Coarse cohomology theories

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Abstract

We propose the notion of a coarse cohomology theory and study the examples of coarse ordinary cohomology, coarse stable cohomotopy and coarse cohomology theories obtained by dualizing coarse homology theories.

Our investigations of coarse stable cohomotopy lead to a solution of J. R. Klein’s conjecture that the dualizing spectrum of a group is a coarse invariant.

We further investigate coarse cohomological $K$-theory functors and explain why (an adaption of) the functor of Emerson–Meyer does not seem to fit into our setting.

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1 Introduction

The goal of the present paper is to propose the notion of a coarse cohomology theory and study several examples.

In [BE16] we have introduced the category \( \text{BornCoarse} \) of bornological coarse spaces and the notion of a C-valued coarse homology theory \( E : \text{BornCoarse} \to C \), where C is a cocomplete stable \( \infty \)-category. If D is a complete stable \( \infty \)-category, then we simply propose to call a functor \( E : \text{BornCoarse}^{\text{op}} \to D \) a coarse cohomology theory if and only if \( E^{\text{op}} : C \to D^{\text{op}} \) is a coarse homology theory.

Example 1.1. The main purpose of the present paper is to present the construction of the following three examples of coarse cohomology theories.

1. To every abelian group \( A \) we associate the coarse ordinary cohomology theory

\[
HA^A : \text{BornCoarse}^{\text{op}} \to \text{Ch}_{\infty}.
\]

Our Definition 3.2 extends the original definition of Roe [Roe93] to the context of bornological coarse spaces, see Lemma 3.12.

2. If \( C \) is a presentable stable \( \infty \)-category, then in Definition 4.1 we associate to every object \( C \) in \( C \) a \( C \)-valued coarse cohomology theory \( Q_C \). For the category of spectra \( C = \text{Sp} \) and for the sphere spectrum \( C = S \) we obtain a coarse version \( QS \) of stable cohomotopy.

3. If \( E \) is a \( C \)-valued coarse homology theory and \( C \) is an object of \( C \), then in the Definition 2.12 we define the dual \( \text{Sp} \)-valued coarse cohomology theory \( DC(E) \) by forming the mapping spectrum with target \( C \). We also consider versions of this construction where we replace the mapping space functor by some internal mapping object functor or a suitable power functor.

Besides of the construction of the functors and the verification of the axioms of a coarse cohomology theory, in Section 2.3 we describe pairings of the cohomology theories with the corresponding coarse homology theories. We further argue that naturality of the pairings implies their compatibility with assembly and coassembly maps.

Note that there are also other approaches to a general framework for coarse cohomology theories. Let us mention exemplary the work of Schmidt [Sch99] and Hartmann [Har17].
The dualizing spectrum of a group  

In Section 4.2 we provide an application of the coarse cohomology $Q_S$ discussed above in Example 1.1.2 for $C = \text{Sp}$ and $C = S$. In order to formulate our result we recall the following.

1. Let $G$ be a group. Under the assumption that the classifying space $BG$ of $G$ is finitely dominated, J. R. Klein [Kle01, Thm. A] has shown that the property of $BG$ being a Poincaré duality space is determined by the equivalence class in $\text{Sp}$ of the dualizing spectrum

$$D_G := S[G]^{hG}$$

of the group $G$.

2. In [BE16] we constructed a universal coarse homology theory

$$\text{Yo}^s : \text{BornCoarse} \to \text{Sp}$$

with values in the stable $\infty$-category of coarse motivic spectra.

3. A group $G$ gives naturally rise to a bornological coarse space $G_{\text{can,min}}$ and therefore to a coarse motivic spectrum $\text{Yo}^s(G_{\text{can,min}})$. Note that $\text{Yo}^s(G_{\text{can,min}})$ is in particular an invariant of the quasi-isometry class of $G$.

We now consider two groups $G$ and $H$. The following result settles a (generalization of a) conjecture stated by Klein [Kle01, Conj. on Page 455].

**Theorem 1.2** (Corollary 4.11). If $G$ and $H$ are finitely generated, torsion-free and we have an equivalence $\text{Yo}^s(G_{\text{can,min}}) \simeq \text{Yo}^s(H_{\text{can,min}})$, then we have an equivalence of dualizing spectra $D_G \simeq D_H$ in $\text{Sp}$.

Note that J. R. Klein asked for the equivalence $D_G \simeq D_H$ only under the assumption of $G$ and $H$ being quasi-isometric to each other (in addition to having homotopy finite classifying spaces). In Example 4.12 we show that the equivalence $\text{Yo}^s(G_{\text{can,min}}) \simeq \text{Yo}^s(H_{\text{can,min}})$ is strictly weaker than being quasi-isometric.

There are, of course, related coarse invariance results in the literature. If the group $G$ has a finitely dominated classifying space $BG$, then by [Bro82, Prop. VIII.6.4] the group $G$ is of the type $FP$. We then define the dualizing $G$-module by $D_G^{Z,n} := H^n(G; Z[G])$. Here we consider $Z[G]$ as a $G$-module with the action induced by the left multiplication on $G$, and the $G$-action on the cohomology is induced from the right $G$-action on $Z[G]$. Moreover, $n$ is the cohomological dimension of $G$.

By $D_G^Z := H(G; Z[G])$ we denote the cohomology complex of $G$ with coefficients in the module $Z[G]$ which we consider as an object of the $\infty$-category $\text{Ch}_{\infty}$ of chain complexes (in particular we forget the $G$-action). The group $G$ is then called a Bieri–Eckmann duality group, if it is of type $FP$ and in addition satisfies $\text{Yo}^s(G_{\text{can,min}}) \simeq \text{Yo}^s(H_{\text{can,min}})$, i.e., $D_G^Z$ has only one non-trivial cohomology group which sits in degree $n$ and is isomorphic to $D_G^{Z,n}$ (with $G$-action forgotten). In this case the four assertions stated in [Bro82, Thm. VIII.10.1] are satisfied. In particular, the group $G$ satisfies a version of Poincaré duality in the sense that there are natural isomorphisms $H^i(G; -) \cong H_{n-i}(G; D_G^Z \otimes -)$ of functors on the
category of $G$-modules. Note that $D^n_G$ does in general not have to be infinite cyclic, but if it is so, then the group $G$ is called a Poincaré duality group.

It is known from work of J. Roe [Roe96, Prop. 2.6] that the cohomology groups $H^*(G; \mathbb{Z}[G])$ are coarse invariants of $G$, because they coincide with the coarse ordinary cohomology groups of $G_{\text{can,min}}$.

**Coarse cohomological $K$-theory functors** In Section 5 we discuss coarse cohomological $K$-theory functors. The probably most important coarse homology theory is the coarse $K$-homology $K\chi^{hlg}$ since it features in the coarse Baum–Connes conjecture. In this case it is appropriate to consider $K\chi^{hlg}$ as a coarse homology theory taking values in the stable $\infty$-category of $KU$-modules. Dualizing $K\chi^{hlg}$ yields a homotopy-theoretic construction of a coarse $K$-cohomology theory, see Section 5.1.

A candidate for an analytic construction of a coarse $K$-cohomology theory was proposed by Emerson–Meyer [EM06]. In Definition 5.10 we adapt the definition of Emerson–Meyer to the context of bornological coarse spaces and define, for every $C^*$-algebra $A$, a corresponding functor $KA\chi$. We then discuss in detail the cohomological properties of the functor $KA\chi$. We argue that this functor is almost a coarse cohomology theory, but we were able to prove excision only under additional conditions. So it remains an open problem to provide an analytic construction of a coarse $K$-cohomology theory.

In Proposition 5.30 we construct, for a bornological coarse space $X$, a pairing between $KA\chi(X)$ and coarse $K$-homology $K\chi^{hlg}(X)$. At the moment it exists only under restrictions on $X$, and we have verified only a weak form of functoriality, see Remark 5.31.

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## 2 Coarse cohomology theories

In Section 2.1 we give the definition of coarse cohomology theories and discuss some basic properties. In Section 2.3 we discuss coassembly maps for coarse cohomology theories and how the naturality of a pairing with a coarse homology theory implies its compatibility with the (co-)assembly maps.

### 2.1 Definition and basic properties

In this section we define coarse cohomology theories (by dualizing the axioms for coarse homology theories given in [BE16]) and then discuss (Corollary 2.10) the corresponding motivic definition.
We will use notions for bornological coarse spaces and coarse homology theories introduced in [BE16], i.e., we assume familiarity of the reader with the material in [BE16, Sec. 2–4].

We start with the description of the axioms of coarse cohomology theories. Let $C$ be a complete stable $\infty$-category.

We equip the two-point set $\{0, 1\}$ with the maximal bornological and coarse structures. The category $\text{BornCoarse}$ has a symmetric monoidal structure $\otimes$ introduced in [BE16]. Since $\{0, 1\}$ is bounded, for every bornological coarse space $X$ the projection $\{0, 1\} \otimes X \to X$ is a morphism of bornological coarse spaces.

We consider a functor $E : \text{BornCoarse}^{\text{op}} \to C$.

**Definition 2.1.** $E$ is called coarsely invariant if for every bornological coarse space $X$ the projection $\{0, 1\} \otimes X \to X$ induces an equivalence $E(X) \to E(\{0, 1\} \otimes X)$.

