\neg\phi \in Inf(\phi \vee \psi)$

With few exceptions, these conditions are well-known from the literature on nonmonotonic reasoning. For a detailed discussion, the reader is referred to Kraus, Lehmann and Magidor (1990), Makinson and Gärdenfors (1991), Makinson (1994), and Gärdenfors and Rott (1995). It is not difficult to see that the postulates for nonmonotonic reasoning are counterparts of the similarly labelled postulates for belief contractions. The key idea, due to Makinson and Gärdenfors (1991), is to identify the inferences $Inf(\phi)$ drawn from a sentence ϕ with the set $Cn((\Delta \dot{-} \neg\phi) \cup \{\phi\})$, that is, the "revision" of a fixed background theory Δ by the premise ϕ. The theory Δ here is supposed to consist of the agent's defaults or expectations rather than his beliefs.

There are also a number of fundamental conditions for non-monotonic reasoning that are not related to choice-theoretic considerations:

$$\phi \in Inf(\phi) \qquad \text{(Reflexivity)}$$

$$\text{If } \psi \in Inf(\phi) \text{ and } \chi \in Cn(\psi), \text{ then } \chi \in Inf(\phi) \qquad \text{(Right Weakening)}$$

$$\text{If } \psi \in Inf(\phi) \text{ and } \chi \in Inf(\phi), \text{ then } \psi \wedge \chi \in Inf(\phi) \qquad \text{(And)}$$

$$\text{If } Cn(\phi) = Cn(\psi), \text{ then } Inf(\phi) = Inf(\psi) \qquad \text{(Left Logical Equivalence)}$$

4. Belief Change and Nonmonotonic Reasoning as Problems of Rational Choice

4.1. How to give it up: Choosing best models

When retracting a sentence ϕ from the set of his beliefs or expectations, the agent takes into account the *most plausible* or *best* models that *falsify* ϕ. If $\mathcal{X}$ is any class of subsets of the set $\mathcal{M}_L$ of models for the propositional language L, a choice function γ with domain $\mathcal{X}$ will be called a *semantic choice function*. In the following, however, we focus on a particular $\mathcal{X}$: the class of all elementary subsets of $\mathcal{M}_L$, i.e., the class of all model sets S such that $S = [\![\phi]\!]$ for some sentence ϕ. The *completion* of a semantic choice function γ is the function γ^+ that returns, for every sentence ϕ, the set of all models that satisfy every sentence that is true throughout $\gamma([\![\phi]\!])$. Clearly, $\gamma([\![\phi]\!]) \subseteq \gamma^+([\![\phi]\!])$, but the converse is not in general true. If γ is identical with γ^+, then γ is called *complete*.

Definition 1. The contraction function $\dot{-}$ over a belief set K is generated by a choice function γ over models, in symbols $\dot{-} = \mathcal{C}(\gamma)$, if and only if for every ψ,

$$\psi \in K \dot{-} \phi \;\text{ iff }\; \psi \in K \text{ and } \gamma(\,]\!]\phi[\![\,) \subseteq [\![\psi]\!]$$

Here $]\!]\phi[\![= \mathcal{M}_L - [\![\phi]\!]$ is the set of all models falsifying ϕ. So a belief ψ of K remains accepted in $K \dot{-} \phi$ if and only if it is satisfied by all the most plausible worlds that falsify ϕ.

4.2. How to give it up: Choosing worst sentences

The second idea of contracting a set of opinions (beliefs or expectations) in the face of counterevidence against ϕ involves choices between (previously accepted) sentences rather than choices between (previously rejected) models. Of course, what needs to be given up depends on ϕ. We now give an argument to the effect that this dependence should not be represented by a change of the choice function used, but rather by a change of the argument to which the one and the same choice function is applied.

When eliminating ϕ, we have to pay attention to the connections that the elements of K have with ϕ. More precisely, we focus on the reasons for ϕ and the consequences of ϕ. *Reasons* for ϕ may be thought of as sentences figuring as premises in non-trivial derivations of ϕ, and *consequences* of ϕ may be thought of as sentences figuring as conclusions in non-trivial derivations that start from a premise set containing ϕ. Now observe that if ψ is a reason for ϕ (or a consequence of ϕ), then $\phi \vee \psi$ is just as well a reason for ϕ (or respectively, a consequence of ϕ). Substituting $\phi \vee \psi$ for ψ in a derivation of any of the above-mentioned sorts will (perhaps with a little modification) result in a similar such derivation. We conclude that if ψ is eliminated because it is a reason for, or a consequence of, ϕ, then $\phi \vee \psi$ should be given up on the same grounds. Since the contraction of the belief set K with respect to ϕ should also be logically closed, we may thus advance the following thesis:

$$\psi \in K \dot{-} \phi \ \text{ iff } \ (\phi \vee \psi) \in K \dot{-} \phi$$

This condition follows directly from the AGM conditions ($\dot{-}$1), ($\dot{-}$2) and ($\dot{-}$5), but we do not want to use the controversial postulate of Recovery for the present argument. Since $\phi \vee \psi$ is an element of $Cn(\phi)$, the condition shows that the contracted belief set can be fully determined by looking only at the set of logical consequences of the sentence to be removed. I indeed suggest to identify the task of contracting K with respect to ϕ with the task of choosing to eliminate at least one element of $Cn(\phi)$. We can employ a single syntactic choice function in order to achieve this, a function suitable to take $Cn(\phi)$ as an argument for *any* sentence ϕ that might happen to be the sentence to be withdrawn. More abstractly, a *syntactic choice function* is a choice function δ the domain of which is a class of subsets of L. Intuitively, δ returns the most *im*plausible or the *worst* ("least entrenched" or "least expected") elements of a given menu, i.e., here: of the set of sentences that are held responsible for deriving ϕ. It has turned out that this set may be taken to be $Cn(\phi)$:

Definition 2. The contraction function $\dot{-}$ over a belief set K is generated by a choice function δ over sentences, in symbols $\dot{-} = \mathcal{C}(\delta)$, if and only if for every ψ,

$$\psi \in K \dot{-} \phi \ \text{ iff } \ \psi \in K \text{ and } \phi \vee \psi \notin \delta(Cn(\phi))$$

Now we take account of the fact that the domain of δ consists of sentences which possess a logical structure. There is no equivalent structure to be considered of the objects of choice in the semantic case since models are essentially unrelated to one another. The following conditions relate choice functions to the monotonic consequence operation Cn.

(LP1) *(Logical intra-menu condition)* If ϕ is selected to be withdrawn from a menu F, then for every subset G of F that logically entails ϕ (i.e., $\phi \in Cn(G)$), at least one element of G must also be withdrawn from the menu F.

(LP2) *(Logical inter-menu condition)* If two menus F and G are logically equivalent (i.e., $Cn(F) = Cn(G)$), then every ϕ which is contained in both F and G is selected to be withdrawn from the menu F just in case it is withdrawn from the menu G.

Choice functions satisfying (LP1) are not always "successful": they may yield $\delta(F) = \emptyset$ even if $F \neq \emptyset$ (consider an F with $F \subseteq Cn(\emptyset)$). If only elements of $Cn(\emptyset)$ are in this way resistant against withdrawal, then we call the syntactic choice function δ *virtually successful*.

(Virtual Success) If $\delta(F) = \emptyset$, then $F \subseteq Cn(\emptyset)$

If δ can take infinite sets as arguments, a simpler and equivalent form of (LP2) is $\delta(F) = F \cap \delta(Cn(F))$. But (LP2) also allows us to give an equivalent finitary reformulation of Definition 2: The contraction function $\dot{-}$ over K is (finitarily) generated by a choice function δ between sentences if and only if for every ψ, $\psi \in K \dot{-} \phi$ just in case $\psi \in K$ and $\phi \vee \psi \notin \delta(\{\phi, \phi \vee \psi\})$.

4.3. Nonmonotonic inference

We define constructions of nonmonotonic inferences which parallel the semantic and syntactic choice-based constructions of belief change.

Definition 3. The inference operation *Inf* is generated by a choice function γ over models, in symbols $Inf = \mathcal{I}(\gamma)$, if and only if for every ψ in L,

$$\psi \in Inf(\phi) \text{ iff } \gamma(\llbracket \phi \rrbracket) \subseteq \llbracket \psi \rrbracket$$

Definition 4. The inference operation *Inf* is generated by a choice function δ over sentences, in symbols $Inf = \mathcal{I}(\delta)$, if and only if for every ψ in L,

$$\psi \in Inf(\phi) \text{ iff } \phi \supset \psi \notin \delta(Cn(\neg\phi))$$

or equivalently (to be used if δ is finitary)

$$\psi \in Inf(\phi) \text{ iff } \phi \supset \psi \notin \delta(\{\phi \supset \psi, \phi \supset \neg\psi\})$$

4.4. Representation theorems

We show that the theory of rational choice can be used to motivate logics of belief change and nonmonotonic reasoning.