Let $\mathcal{Y} := (Y_i)_{i \in I}$ be a big family on a bornological coarse space $X$ [BE16, Def. 3.2]. Set

$$E(\mathcal{Y}) := \lim_{i \in I} E(Y_i)$$

and note that have a natural morphism

$$E(X) \to E(\mathcal{Y}).$$

The definition of excisiveness involves the notion of complementary pairs [BE16, Def. 3.5].

**Definition 2.2.** $E$ is excisive, if for every complementary pair $(\mathcal{Y}, Z)$ on a bornological coarse space $X$ the square

$$\begin{array}{ccc}
E(X) & \longrightarrow & E(Z) \\
\downarrow & & \downarrow \\
E(\mathcal{Y}) & \longrightarrow & E(Z \cap \mathcal{Y})
\end{array}$$

is cartesian.

Recall the notion of a flasque bornological coarse space [BE16, Def. 3.21].

**Definition 2.3.** $E$ vanishes on flasques if $E(X) \simeq 0$ for every flasque bornological coarse space $X$.

In the following let $C$ denote the coarse structure of a bornological coarse space $X$. For an entourage $U$ in $C$ we let $X_U$ be the bornological coarse space obtained from $X$ by replacing the coarse structure $C$ by the coarse structure $C(U)$ generated by $U$. The identity map of the underlying set of $X$ is a morphism $X_U \to X$ of bornological coarse spaces. We get a natural morphism

$$E(X) \to \lim_{U \in C} E(X_U).$$
Definition 2.4. \( E \) is \( u \)-continuous if for every bornological coarse space \( X \) the natural morphism

\[
E(X) \to \lim_{U \in C} E(X_U)
\]

is an equivalence.

We are now ready to define the notion of a classical coarse cohomology theory. Let \( C \) be a complete stable \( \infty \)-category and consider a functor \( E : \text{BornCoarse}^{\text{op}} \to C \).

Definition 2.5. \( E \) is a classical coarse cohomology theory if it has the following properties:

1. \( E \) is coarsely invariant.
2. \( E \) is excisive.
3. \( E \) vanishes on flasques.
4. \( E \) is \( u \)-continuous.

Remark 2.6. In this remark we compare Definition 2.5 with Fukaya and Oguni’s definition of a coarse cohomology theory [FO13, Def. 3.3].

As usual in the coarse geometry literature they only consider proper metric spaces. In order to encode coarse invariance they introduce the coarse category. It is obtained from the full subcategory of \( \text{BornCoarse} \) of proper metric spaces (where the coarse and bornological structures are induced from the metric) by identifying morphisms which are close to each other. Then a coarse cohomology theory in the sense of Fukaya and Oguni is a contravariant, \( \mathbb{Z} \)-graded, group-valued functor on the coarse category which vanishes on all spaces of the form \( X \otimes \mathbb{N} \) (where \( \mathbb{N} \) has the canonical metric structures) and satisfies a Mayer–Vietoris sequence for coarsely excisive decompositions.

If \( E \) is an \( \text{Sp} \)-valued classical coarse cohomology theory as in Definition 2.5, then we can derive a coarse cohomology theory in the sense of Fukaya–Oguni by taking homotopy groups and restricting to proper metric spaces. Condition 2.5.1 ensures that the resulting \( \mathbb{Z} \)-graded group-valued functor factorizes over the coarse category. The excisiveness Condition 2.5.2 is stronger than satisfying a Mayer–Vietoris sequence for coarsely excisive decompositions, cf. [BE16, Lem. 3.38]. Finally, the Condition 2.5.4 is actually equivalent to the requirement that \( E(X \otimes \mathbb{N}_{\text{can}}) \simeq 0 \), and \( u \)-continuity is not part of Fukaya and Oguni’s axioms.

If the \( \infty \)-category \( C \) is complete and stable, then the opposite \( \infty \)-category \( C^{\text{op}} \) is cocomplete and stable. If \( E : \text{BornCoarse}^{\text{op}} \to C \) is a functor, then we let \( E^{\text{op}} : \text{BornCoarse} \to C^{\text{op}} \) denote the induced functor between the opposite categories.

Recall the definition of a classical \( C^{\text{op}} \)-valued coarse homology theory from [BE16, Def. 4.5]. The following corollary immediately follows from a comparison of the definitions.

Corollary 2.7. \( E \) is a classical \( C \)-valued coarse cohomology theory if and only if \( E^{\text{op}} \) is a classical \( C^{\text{op}} \)-valued coarse homology theory.
Using the correspondence between coarse homology theories and coarse cohomology theories we can transfer the results and definitions concerning coarse homology theories shown or stated in [BE16] and [BE17] to the case of coarse cohomology theories. Here are two examples of such a transfer of definitions.

Recall the notion of a weakly flasque bornological coarse space [BEKW17, Def. 4.17].

**Definition 2.8.** A coarse cohomology theory $E$ is called strong if $E(X) \simeq 0$ for every weakly flasque bornological coarse space $X$.

Thus a coarse cohomology theory $E$ is strong if and only if $E^{\text{op}}$ is a strong coarse homology theory.

Recall from [BE16, Defn. 6.7] that a coarse homology theory $E^{\text{hlg}}$ is called strongly additive if for every family $(X_i)_{i \in I}$ of bornological coarse spaces we have an equivalence

$$
E^{\text{hlg}}\left( \bigcup_{i \in I} X_i \right) \simeq \prod_{i \in I} E^{\text{hlg}}(X_i)
$$

(induced by the collection of projection maps which exist by excision). Hence for coarse cohomology theories we get the following definition of additivity.

**Definition 2.9.** A coarse cohomology theory $E$ is called strongly additive if for every family $(X_i)_{i \in I}$ of bornological coarse spaces we have an equivalence

$$
\bigoplus_{i \in I} E(X_i) \simeq E\left( \bigcup_{i \in I} X_i \right)
$$

induced by the natural map.

We say that $E$ is additive, if the equivalence above is satisfied for all families of one-point spaces.

In Section 2.2 we use the relation between classical coarse homology and coarse cohomology theories to construct coarse cohomology theories by dualizing coarse homology theories.

In [BE16, Sec. 4] we have introduced the stable $\infty$-category of coarse motivic spectra $\text{Sp}\mathcal{X}$ and the universal classical coarse homology theory

$$
\text{Yo}^* : \text{BornCoarse} \to \text{Sp}\mathcal{X}.
$$

For a complete stable $\infty$-category $\mathcal{C}$ precomposition with $\text{Yo}^*$ induces by [BE16, Cor. 4.6] an equivalence between the $\infty$-categories of colimit preserving functors from $\text{Sp}\mathcal{X}$ to $\mathcal{C}^{\text{op}}$ and classical $\mathcal{C}^{\text{op}}$-valued coarse homology theories. By Corollary 2.7 we have the analogous statement for coarse cohomology theories.

Let $\mathcal{C}$ be a complete stable $\infty$-category.

**Corollary 2.10.** Precomposition by $\text{Yo}^{*\text{op}}$ induces an equivalence between the $\infty$-categories of limit-preserving functors $\text{Sp}\mathcal{X}^{\text{op}} \to \mathcal{C}$ and classical $\mathcal{C}$-valued coarse cohomology theories.
2.2 Coarse cohomology theories by duality

In this section we explain the construction of coarse cohomology theories by dualizing coarse homology theories.

Let $C$ be a stable $\infty$-category and $C$ be an object of $C$. We assume one of the following:

1. $C$ is cocomplete and $E^{hlg}: \text{BornCoarse} \to C$ is a coarse homology theory. Then we use the notation
   \[ C(-) := \text{map}(-, C): C^{op} \to \text{Sp} \]
   for the mapping space functor and set $D := \text{Sp}$.

2. $E^{hlg}: \text{BornCoarse} \to \text{Sp}$ is a coarse homology theory. In this case we assume that
   $C$ is complete and powered over $\text{Sp}$ (e.g., if $C$ is presentable). We write
   \[ C(-): \text{Sp}^{op} \to C, \quad A \mapsto C^A \]
   for the power functor and set $D := C$.

3. $C$ is complete and cocomplete and $E^{hlg}: \text{BornCoarse} \to C$ is a coarse homology theory. Furthermore,
   $C$ is closed symmetric monoidal and in particular admits a limit-preserving internal mapping object functor
   \[ \text{map}(-, C): C^{op} \to C. \]
   In this case we write $C(-) := \text{map}(-, C)$ and set $D := C$.

**Remark 2.11.** If $C$ is enriched over some complete stable $\infty$-category $V$, then one can generalize Case 1 further and define a $V$-valued dual $D_C(E)$ of $E$. An example will be indicated in Remark 5.1.

**Definition 2.12.** We define the functor

\[ D_C(E^{hlg}): \text{BornCoarse}^{op} \to D, \quad D_C(E^{hlg}) := C(-) \circ E^{hlg, op}. \]

We consider $D_C(E^{hlg})$ as the dual of $E^{hlg}$ with respect to $C$.

**Theorem 2.13.** $D_C(E^{hlg})$ is a classical $D$-valued coarse cohomology theory.

**Proof.** The functor $C(-)^{op}$ is colimit-preserving. The composition
\[ D_C(E^{hlg})^{op} = C(-)^{op} \circ E^{hlg}: \text{BornCoarse} \to D^{op} \]
is therefore a classical $D^{op}$-valued coarse homology theory. Hence $D_C(E^{hlg})$ is a classical $D$-valued coarse cohomology theory by Corollary 2.7.

The proof of the following lemma is straightforward:

**Lemma 2.14.** If $E^{hlg}$ is strong, then so is $D_C(E^{hlg})$. 

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In general $D_C(E^{hlg})$ is not additive even if $E^{hlg}$ is additive, as the next example shows.

**Example 2.15.** We consider the additive $\text{Sp}$-valued coarse $K$-homology theory $K\chi^{hlg}$ (denoted by $K\chi$ in $[BE16]$) and the complex $K$-theory spectrum $KU$ in $\text{Sp}$. Then, for every infinite set $I$,

$$D_{KU}(K\chi^{hlg})(\bigvee_I *) \simeq \text{map}(K\chi^{hlg,op}(\bigvee_I *), KU) \simeq \text{map}(\prod_I KU^{op}, KU) \not\simeq \bigoplus_I KU$$

showing that $D_{KU}(K\chi^{hlg})$ is not additive. Here the notation $(\prod_I KU)^{op}$ indicates that we first form the product in $\text{Sp}$ and then consider the result as an object of $\text{Sp}^{op}$. ♦

### 2.3 Pairings and (co-)assembly maps

The coarse cohomology theories that we construct in this paper all admit a pairing with a corresponding coarse homology theory. In this section we give a precise definition of the notion of a natural pairing. We further discuss assembly and coassembly maps and state a formula which expresses the compatibility of these morphisms with the pairing.

Let $C$ be a stable $\infty$-category and let $C$ be an object of $C$. The following three cases correspond to the cases considered in Section 2.2. We assume one of the following:

1. a) $C$ is cocomplete.
   
   b) $E : \text{BornCoarse}^{op} \to \text{Sp}$ is a coarse cohomology theory.
   
   c) $E^{hlg} : \text{BornCoarse} \to C$ is a coarse homology theory.

   In this case we use the notation $C(-) := \text{map}(-, C) : C^{op} \to \text{Sp}$ for the mapping space functor and set $D := \text{Sp}$.

2. a) The $\infty$-category $C$ is complete and powered over $\text{Sp}$, e.g., that $C$ is presentable.

   b) $E : \text{BornCoarse}^{op} \to C$ is a coarse cohomology theory.

   c) $E^{hlg} : \text{BornCoarse} \to \text{Sp}$ is a coarse homology theory.

   In this case we write

   $$C(-) : \text{Sp}^{op} \to C, \quad A \mapsto C^A$$

   for the power functor and set $D := C$.

3. a) $C$ is complete and cocomplete.

   b) $C$ is closed symmetric monoidal and in particular admits a limit-preserving $C$-valued mapping object functor $\text{map}(-, C) : C^{op} \to C$.

   c) $E : \text{BornCoarse}^{op} \to C$ is a coarse cohomology theory.

   d) $E^{hlg} : \text{BornCoarse} \to C$ is a coarse homology theory.

   In this case we write $C(-) := \text{map}(-, C)$ and set $D := C$.
In all three cases we can form the dual cohomology theory

\[ D_C(E^{hlg}) : \text{BornCoarse}^{op} \to \text{D} \]

according to Definition 2.12.

**Definition 2.16.** A pairing between \( E \) and \( E^{hlg} \) with values in \( C \) is a morphism of coarse cohomology theories \( p : E \to D_C(E^{hlg}) \).

**Example 2.17.** The identity morphism is a paring between \( D_C(E^{hlg}) \) and \( E^{hlg} \).

By Corollary 2.10 we can equivalently interpret \( E \) and \( D_C(E^{hlg}) \) as limit-preserving functors \( \text{Sp}X^{op} \to \text{D} \) and the pairing \( p \) as a morphism between such functors.

We call \( p \) a \( C \)-valued pairing because of the following construction. Given such a pairing, using the evaluation morphism \( \text{ev}_A : C^A \otimes A \to C \), we can define for every motivic coarse spectrum \( M \) in \( \text{Sp}X \) a morphism

\[ P_M : E(M) \otimes E^{hlg}(M) \xrightarrow{p \otimes \text{id}_{E^{hlg}(M)}} D_C(E^{hlg})(M) \otimes E^{hlg}(M) \xrightarrow{\text{ev}_{E^{hlg}(M)}} C . \]  

(2.1)

If \( f : M \to M' \) is a morphism of motivic coarse spectra, then the naturality of the pairing \( p \) leads to the relation

\[ P_M \circ (f^* \circ \text{id}_{E^{hlg}(M)}) \simeq P_{M'}(\text{id}_{E(M')} \otimes f) : E(M') \otimes E^{hlg}(M) \to C . \]  

(2.2)

We now turn to the assembly maps.

In [BE17] we introduced the category of uniform bornological coarse spaces \( \text{UBC} \), the notion of a local homology theory, and the universal local homology theory

\[ \text{Yo}^*B : \text{UBC} \to \text{Sp}B . \]

Furthermore, in [BEKW17, Sec. 4.4] we introduced the universal strong coarse homology theory

\[ \text{Yo}^*wfl : \text{BornCoarse} \to \text{Sp}X_{wfl} . \]

We let

\[ \mathcal{O}^\infty : \text{UBC} \to \text{Sp}X \]

denote the germs-at-\( \infty \) of the cone functor ([BE17, Sec. 8] and [BEKW17, Sec. 9.5]). By [BE17, Lem. 9.5] the composition

\[ \mathcal{O}^\infty_{wfl} : \text{UBC} \xrightarrow{\mathcal{O}^\infty} \text{Sp}X \to \text{Sp}X_{wfl} \]

is a local homology theory and therefore extends essentially uniquely to a colimit-preserving functor (denoted by the same symbol)

\[ \mathcal{O}^\infty_{wfl} : \text{Sp}B \to \text{Sp}X_{wfl} . \]
By [BE17 Prop. 5.2] the Rips complex construction yields a classical coarse homology theory

\[ P : \text{BornCoarse} \to \text{Sp} \mathcal{B} \, . \]

Therefore the composition

\[ \text{BornCoarse} \xrightarrow{\quad} \text{Sp} \mathcal{B} \xrightarrow{\mathcal{O}^\infty_{\text{wfl}}} \text{Sp} \mathcal{X}_{\text{wfl}} \]

is a classical coarse homology theory.

We assume now that \( E^{\text{hlg}} \) is a strong coarse homology theory. We can interpret \( E^{\text{hlg}} \) as a colimit-preserving functor defined on \( \text{Sp} \mathcal{X}_{\text{wfl}} \). The composition

\[ E^{\text{hlg}} \circ \mathcal{O}^\infty_{\text{wfl}} \circ P \]

is a new classical coarse homology theory with the same target as \( E^{\text{hlg}} \).

**Definition 2.18.** The coarse homology theory \( E^{\text{hlg}} \circ \mathcal{O}^\infty_{\text{wfl}} \circ P \) is called the coarsification of \( E^{\text{hlg}} \).

The coarsification of \( E^{\text{hlg}} \) is related with \( E^{\text{hlg}} \) by an assembly map, a natural transformation of functors

\[ \mu_{E^{\text{hlg}}} : E^{\text{hlg}} \circ \mathcal{O}^\infty_{\text{wfl}} \circ P \xrightarrow{} \Sigma E^{\text{hlg}} \]

defined in [BE17 Def. 9.6].

Let now \( E \) be a strong \( D \)-valued coarse cohomology theory. We can define a new coarse cohomology theory by

\[ E \circ \mathcal{O}^\infty_{\text{wfl}} \circ P := (E^{op} \circ \mathcal{O}^\infty_{\text{wfl}} \circ P)^{op} : \text{BornCoarse}^{op} \to D \]

with the same target as \( E \).

**Definition 2.19.** The coarse cohomology theory \( E \circ \mathcal{O}^\infty_{\text{wfl}} \circ P \) is called the coarsification of \( E \).

**Definition 2.20.** The coarse coassembly map is defined by

\[ \mu^E := (\mu_{E^{op}})^{op} : \Sigma^{-1} E \xrightarrow{} E \circ \mathcal{O}^\infty_{\text{wfl}} \]

Hence the coassembly map is the morphism obtained by interpreting \( E \) as a \( D^{op} \)-valued strong coarse homology theory \( E^{op} \) and then using [BE17 Def. 9.6].