Observation 2. For every semantic choice function γ which satisfies

$$\left\{\begin{array}{c} \text{—} \\ \text{(I)} \\ \text{(I}^-\text{)} \\ \text{(II) and which is complete} \\ \text{(II}^+\text{)} \\ \text{(III)} \\ \text{(IV)} \\ \text{(IV}^+\text{)} \\ \text{(Faith1) wrt } [\![K]\!] \\ \text{(Success)} \\ (\emptyset 1) \\ (\emptyset 2) \end{array}\right\}$$

the contraction function $\dot{-} = \mathcal{C}(\gamma)$ over K generated by γ satisfies (Closure), (Recovery), (Extensionality), (BFaith2) wrt K and $\left\{\begin{array}{c} \text{—} \\ \text{(BI)} \\ \text{(BI}^-\text{)} \\ \text{(BII}^\gamma\text{)} \\ \text{(BII}^+\text{)} \\ \text{(BIII)} \\ \text{(BIV)} \\ \text{(BIV}^+\text{)} \\ \text{(BFaith1) wrt } K \\ \text{(BSuccess)} \\ (\text{B}\emptyset 1) \\ (\text{B}\emptyset 2) \end{array}\right\}$, respectively.

Given a contraction function $\dot{-}$ over K, we can derive from it a semantic choice function $\gamma = \mathcal{G}(\dot{-})$ over the elementary sets of models. The idea is that a model falsifying ϕ is a best element of $]\!]\phi[\![$ if it satisfies everything that is contained in the contraction of K with respect to ϕ. So we define, for every sentence ϕ

$$\gamma(\,]\!]\phi[\![\,) = \,]\!]\phi[\![\cap [\![K\dot{-}\phi]\!] = \{m \in \,]\!]\phi[\![: K\dot{-}\phi \subseteq \widehat{m}\}$$

We say that the contraction function $\dot{-}$ *reveals*, or is a *manifestation* of, the underlying semantic choice function $\mathcal{G}(\dot{-})$. The following completeness theorem draws on this construction and shows that interesting properties of $\dot{-}$ lead to corresponding interesting properties of γ.

Observation 3. Every contraction function $\dot{-}$ over a theory K which satisfies (Closure), (Recovery), (Extensionality), (BFaith2) wrt K and $\left\{\begin{array}{c} \text{—} \\ \text{(BI)} \\ \text{(BI}^-\text{)} \\ \text{(BII}^\gamma\text{)} \\ \text{(BII}^+\text{)} \\ \text{(BIII)} \\ \text{(BIV)} \\ \text{(BIV}^+\text{)} \\ \text{(BFaith1) wrt } K \\ \text{(BSuccess)} \\ \text{(B}\emptyset\text{1)} \\ \text{(B}\emptyset\text{2)} \end{array}\right\}$ can be represented as a contraction function $\mathcal{C}(\gamma)$ generated by a semantic choice function γ which is complete and satisfies (Faith2) wrt $[\![K]\!]$ and $\left\{\begin{array}{c} \text{—} \\ \text{(I)} \\ \text{(I}^-\text{)} \\ \text{(II)} \\ \text{(II}^+\text{)} \\ \text{(III)} \\ \text{(IV)} \\ \text{(IV}^+\text{)} \\ \text{(Faith1) wrt } [\![K]\!] \\ \text{(Success)} \\ \text{(}\emptyset\text{1)} \\ \text{(}\emptyset\text{2)} \end{array}\right\}$, respectively.

Almost, but not quite the same results can be obtained if we use syntactic rather than semantic choice functions.

Observation 4. For every syntactic choice function δ which satisfies (LP1), (LP2) and

$$\left\{\begin{array}{c} \text{—} \\ \text{(I)} \\ \text{(I}^-\text{)} \\ \text{(II)} \\ \text{(II}^+\text{)} \\ \text{(III)} \\ \text{(IV)} \\ \text{(IV}^+\text{)} \\ \text{(Faith1) wrt } L - K \\ \text{(Virtual Success)} \\ (\emptyset 1) \\ (\emptyset 2) \end{array}\right\}$$

, the contraction function $\dot{-} = \mathcal{C}(\delta)$ over K generated by δ satisfies (Closure), (Recovery), (Extensionality), (BFaith2) wrt K and

$$\left\{\begin{array}{c} \text{—} \\ \text{(BI)} \\ \text{(BI}^-\text{)} \\ \text{(BII}^\delta\text{)} \\ \text{(BII}^+\text{)} \\ \text{(BIII)} \\ \text{(BIV)} \\ \text{(BIV}^+\text{)} \\ \text{(BFaith1) wrt } K \\ \text{(BSuccess)} \\ (\text{B}\emptyset 1) \\ (\text{B}\emptyset 2) \end{array}\right\},$$

respectively.