**Remark 2.21.** In [BE17] we discussed various conditions implying that the coarse assembly map is an equivalence. Using the relation between coarse cohomology theories and coarse homology theories those results yield conditions on the bornological coarse space \( X \) and the coarse cohomology theory \( E \) which imply that the coarse coassembly map

\[ \mu_{E,X} : \Sigma^{-1} E(X) \to E \circ \mathcal{O}^\infty_{\text{wfl}}(X) \]

is an equivalence. Because this translation is straightforward we will not spell out those statements in detail here.
Note that most of the theorems stated in the introduction [BE17, Sec. 1.2] assume that the coarse homology theory takes values in a presentable, stable ∞-category. If \( E \) is a \( C \)-valued coarse cohomology theory, then the coarse homology theory \( E^{op} \) is \( C^{op} \)-valued. But if \( C \) is presentable, then in general \( C^{op} \) is not presentable. Hence the results of [BE17] are not applicable in a formal sense. But fortunately, the assumption of presentability made in the theorems in [BE17] is actually not necessary: the assumption really needed in those theorems is that the stable ∞-category is complete, cocomplete and tensored and powered over \( \mathbf{Sp} \). If \( C \) is presentable, then \( C^{op} \) has these properties and hence the results of [BE17] are applicable.

Assume now that we have a paring \( p \) between \( E \) and \( E^{hlg} \), see Definition 2.16. Let \( X \) be a bornological coarse space. Since we assume that \( E \) and \( E^{hlg} \) are strong we can interpret \( E \) and \( D_{C}(E^{hlg}) \) as limit-preserving functors \( \mathbf{Sp}X \text{wfl} \rightarrow D_{C} \), and the pairing \( p \) as a natural transformation between such functors. Similarly as in (2.1), we define for every motivic coarse spectrum \( M \) in \( \mathbf{Sp}X \text{wfl} \) a morphism

\[
P_{\text{wfl}}^M : E(M) \otimes E^{hlg}(M) \xrightarrow{\partial \otimes \text{id}_{E^{hlg}(M)}} D_{C}(E^{hlg})(M) \otimes E^{hlg}(M) \xrightarrow{\text{ev}_{E^{hlg}(M)}} C .
\]

Moreover, if \( f : M \rightarrow M' \) is a morphism of motivic coarse spectra in \( \mathbf{Sp}X \text{wfl} \), then the naturality of the pairing \( p \) leads to the relation

\[
P_{\text{wfl}}^M \circ (f^{*} \circ \text{id}_{E^{hlg}(M)}) \simeq P_{M'}(\text{id}_{E(M')} \otimes f_{*}) : E(M') \otimes E^{hlg}(M) \rightarrow C .
\]

**Proposition 2.22.** Under the assumptions described above we have the relation

\[
P_{Y_{\text{wfl}}^{*}O^{\infty}P(M)}^{\text{wfl}} \circ (\mu^{E} \otimes \text{id}_{E^{hlg}O^{\infty}P(M)}) \simeq P_{\Sigma Y_{\text{wfl}}^{*}O^{\infty}P(M)}^{\text{wfl}} \circ (\text{id}_{\Sigma^{-1}E(M)} \otimes \mu_{E^{hlg}}) : \Sigma^{-1}E(M) \otimes E^{hlg}O^{\infty}P(M) \rightarrow C .
\]

**Proof.** The coarse assembly maps for \( E \) and \( E^{hlg} \) are obtained by applying \( E \) or \( E^{hlg} \) to the motivic version of the assembly map

\[
\mu^{\text{Mot}} := \mu_{Y_{\text{wfl}}^{*}O^{\infty}P} : Y_{\text{wfl}}^{*}O^{\infty}P \rightarrow \Sigma Y_{\text{wfl}}^{*} .
\]

The assertion of the proposition is now the special case of (2.3) for \( f = \mu^{\text{Mot}}(M) \).

**Remark 2.23.** If \( C = \mathbf{Sp} \) and \( E \) is the coarse stable homotopy theory \( Q \) introduced in [BE16, Def. 6.23], then the coarse cohomology theory \( D_{C}(Q) \) constructed in the present section coincides with the coarse cohomology \( Q_{C} \) in Definition 4.1. In order to prove Theorem 4.2 instead of verifying the axioms directly, one could also appeal to Theorem 2.13. In that case one must use the fact (shown in [BE16 Thm. 6.25]) that \( Q \) is a coarse homology theory. Our motivation for a direct approach in Section 4.1 is that the details of the construction are used in the proof of Proposition 4.13.

\[\Diamond\]
3 Coarse ordinary cohomology

In this section we construct coarse cohomology with coefficients in an abelian group $A$:

\[ H\mathcal{A}\mathcal{X} : \text{BornCoarse} \to \text{Ch}_\infty. \]

Its target is the $\infty$-category of chain complexes.

**Remark 3.1.** In this remark we recall the construction of $\text{Ch}_\infty$. We start with the category $\text{Ch}$ of chain complexes of abelian groups and let $W$ denote the quasi-isomorphisms in $\text{Ch}$. Then we form the $\infty$-category $\text{Ch}_\infty := \text{Ch} [W^{-1}]$ which turns out to be stable. The $\infty$-category $\text{Ch}_\infty$ is complete. We let

\[ \iota : \text{Ch} \to \text{Ch}_\infty \] (3.1)

denote the natural localization functor.

To a set $X$ we can functorally associate a simplicial set $\hat{X}$, the Čech nerve of the projection $X \to \ast$. For $n$ in $\mathbb{N}$ its set of $n$-simplices is given by $\hat{X}[n] := X^{\times (n+1)}$.

Let $X$ be a bornological coarse space, $U$ be a coarse entourage of $X$, and $B$ be a bounded subset of $X$. An $n$-simplex $(x_0, \ldots, x_n)$ in $\hat{X}$ is called $U$-controlled if $(x_i, x_j) \in U$ for all $i, j$ in $[n]$. We say that this simplex is contained in $B$ if $x_i \in B$ for all $i$ in $[n]$.

If the entourage $U$ contains the diagonal, then the $U$-controlled simplices form a simplicial subset $\hat{X}_U$ of $\hat{X}$.

To any simplicial set $S$ and abelian group $A$ we can functorially associate a chain complex $C(S; A)$ in $\text{Ch}$. It is defined as the chain complex associated to the cosimplicial abelian group $A^S$. For $n$ in $\mathbb{Z}$ the group of $n$-chains is given by $A^S[n]$, and the boundary operator $d : C^n(S; A) \to C^{n+1}(S; A)$ is given by $\sum_{i=0}^{n+1} (-1)^i d_i$, where $d_i$ is induced by the $i$th face map $\partial_i : S[n+1] \to S[n]$. For example, $\partial_0(x_0, \ldots, x_{n+1}) := (x_1, \ldots, x_{n+1})$.

We let $C_U(X, B; A)$ be the $\mathbb{Z}$-graded subgroup of $C(\hat{X}_U; A)$ of functions which vanish on all simplices which are not contained in $B$. Observe that the $\mathbb{Z}$-graded subgroup $C_U(X, B; A)$ is not a subcomplex. Indeed, the differential of $C(\hat{X}_U; A)$ restricts to maps

\[ d : C_U(X, B; A) \to C_U(X, U[B]; A) \] (3.2)

for all $B$ in the bornology $B$ of $X$. Hence, if we form the union of these $\mathbb{Z}$-graded subgroups over the bounded subsets $B$ of $X$, then we obtain a subcomplex of $C(\hat{X}_U; A)$

\[ C_U(X; A) := \colim_{B \in B} C_U(X, B; A). \] (3.3)

Note that we consider $C_U(X; A)$ as an object of $\text{Ch}$.

Let $f : X' \to X$ be a morphism between bornological coarse spaces. The map $f$ induces a map of simplicial sets $\hat{f} : \hat{X}' \to \hat{X}$. Assume that $U'$ is an entourage of $X'$, $U$ is an entourage
of $X$, and that $f(U') \subseteq U$. Then $\hat{f}$ restricts to a map of simplicial sets $\hat{X}'_U \to \hat{X}_U$. Since $f$ is proper, pull-back along this map induces a morphism of chain complexes

$$f^*: C_U(X; A) \to C_{U'}(X'; A).$$

We consider the following category $\text{BornCoarse}^C$:

1. The objects of $\text{BornCoarse}^C$ are pairs $(X, U)$ of a bornological coarse space $X$ and an entourage $U$.
2. The morphisms $(X', U') \to (X, U)$ in $\text{BornCoarse}^C$ are morphisms of bornological coarse spaces $f: X' \to X$ such that $f(U') \subseteq U$.

We have functors

$$p: \text{BornCoarse}^C \to \text{BornCoarse}, \quad (X, U) \mapsto X$$

and (recall $\iota$ from (3.1))

$$\iota C(A): (\text{BornCoarse}^C)^{op} \to \text{Ch}_\infty, \quad (X, U) \mapsto \iota C_U(X; A).$$

**Definition 3.2.** We define the functor $HAX: \text{BornCoarse}^{op} \to \text{Ch}_\infty$ as the right Kan extension

$$\begin{array}{ccc}
\text{BornCoarse}^{op} & \xrightarrow{\eta C(A)} & \text{Ch}_\infty \\
p^{op} \downarrow & & \downarrow HAX \\
(p^C)^{op} & & \\
\end{array}$$

**Remark 3.3.** The point-wise formula for the right Kan extension provides the formula

$$HAX(X) \simeq \lim_{U \in \mathcal{C}} \iota C_U(X; A)$$

for the evaluation of the functor $HAX$ on the bornological coarse space $X$, where $\mathcal{C}$ denotes the coarse structure of $X$.

It is crucial to apply $\iota$ before taking the limit. This ensures that the limit is derived. ♦

**Theorem 3.4.** $HAX$ is a coarse cohomology theory.

**Proof.** The axioms given in Definition 2.5 for a coarse cohomology theory will be verified in the following four Lemmas 3.5, 3.6, 3.7, and 3.9.

**Lemma 3.5.** $HAX$ is $u$-continuous

**Proof.** Let $X$ be a bornological coarse space with coarse structure $\mathcal{C}$. Using (3.5) and a cofinality consideration we get the chain of canonical equivalences

$$HAX(X) \simeq \lim_{U \in \mathcal{C}} \iota C_U(X; A) \simeq \lim_{U \in \mathcal{C}} \lim_{V \in \mathcal{C}(U)} \iota C_V(X; A) \simeq \lim_{U \in \mathcal{C}} HAX(X_U).$$
Lemma 3.6. $\text{HAX}$ is coarsely invariant.

Proof. Let $f, g : X \to X'$ be two maps of bornological coarse spaces which are close to each other. If $U$ is an entourage of $X$, then we choose an entourage $U'$ of $X'$ so large that $f(U) \subseteq U'$, $g(U) \subseteq U'$ and $(f,g)(\text{diag}(X)) \subseteq U'$. Then for all $n \in \mathbb{N}$ and $i$ in $[n]$ we can form the maps $h_i : \hat{X}_U[n] \to \hat{X}_U'[n+1]$ given by

$$(x_0, \ldots, x_n) \mapsto (f(x_0), \ldots, f(x_i), g(x_i), \ldots, g(x_n)).$$

Pull-back along $h_i$ induces a map

$$h_i^{n*} : C_{U'}^{n+1}(X'; A) \to C_U^n(X; A).$$

We form $h_n := \sum_{i=0}^n (-1)^i h_i^{n*}$ and the map $h := \bigoplus_{n \in \mathbb{Z}} h^n$ of degree $-1$. Then one checks directly that

$$d \circ h + h \circ d = g^* - f^* : C_{U'}(X'; A) \to C_U(X; A).$$

Let $X$ be a bornological coarse space. We consider the maps

$$p : \{0, 1\} \otimes X \to X, \quad i : X \to \{0, 1\} \otimes X$$

given by the projection and the inclusion of the point 0, respectively. Then $p \circ i = \text{id}$ and $i \circ p$ is close to the identity of $\{0, 1\} \otimes X$. For an entourage $U$ of $X$ let $\tilde{U} := \{0, 1\}^2 \times U$ be the corresponding entourage of $\{0, 1\} \otimes X$. The above construction then shows that $(i \circ p)^*$ is chain homotopic to the identity on $C(\{0, 1\} \otimes X; A)$. This implies that

$$p^* : C_{U}(X; A) \to C_{\tilde{U}}(\{0, 1\} \otimes X; A)$$

is an equivalence for every entourage $U$ of $X$. We conclude that

$$\text{HAX}(p) : \text{HAX}(X) \to \text{HAX}(\{0, 1\} \otimes X)$$

is an equivalence. \hfill \qed

Lemma 3.7. $\text{HAX}$ is excisive.

Proof. Let $X$ be a bornological coarse space $X$. For an entourage $U$ and a subset $Y$ of $X$ we write $U_Y := U \cap (Y \times Y)$. We have a surjective restriction

$$C_U(X; A) \to C_{U_Y}(Y; A).$$

We denote its kernel by $C_U(X, Y; A)$.

Let $(Z, \mathcal{Y})$ be a complementary pair on $X$ with $\mathcal{Y} = (Y_i)_{i \in I}$. For every $i$ in $I$ we consider the map of exact sequences

$$
\begin{array}{cccccccc}
0 & \longrightarrow & C_U(X, Y_i; A) & \longrightarrow & C_U(X; A) & \longrightarrow & C_{U_Y}(Y_i; A) & \longrightarrow & 0 \\
& & \downarrow \tau_i & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{U_Z}(Z, Z \cap Y_i; A) & \longrightarrow & C_{U_Z}(Z; A) & \longrightarrow & C_{U_{Z \cap Y_i}}(Z \cap Y_i; A) & \longrightarrow & 0
\end{array}
$$

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We claim that the morphism \((r_i)\) is an isomorphism of pro-systems indexed by \(I\). Indeed we can define a map of complexes \(s_i : C_{U_Z}(Z, Z \cap Y_j; A) \to C_U(X, Y_i; A)\) by extension by zero as long as \(j\) in \(I\) satisfies \(U[Y_i] \subseteq Y_j\). Since \(Y\) is a big family, for every \(i\) in \(I\) we can choose such an index \(j\) in \(I\). Then the resulting family \((s_i)_{i \in I}\) is an inverse of \((r_i)_{i \in I}\).

The localization \(\iota\) sends short exact sequences of chain complexes to fibre sequences. We apply \(\iota\) and \(\lim_{\iota \in C}\) and \(\lim_{U \in C}\) in order to get the morphism between fibre sequences

\[
\begin{array}{ccc}
\lim_{U \in C} \lim_{i \in I} \iota C_U(X, Y_i; A) & \longrightarrow & HAX(X) \\
\| & & \| \\
\lim_{U \in C} \lim_{i \in I} \iota C_U(Z, Z \cap Y_i; A) & \longrightarrow & HAX(Z) \\
\| & & \| \\
\end{array}
\]

In view of the outer vertical equivalences the middle square is cartesian.

**Remark 3.8.** It is important for the proof of Lemma 3.7 that we consider the limits after applying \(\iota\). Limits in \(\text{Ch}_\infty\) preserve fibre sequences. In contrast, limits in \(\text{Ch}\) in general do not preserve short exact sequences. ♦

**Lemma 3.9.** \(HAX\) vanishes on flasques.

**Proof.** Let \(X\) be a flasque bornological coarse space with flasqueness implemented by the morphism \(f : X \to X\). We define a map of chain complexes

\[
S : C_V(X; A) \to C_U(X; A)
\]

by

\[
S(\phi) := \sum_{n=0}^{\infty} f^{n*} \phi ,
\]

where \(U\) is an entourage of \(X\) and \(V := \bigcup_{n \in \mathbb{N}} f^n(U)\). Since \(\phi\) is supported on some bounded subset of \(X\) almost all summands vanish and the sum has a well-defined interpretation. One furthermore checks that

\[
f^* \circ S + r = S ,
\]

where \(r : C_V(X; A) \to C_U(X; A)\) is the restriction. Applying \(\iota\) and \(\lim_{U \in C}\), the morphisms \(S\) for various \(U\) induce a morphism of chain complexes

\[
\tilde{S} : HAX(X) \to HAX(X)
\]

satisfying

\[
HAX(f) \circ \tilde{S} + \text{id}_{HAX(X)} \simeq \tilde{S} .
\]  

(3.6)

Since \(HAX\) is coarsely invariant by Lemma 3.6 we get the equivalence

\[
\tilde{S} + \text{id}_{HAX(X)} \simeq \tilde{S} .
\]  

(3.7)

It implies \(HAX(X) \simeq 0\).
We have thus verified all four axioms for coarse cohomology theories and hence finished the proof of Theorem 3.3.

In the remaining part of this section we will verify that \( \text{HA} \) is a strong coarse cohomology theory that satisfies additivity, compare our definition to the original one of Roe [Roe93], and finally describe a natural pairing with coarse homology.

**Lemma 3.10.** \( \text{HA} \) is strong.

*Proof.* We assume that \( X \) is weakly flasque. We repeat now the argument for Lemma 3.9. Because we now already know that \( \text{HA} \) is a coarse cohomology theory, to see that (3.6) implies (3.7) it suffices to assume that \( \text{Yo}^i(f) \simeq \text{id}_{\text{Yo}^i(X)} \).

Recall Definition 2.9 of strong additivity.

**Lemma 3.11.** \( \text{HA} \) is strongly additive.

*Proof.* We must show that for every family \( (X_i)_{i \in I} \) of bornological coarse spaces we have the equivalence

\[
\text{HA}( \biguplus_{i \in I} X_i ) \simeq \bigoplus_{i \in I} \text{HA}(X_i).
\]

(see [BE16, Def. 2.25] for the definition of the free union). We abbreviate \( X := \bigcup_{i \in I} X_i \). The set of entourages of \( X \) of the form \( \bigcup_{i \in I} U_i \) for families \( (U_i)_{i \in I} \) in \( \prod_{i \in I} C_i \) (where \( C_i \) denotes the coarse structure of \( X_i \)) is cofinal in the set \( C \) of entourages of \( X \). Let \( B \) denote the bornology of \( X \) and \( B_i \) denote the bornology of \( X_i \) for all \( i \in I \). For \( B \) in \( B \) we have \( B \cap X_i \in B_i \) for all \( i \in I \), and \( B \cap X_i = \emptyset \) for all but finitely many \( i \) in \( I \). Consequently we get the chain of equivalences

\[
\text{HA}(X) \simeq \lim_{U_i \in C} \lim_{B \in B} \text{colim}_C U(X, B; A)
\]

\[
\simeq \lim_{(U_i)_{i \in I} \in \prod_{i \in I} C_i} \lim_{B \in B} \text{colim}_C U(X, B; A)
\]

\[
\simeq \lim_{(U_i)_{i \in I} \in \prod_{i \in I} C_i} \text{colim}_{B \in B} \bigoplus_{i \in I} C(U(X, B \cap X_i; A))
\]

\[
\simeq \lim_{(U_i)_{i \in I} \in \prod_{i \in I} C_i} \bigoplus_{i \in I} \text{colim}_{B_i \in B_i} C(U(X_i, B_i; A))
\]

\[
\simeq \bigoplus_{i \in I} \text{colim}_{U_i \in C_i} C(U(X_i; A))
\]

\[
\simeq \bigoplus_{i \in I} \text{HA}(X_i)
\]
In addition to the properties of the bounded subsets of \( X \) mentioned above, for the marked equivalence we also use the fact that the chain boundary operator does not mix the different coarse components of \( X \).