Given a contraction function $\dot{-}$ over K, we can derive from it a *finitary and ω-covering* syntactic choice function $\delta = \mathcal{D}(\dot{-})$ for K.[1] The idea is that a sentence ϕ_i is chosen as a "best" (or more properly, *worst*) element of $\{\phi_1, \ldots, \phi_n\}$ if it is in fact withdrawn in the contraction of K with respect to $\phi_1 \wedge \ldots \wedge \phi_n$. The task of giving up at least one element of a finite set $\{\phi_1, \ldots, \phi_n\}$ is now identified with the task of giving up the conjunction $\phi_1 \wedge \ldots \wedge \phi_n$. So we define, for all sentences $\phi_1, \ldots, \phi_n$

$$\delta(\{\phi_1, \ldots, \phi_n\}) = \{\phi_i : \phi_i \notin K \dot{-} (\phi_1 \wedge \ldots \wedge \phi_n)\}$$

The choice set $\delta(\{\phi_1, \ldots, \phi_n\})$ contains just those ϕ_i's which are *not* retained in $K \dot{-} (\phi_1 \wedge \ldots \wedge \phi_n)$.[2] Let us say that the contraction function $\dot{-}$ *reveals*, or is a *manifestation of*, the underlying syntactic choice function $\mathcal{D}(\dot{-})$.

The following completeness theorem draws on this construction and shows that interesting properties of $\dot{-}$ lead to corresponding interesting properties of δ.

[1]This restriction to finitary arguments in the syntactic case corresponds to the restriction to elementary arguments in the semantic case.

[2]Or that are not in K in the first place.

Observation 5. Every contraction function $\dot{-}$ over a theory K which satisfies (Closure), (Recovery), (Extensionality), (BFaith2) wrt K and

$$\left\{ \begin{array}{c} \text{---} \\ \text{(BI)} \\ \text{(BI}^{-}\text{)} \\ \text{(BII}^{\delta}\text{)} \\ \text{(BII}^{+}\text{)} \\ \text{(BIII)} \\ \text{(BIV)} \\ \text{(BIV}^{+}\text{)} \\ \text{(BFaith1) wrt } K \\ \text{(BSuccess)} \\ \text{(B}\emptyset\text{1)} \\ \text{(B}\emptyset\text{2)} \end{array} \right\}$$

can be represented as a contraction function $\mathcal{C}(\delta)$ finitarily generated by a syntactic choice function δ which satisfies (LP1), (LP2), (Faith2) wrt $L - K$ and

$$\left\{ \begin{array}{c} \text{---} \\ \text{(I)} \\ \text{(I}^{-}\text{)} \\ \text{(II)} \\ \text{(II}^{+}\text{)} \\ \text{(III)} \\ \text{(IV)} \\ \text{(IV}^{+}\text{)} \\ \text{(Faith1) wrt } L - K \\ \text{(Virtual Success)} \\ (\emptyset 1) \\ (\emptyset 2) \end{array} \right\}$$

, respectively.

Soundness and completeness theorems for nonmonotonic reasoning are exactly parallel to those for belief contractions. The role of the belief set K is here played by the set $Inf(\top)$, that is, the defaults or expectations that are active when no factual information is available at all.

Inspection of the above theorems shows that the logical postulate corresponding to a certain choice-theoretic postulate in the syntactic approach is almost always the same as that in the semantic approach. This can be explained by two mappings which establish a direct bridge between the two approaches.

First, when at least one of the sentences in F has to be given up, ϕ from F is among the discarded sentences if and only if at least one of the most plausible models falsifying (at least one element of) F also falsifies ϕ:

Definition 5. Let a choice function γ over the class of Σ-elementary (elementary) sets of models be given. Then we can define a corresponding (finite) choice function $\delta = \mathcal{D}(\gamma)$ over sentences by putting $\phi \in \delta(F)$ iff $\phi \in F$ and $\gamma(\,]\!]F[\![\,) \cap\,]\!]\phi[\![\neq \emptyset$.

Conversely, a model m is among the most plausible models falsifying (at least one element of) F if and only if m satisfies all the sentences in $Cn(F)$ which are not given up when the worst sentences in $Cn(F)$ are discarded.