One easily checks that the morphism inducing this equivalence is the one given by excision for the complementary pairs \( (X_i, \{X \setminus X_i\}) \) on \( X \) for all \( i \) in \( I \).  

If \( X \) is a metric space, then Roe [Roe93] has defined coarse cohomology groups \( H\mathcal{X}_{\text{Roe}}(X; A) \). Roe’s coarse cohomology groups are defined as the cohomology groups of the complex \( C\mathcal{X}_{\text{Roe}}(X; A) \) of locally bounded, \( A \)-valued Borel functions on the simplicial space \( \hat{X} \) whose restrictions to \( \hat{X}_{U_r} \) have bounded support for every entourage \( U_r := \{ x, y \in X \mid d(x, y) < r \} \). In our notation

\[
C\mathcal{X}_{\text{Roe}}(X; A) := \lim_{U \in \mathcal{C}} C_{U, \text{Roe}}(X; A),
\]

where \( C_{U, \text{Roe}}(X; A) \) is the subcomplex of \( C_U(X; A) \) of locally bounded Borel functions.

We have a natural morphism

\[
\iota C\mathcal{X}_{\text{Roe}}(X; A) \to H\mathcal{X}(X).
\]

**Lemma 3.12.** If \( X \) is a proper metric space, then the morphism \( (3.8) \) is an equivalence.

**Proof.** It suffices to show that \( (3.8) \) induces a quasi-isomorphism. Since the domain and the target of this morphism are coarsely invariant we can replace \( X \) by a locally finite, discrete subset which is coarsely equivalent to \( X \). Furthermore, we can replace the limit over \( U \) in \( \mathcal{C} \) by the limit over the family of entourages \( U_n := \{ x, y \in X \mid d(x, y) < n \} \) indexed by \( n \in \mathbb{N} \).

If \( X \) is a locally finite, discrete metric space, then the conditions of being a Borel function and of being locally bounded are vacuous. In this case the only difference between Roe’s complex and our complex is the order of the limit \( \lim_{U \in \mathcal{C}} \) and the localization \( \iota \). In Roe’s case the limit is not derived.

We now observe that the restriction maps \( C_{U_{n+1}}(X; A) \to C_{U_n}(X; A) \) are surjective for all \( n \) in \( \mathbb{N} \). This condition ensures that one can interchange the order of taking the limit and the localization. The assertion follows from this.

Therefore our construction extends Roe’s coarse cohomology from proper metric spaces to all bornological coarse spaces.

**Remark 3.13.** In this remark we describe a natural pairing between \( H\mathcal{X} \) and \( H\mathcal{X}_{\text{hlg}} \). In this example we are in the Case 3 of the list described in the beginning of Section 2.3.

We use the closed symmetric monoidal structure of the \( \infty \)-category \( \mathbf{Ch}_\infty \). The dualizing object (denoted by \( C \) in Section 2.3) is the object \( \iota A[0] \) in \( \mathbf{Ch}_\infty \), where \( A[0] \) in \( \mathbf{Ch} \) is the chain complex with \( A \) in degree zero.

Recall that the coarse homology of the bornological coarse space \( X \) is given by

\[
H\mathcal{X}_{\text{hlg}}(X) \simeq \iota C\mathcal{X}_{\text{hlg}}(X),
\]
where $C\mathcal{X}^{hlg}(X)$ is the complex of locally finite and controlled chains [BE16, Def. 6.13]. Let $U$ be a coarse entourage of $X$. Then we let

$$\widetilde{C\mathcal{X}}^{hlg}(X,U) := C\mathcal{X}^{hlg}_{U}(X)$$

be the subcomplex of $C\mathcal{X}^{hlg}(X)$ of locally finite, $U$-controlled chains. We thus get a functor

$$\widetilde{C\mathcal{X}}^{hlg} : \text{BornCoarse}^C \to \text{Ch}, \quad (X,U) \mapsto \widetilde{C\mathcal{X}}^{hlg}(X,U).$$

Furthermore, recall the functor

$$C(A) : (\text{BornCoarse}^C)^{op} \to \text{Ch}, \quad (X,U) \mapsto C(A)(X,U) := C_U(X;A).$$

We first define a natural transformation of $\text{Ch}$-valued functors

$$\tilde{p} : C(A) \to \text{Hom}(\widetilde{C\mathcal{X}}^{hlg,op},A[0]) : (\text{BornCoarse}^C)^{op} \to \text{Ch}$$

as follows. Let $(X,U)$ be an object of $\text{BornCoarse}^C$, fix $n \in \mathbb{N}$, and let $\phi$ be an element of $C_A(X,U)^n$. Then we define the homomorphism $\tilde{p}_{(X,U)}(\phi)$ in $\text{Hom}(\widetilde{C\mathcal{X}}^{hlg,op}(X,U),A[0])^n$ as the $\mathbb{Z}$-linear extension of the map which sends the simplex $(x_0,\ldots,x_n)$ in $\hat{X}_U[n]$ to $\phi((x_0,\ldots,x_n))$ and vanishes on simplices of dimensions different from $n$. One easily checks that $\tilde{p}_{(X,U)}$ is a map of chain complexes, and that the collection of maps $\tilde{p}_{(X,U)}$ for all $(X,U)$ in $\text{BornCoarse}^C$ defines a natural transformation of functors $\tilde{p}$.

The natural transformation $\tilde{p}$ induces a morphism

$$\iota \tilde{p} : \iota C(A) \to \iota \text{Hom}(\widetilde{C\mathcal{X}}^{hlg,op},A[0])$$

between functors from $(\text{BornCoarse}^C)^{op}$ to $\text{Ch}_{\infty}$. We derive the desired pairing

$$p : H\mathcal{X} \to D_{\iota A[0]}(H\mathcal{X}^{hlg})$$

by a right Kan extension of $\iota \tilde{p}$ along the forgetful functor (3.4) from $(\text{BornCoarse}^C)^{op}$ to $\text{BornCoarse}^{op}$. To this end we must check that the domain and target of this extension are the correct functors.

By Definition [3.2] the domain of the Kan extension of $\iota \tilde{p}$ is $H\mathcal{X}$. We now evaluate the target on $X$ in $\text{BornCoarse}$. Using the objectwise formula for the Kan extension this evaluation is given by

$$\lim_{U \in \mathcal{C}} \text{Hom}(\widetilde{C\mathcal{X}}^{hlg}(X,U),A[0]) \simeq \lim_{U \in \mathcal{C}} \text{map}(\iota \widetilde{C\mathcal{X}}^{hlg}(X,U),\iota A[0]) \simeq \text{map}(\text{colim}_{U \in \mathcal{C}} \iota \widetilde{C\mathcal{X}}^{hlg}(X,U),\iota A[0]).$$

We now use the chain of equivalences

$$\text{colim}_{U \in \mathcal{C}} \iota \widetilde{C\mathcal{X}}^{hlg}(X,U) \simeq \iota \text{colim}_{U \in \mathcal{C}} \widetilde{C\mathcal{X}}^{hlg}(X,U) \simeq \iota C\mathcal{X}^{hlg}(X) \simeq H\mathcal{X}^{hlg}(X)$$

(where the first equivalence follows from the fact that the poset $\mathcal{C}$ of entourages of $X$ is filtered and $\iota$ commutes with filtered colimits). We therefore obtain the following formula for the target:

$$\text{map}(H\mathcal{X}^{hlg}(X),\iota A[0]) \simeq D_{\iota A[0]}(H\mathcal{X}^{hlg})(X).$$

The right-Kan extension of $\iota \tilde{p}$ therefore is a morphism as in (3.9). This is the pairing. ◆
4 The coarse cohomology theory $Q_C$

4.1 Definition, verification of the axioms, and a pairing

In this section we introduce the $C$-valued coarse cohomology theory $Q_C$ for $C$ an object of the presentable stable $\infty$-category $C$. This coarse cohomology theory can be thought of as a generalized version of coarse stable cohomotopy. We prove that $Q_C$ is a strong coarse cohomology theory, and at the end of this section we discuss a natural pairing with coarse stable homotopy (which was introduced in [BE16, Sec. 6.4]).

If $X$ is a bornological coarse space and $U$ is a coarse entourage of $X$, then $P_U(X)$ denotes the space of probability measures on (the discrete measurable space) $X$ which have finite, $U$-bounded support. This space has the structure of a simplicial complex. It is a (quasi-)metric space with the path (quasi-)metric induced by the spherical metric on the simplices.

We actually have a functor $\text{BornCoarse}^C \to \text{Top}$, $(X,U) \mapsto P_U(X)$ (see Section 3 for the definition of $\text{BornCoarse}^C$). Let $\iota: \text{Top} \to \text{Spc}$ be the canonical functor, where $\text{Spc}$ is the $\infty$-category of spaces.

Assume that $C$ is a presentable stable $\infty$-category. Then $C$ is tensored and powered over $\text{Spc}$. In particular, any object $C$ of $C$ gives rise to a functor

$$C^(-): \text{Spc}^{\text{op}} \to C, \quad A \mapsto C^A.$$ (4.1)

We shall define a functor $Q_C: (\text{BornCoarse})^{\text{op}} \to C$ whose evaluation on objects is given by

$$X \mapsto \lim_{U \in C} \colim_{B \in B} \text{Fib}(C^{n^P_U(X)} \to C^{n^P_U(X \setminus B)}).$$ (4.2)

To this end we consider the category $\text{BornCoarse}^{C,B}$:

1. An object of $\text{BornCoarse}^{C,B}$ is a triple $(X,U,B)$ of a bornological coarse space $X$, a coarse entourage $U$ of $X$, and a bounded subset $B$ of $X$.

2. A morphism $f: (X',U',B') \to (X,U,B)$ is a morphism of bornological coarse spaces $f: X' \to X$ such that $(f \times f)(U') \subseteq U$ and $f^{-1}(B) \subseteq B'$.

We have forgetful functors

$$\text{BornCoarse}^{C,B} \xrightarrow{q} \text{BornCoarse}^C \xrightarrow{p} \text{BornCoarse}, \quad (X,U,B) \mapsto (X,U) \mapsto X.$$ (4.3)

We furthermore have a functor

$$W: \text{BornCoarse}^{C,B} \to \text{Top}^{\Delta^I}, \quad W(X,U,B) := (P_U(X \setminus B) \to P_U(X)).$$ (4.4)

It induces the functor

$$\hat{Q}_C: (\text{BornCoarse}^{C,B})^\text{op} \to C, \quad \hat{Q}_C(X,U,B) := \text{Fib}(C^{nW}).$$ (4.5)
Definition 4.1. We define the functor $Q_C$ as the composition of a left and a right Kan extension

$$
\begin{array}{c}
\text{(BornCoarse}^C \text{)}^\text{op} \\
\downarrow p \\
\text{BornCoarse}^\text{op}
\end{array}
\xrightarrow{\hat{Q}_C} 
\begin{array}{c}
\text{(BornCoarse}^C \text{)}^\text{op} \\
\downarrow q \\
C
\end{array}
\xrightarrow{Q_C}
\begin{array}{c}
\text{(BornCoarse}^C \text{)}^\text{op} \\
\downarrow p \\
\text{BornCoarse}^\text{op}
\end{array}
$$

Theorem 4.2. $Q_C$ is a $C$-valued coarse cohomology theory.

Proof. In the following four Lemmas 4.3, 4.4, 4.5 and 4.6 we verify the four axioms from Definition 2.5 on coarse cohomology theories.

Lemma 4.3. $Q_C$ is $u$-continuous.

Proof. We have $\hat{Q}_C(X,U) \simeq \operatorname{colim}_{B \in B} \hat{Q}_C(X,U,B)$. Then, by (4.2),

$$
Q_C(X) \simeq \varprojlim_{U \in \mathcal{C}} \hat{Q}_C(X,U) \simeq \varprojlim_{U \in \mathcal{C}} \varprojlim_{n \in \mathbb{N}} \hat{Q}_C(X,U^n) \simeq \varprojlim_{U \in \mathcal{C}} Q_C(X,U).
$$

Lemma 4.4. $Q_C$ is coarsely invariant.

Proof. For a coarse entourage $U$ of $X$ we form the entourage $\tilde{U} := \{0,1\} \times U$ of $\{0,1\} \times X$. The projection $P_U(\{0,1\} \otimes Y) \to P_U(Y)$ is a homotopy equivalence for every subset $Y$ of $X$. For every bounded subset $B$ of $X$ we define the bounded subset $\tilde{B} := \{0,1\} \times B$ of $\{0,1\} \otimes X$. Then $\hat{Q}_C(X,U,B) \to \hat{Q}_C(\{0,1\} \otimes X, \tilde{U}, \tilde{B})$ is an equivalence for every $B$ in $\mathcal{B}$ and $U$ in $\mathcal{C}$. We get an equivalence after applying $\varprojlim_{U \in \mathcal{C}} \operatorname{colim}_{B \in \mathcal{B}}$. Since the bounded subsets of the form $\tilde{B}$ for $B$ in $\mathcal{B}$ and the entourages of the form $\tilde{U}$ for $U$ in $\mathcal{C}$ are cofinal in the bounded subsets or entourages, respectively, of $\{0,1\} \otimes X$ we get the desired equivalence $Q_C(X) \to Q_C(\{0,1\} \otimes X)$.

Lemma 4.5. $Q_C$ is excisive.

Proof. Let $\mathcal{Y} := (Y_i)_{i \in I}$ be a big family on $X$ and let $(\mathcal{V}, Z)$ be a complementary pair. Let $W$ be a subset of $X$. If $i$ is sufficiently large, then $(Y_i, Z)$ is a $U$-covering of $X$, i.e., every $U$-bounded subset of $X$ is contained in at least one of $Y_i$ or $Z$. In this case

$$
\begin{array}{c}
P_U(W \cap Z \cap Y_i) \longrightarrow P_U(W \cap Z) \\
\downarrow \\
P_U(W \cap Y_i) \longrightarrow P_U(W)
\end{array}
$$
is a homotopy cocartesian diagram since it is cocartesian and all maps are inclusions of subcomplexes. It follows that

\[
\begin{array}{ccc}
C^n P_U(W) & \to & C^n P_U(W \cap Z) \\
\downarrow & & \downarrow \\
C^n P_U(W \cap Y_i) & \to & C^n P_U(W \cap Z \cap Y_i)
\end{array}
\]

is cartesian, from which we conclude that

\[
\begin{array}{ccc}
\tilde{Q}_C(X, U, B) & \to & \tilde{Q}_C(Z, U, B) \\
\downarrow & & \downarrow \\
\tilde{Q}_C(Y_i, U, B) & \to & \tilde{Q}_C(Z \cap Y_i, U, B)
\end{array}
\]

is cartesian. We apply \(\lim_{i \in I} \lim_{U \in C} \text{colim}_{B \in B} \) and get a square

\[
\begin{array}{ccc}
Q_C(X) & \to & Q_C(Z) \\
\downarrow & & \downarrow \\
Q_C(Y) & \to & Q_C(Z \cap Y)
\end{array}
\]

in \(C\). We can interchange the order of taking the limits, i.e., apply \(\lim_{U \in C} \lim_{i \in I} \text{colim}_{B \in B} \) without changing the result. For every \(U\) in \(C\) let \(I(U)\) be the subset of those \(i\) in \(I\) such that \((Y_i, Z)\) is a \(U\)-covering. By cofinality, we can restrict the limit to \(\lim_{U \in C} \lim_{i \in I(U)} \text{colim}_{B \in B} \). Then the square above is obtained by applying this operation to a diagram of cartesian squares and, by stability of \(C\) in order to deal with the colimit, is itself cartesian. \(\square\)

**Lemma 4.6.** \(Q_C\) vanishes on flasques.

**Proof.** Let \(X\) be a flasque bornological coarse space with flasqueness implemented by the morphism \(f : X \to X\). We write

\[
F_U(B) := \tilde{Q}_C(X, U, B).
\]

Note that by definition

\[
Q_C(X) \simeq \lim_{U \in C} \text{colim}_{B \in B} F_U(B).
\]
For an entourage $U$ of $X$ we define $\tilde{U} := \bigcup_{n \in \mathbb{N}} f^n(U)$. We then have the diagram

$$
\begin{array}{ccc}
\colim_{B \in B, B \cap f^1(X) = \emptyset} F_{\tilde{U}}(B) & \xrightarrow{f^0_*} & \colim_{B \in B} F_{U}(B) \\
\downarrow & & \downarrow \\
\colim_{B \in B, B \cap f^2(X) = \emptyset} F_{\tilde{U}}(B) & \xrightarrow{f^0_* + f^1_*} & \colim_{B \in B} F_{U}(B) \\
\downarrow & & \downarrow \\
\colim_{B \in B, B \cap f^3(X) = \emptyset} F_{\tilde{U}}(B) & \xrightarrow{f^0_* + f^1_* + f^2_*} & \colim_{B \in B} F_{U}(B) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\colim_{B \in B} F_{\tilde{U}}(B) & \xrightarrow{s_U} & \colim_{B \in B} F_{U}(B)
\end{array}
$$

The squares commute since the composition

$$
\colim_{B \in B, B \cap f^n(X) = \emptyset} F_{\tilde{U}}(B) \to \colim_{B \in B, B \cap f^n(X) = \emptyset} F_{\tilde{U}}(B) \xrightarrow{f^{n-1}_*} \colim_{B \in B} F_{U}(B)
$$

has a preferred equivalence to zero. The map $s_U$ is induced. If $U'$ is a second entourage of $X$ such that $U \subseteq U'$, then we have a natural commuting diagram

$$
\begin{array}{ccc}
\colim_{B \in B} F_{U'}(B) & \xrightarrow{s_{U'}} & \colim_{B \in B} F_{U'}(B) \\
\downarrow & & \downarrow \\
\colim_{B \in B} F_{\tilde{U}}(B) & \xrightarrow{s_U} & \colim_{B \in B} F_{U}(B)
\end{array}
$$

More precisely, one can perform the construction above in diagrams indexed by the poset $C$. The construction then yields an interpretation of the family of morphisms $(s_U)_{U \in C}$ as a morphism between diagrams. By applying $\lim_{U \in C}$ we get a morphism

$$
s : Q_C(X) \to Q_C(X).
$$

By construction it satisfies

$$
Q_C(f) \circ s + \text{id}_{Q_C(X)} \simeq s .
$$

Since $Q_C$ is coarsely invariant we conclude that

$$
s + \text{id}_{Q_C(X)} \simeq s
$$

and therefore $Q_C(X) \simeq 0.