Definition 6. Let a choice function δ over (arbitrary) sets of sentences be given. Then we can define a corresponding choice function $\gamma = \mathcal{G}(\delta)$ over (Σ-elementary sets of) models by putting $m \in \gamma(\rrbracket F \llbracket)$ iff $m \in \rrbracket F \llbracket$ and $m \in \llbracket Cn(F) - \delta(Cn(F)) \rrbracket$.[3]

It makes sense indeed to talk of semantic and syntactic choice functions that "correspond to each other":

Observation 6. (i) For every semantic choice function γ, the corresponding syntactic choice function $\delta = \mathcal{D}(\gamma)$ satisfies (LP1) and (LP2).
(ii) For every syntactic choice function δ that satisfies (LP1) and (LP2), the corresponding semantic choice function $\gamma = \mathcal{D}(\delta)$ is complete.
(iii) For every syntactic choice function δ that satisfies (LP1) and (LP2), the syntactic choice function corresponding to the semantic choice function corresponding to δ is identical with δ, that is, $\mathcal{D}(\mathcal{G}(\delta)) = \delta$.
(iv) For every semantic choice function γ, the semantic choice function corresponding to the syntactic choice function corresponding to γ is the completion γ^+ of γ, that is, $\mathcal{G}(\mathcal{D}(\gamma)) = \gamma^+$.
(v) Corresponding choice functions lead to identical contraction functions, that is, it holds that both $\mathcal{C}(\mathcal{D}(\gamma)) = \mathcal{C}(\gamma)$ and $\mathcal{C}(\mathcal{G}(\delta)) = \mathcal{C}(\delta)$.

It is surprising that choice-theoretic "rationality profiles" are fully preserved in the transition from the syntactic to the semantic level.

Observation 7. Let δ be a syntactic choice function satisfying (LP1), (LP2). If δ satisfies (I) (or (I$^-$), (II), (II$^+$), (III), (IV), (IV$^+$), ($\emptyset$1), ($\emptyset$2)), so does the associated semantic choice function $\gamma = \mathcal{G}(\delta)$. If δ satisfies (Faith1) or (Faith2) wrt $\delta(L)$, then γ satisfies (Faith1) or (Faith2) wrt $\gamma(\mathcal{M}_L)$. If δ satisfies (Virtual Success), then γ satisfies (Success).

The transfer from the semantic to the syntactic level works almost, but not quite as smooth. There is a single remarkable exception concerning condition (II) which explains why (BII$^\delta$) and (NII$^\delta$) are stronger than (BII$^\gamma$) and (NII$^\gamma$), respectively.

Observation 8. Let γ be a semantic choice function. Then the associated syntactic choice function $\delta = \mathcal{D}(\gamma)$ satisfies (LP1) and (LP2). If γ satisfies (I) (or (I$^-$), (II$^+$), (III), (IV), (IV$^+$), ($\emptyset$1), ($\emptyset$2)), so does δ. If γ satisfies (Faith1) or (Faith2) wrt $\gamma(\mathcal{M}_L)$, then δ satisfies (Faith1) or (Faith2) wrt $\delta(L)$. If γ satisfies (Success), then δ satisfies (Virtual Success). *However*, if γ satisfies (II), the associated syntactic choice function $\delta = \mathcal{D}(\gamma)$ does not necessarily satisfy (II), even if the language is finitary and if γ in addition satisfies (I) and (III).

[3] If δ can take only finite sets of sentences, we can put $m \in \gamma(\rrbracket \phi \llbracket)$ iff $m \in \rrbracket \phi \llbracket$ and $m \in \llbracket \{\psi \in Cn(\phi) : \psi \notin \delta(\{\phi, \psi\})\} \rrbracket$ Given (LP1) and (LP2), this is the restriction of Definition 6 to the case of elementary model sets.

5. Summary

We have presented a number of representation theorems that establish a surprising 1-1-correspondence between sets of postulates that have been suggested and motivated entirely independently in different areas of research. Choice-theoretic constraints can be applied on a semantic and on a syntactic level, and one gets a very far-reaching parallelism in the logical behaviour of the two kinds of modellings. Theories of belief revision and nonmonotonic reasonig can thus be decomposed into (monotonic) logic on the one hand and the theory of choice on the other. Into the general choice-theoretic framework, one can fit in more specific concepts and results of decision and game theory.

The present approach is, I believe, attractive to economists who will find yet another field of application for their theories. It is even more attractive for logicians and computer scientists who can utilize the refined techniques and results of the general theory of choice in their studies. For instance, the theory of epistemic entrenchments *alias* expectation orderings (Gärdenfors and Makinson 1988, 1994) can be reconstructed as a theory of *revealed preferences* (Rott 1994, 1996). More generally, methods of actually arriving at and motivating decisions can be studied in their potential for application to processes of thinking and deliberation. The new logical applications can be checked against the whole historical development of rational choice theory, its good as well as its bad parts (Green and Shapiro 1994). This generates a research programme of cognitive choice or decision theory at the interface between practical and theoretical reason—fields that have so far been developing largely independently of each other.

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