In the next lemmas we establish that $Q_C$ is strong and strongly additive. In Remark 4.9 we describe the natural pairing with coarse stable homotopy.

**Lemma 4.7.** $Q_C$ is strong.
Proof. Let \( X \) be weakly flasque. We repeat the argument for Lemma 4.6. Since we already know that \( Q_C \) is a coarse cohomology theory, in order to see that (4.3) implies (4.4) we only need that \( \text{Yo}^s(f) \simeq \text{id}_{\text{Yo}^s(X)} \).

Recall the Definition 2.9 of strong additivity.

**Lemma 4.8.** \( Q_C \) is strongly additive.

**Proof.** Let \((X_i)_{i \in I}\) be a family of bornological coarse spaces and \( U := \bigsqcup_{i \in I} U_i \) (4.5) be an entourage of the free union (see [BE16, Def. 2.25] for the definition of the free union)

\[
X := \bigfreebigcup_{i \in I} X_i.
\]

Then we have an isomorphism of topological spaces

\[
P_U(X) \cong \bigsqcup_{i \in I} P_{U_i}(X_i).
\]

A subset \( B \) of \( X \) is bounded if and only if \( B_i := B \cap X_i \) is bounded for all \( i \) in \( I \) and empty for all but finitely many \( i \) in \( I \). We conclude that

\[
\hat{Q}_C(X, U, B) \simeq \text{Fib}(C^n_{P_U(X)} \to C^n_{P_U(X \setminus B)}) \simeq \bigoplus_{i \in I} \text{Fib}(C^n_{P_{U_i}(X_i)} \to C^n_{P_{U_i}(X_i \setminus B_i)}) \simeq \bigoplus_{i \in I} \hat{Q}_C(X_i, U_i, B_i).
\]

We get the equivalence

\[
\hat{Q}_C(X, U) \simeq \varinjlim_{B \in B} \hat{Q}_C(X, U, B) \simeq \varinjlim_{B \in B} \bigoplus_{i \in I} \hat{Q}_C(X_i, U_i, B_i) \simeq \bigoplus_{i \in I} \hat{Q}_C(X_i, U_i),
\]

where \( B \) is the poset of bounded subsets of \( X \). The subset of entourages of the form (4.5) is cofinal in the coarse structure \( C \) of \( X \). In the definition of \( Q_C(X) \) we can therefore restrict the limit over \( C \) to this set and get the equivalence

\[
Q_C(X) \simeq \varprojlim_{U \in \mathcal{C}} \hat{Q}_C(X, U) \simeq \varprojlim_{(U_i), i \in I \in \prod_{i \in I} C_i} \bigoplus_{i \in I} \hat{Q}_C(X_i, U_i) \simeq \bigoplus_{i \in I} \lim_{U_i \in \mathcal{C}} \hat{Q}_C(X_i, U_i) \simeq \bigoplus_{i \in I} Q_C(X_i).
\]

One quickly checks that this equivalence is indeed induced by the collection of morphisms \( Q_C(X_i) \to Q_C(X) \) for all \( i \) in \( I \) given by excision for the complementary pair \( (X_i, \{X \setminus X_i\}) \) on \( X \).
Remark 4.9. In this remark we describe the natural pairing
\[ p : Q_C \to D_C(Q^{hlg}) , \]
where \( Q^{hlg}(X) \) is the coarse stable homotopy theory of \( X \). We are in the Case 2 of the list described in the beginning of Section 2.3.

We first recall the definition of \( Q^{hlg} \) from [BE16, Def. 6.23]. We start with the functor
\[ \text{BornCoarse}^{C,B} \to \text{Top}^{\Delta^1} , \quad (X,U,B) \mapsto (P_U(X \setminus B) \to P_U(X)) . \]

We apply the localization functor \( \iota : \text{Top} \to \text{Spc} \), the stabilization functor \( \Sigma^\infty_+ : \text{Spc} \to \text{Sp} \), and finally the cofibre functor in order to get the functor
\[ \tilde{Q}^{hlg} : \text{BornCoarse}^{C,B} \to \text{Sp} , \quad \tilde{Q}^{hlg}(X,U,B) \simeq \text{Cofib}(\Sigma^\infty_+ \iota P_U(X \setminus B) \to \Sigma^\infty_+ \iota P_U(X)) . \]

Similarly as in Definition 4.1, the coarse homology theory \( Q^{hlg} \) is obtained as the composition of a right and a left Kan extension

\[
\begin{array}{ccc}
\text{BornCoarse}^{C,B} & \xrightarrow{\tilde{Q}^{hlg}} & \text{Sp} \\
\downarrow & & \downarrow \\
\text{BornCoarse}^C & \xrightarrow{Q^{hlg}} & \text{Sp}
\end{array}
\]

Since \( C \) is stable, the power structure of \( C \) over \( \text{Spc} \) extends to a power structure over \( \text{Sp} \). If we fix the object \( C \) in \( C \), then in analogy with (4.1) we have a functor
\[ C(\cdot) : \text{Sp}^{op} \to C , \quad W \mapsto C^W . \quad (4.6) \]

For a space \( A \) we have the natural equivalence \( C^A \simeq C^{\Sigma^\infty_+ A} \).

We now construct the pairing. We first observe that we have an equivalence of functors
\[ \tilde{Q}_C \simeq C^{Q^{hlg}} : (\text{BornCoarse}^{C,B})^{op} \to C . \]

Indeed, for \( (X,U,B) \) in \( \text{BornCoarse}^{C,B} \) we have the natural equivalences
\[
\begin{align*}
\tilde{Q}_C(X,U,B) & \simeq \text{Fib}(C^n P_U(X) \to C^n P_U(X \setminus B)) \\
& \simeq \text{Fib}(C^{\Sigma^\infty_+ \iota P_U(X) \to \Sigma^\infty_+ \iota P_U(X \setminus B)} \\
& \simeq C^{\text{Fib}(\Sigma^\infty_+ \iota P_U(X \setminus B) \to \Sigma^\infty_+ \iota P_U(X))} .
\end{align*}
\]

We now form the left Kan extension of this equivalence along the functor
\[ (\text{BornCoarse}^{C,B})^{op} \to (\text{BornCoarse}^C)^{op} \]
and get the natural transformation

\[ \hat{Q}_C \simeq LK(C^{\hat{Q}_{hlg}}) \xrightarrow{\sim} C^{RK(\hat{Q}_{hlg})} \simeq C^{\hat{Q}_{hlg}}. \]

Here \( LK \) and \( RK \) stand for the left, resp. right Kan extension. In general, the marked transformation is not an equivalence since the functor (4.6) in general does not preserve colimits. We now form the right Kan extension of this morphism along the functor

\((\text{BornCoarse}^C)^{op} \to \text{BornCoarse}^{op}\)

and get the morphism

\[ p : Q_C \simeq RK(\hat{Q}_C) \to RK(C^{\hat{Q}_{hlg}}) \xrightarrow{\sim} C^{LK(\hat{Q}_{hlg})} \simeq C^{Q_{hlg}} \simeq D_C(\hat{Q}_{hlg}) \]

which is the desired pairing. Note that here the marked morphism is an equivalence, since (4.6) preserves limits.

\[ \diamondsuit \]

4.2 The dualizing spectrum of a group

To a group \( G \) we can associate the \( G \)-spectrum \( S[G] \) in \( GSp \), where \( S \) denotes the sphere spectrum (see Remark 4.14 for a detailed definition of \( S[G] \)). Following Klein [Kle01] we then define the dualizing spectrum of \( G \) by

\[ D_G := \varprojlim_{BG} S[G] \]

in \( Sp \).

In this section we settle a conjecture formulated by Klein at the end of [Kle01]. He asked whether the dualizing spectra of two quasi-isometric groups admitting finite classifying spaces are equivalent. In fact, we prove something stronger (see Corollary 4.11 below). But before we can state our result, we first need a definition.

We consider two groups \( G \) and \( H \).

**Definition 4.10.** We say that \( G \) and \( H \) are coarse motivically equivalent if there exists an equivalence \( Yo^*(G_{\text{can,min}}) \simeq Yo^*(H_{\text{can,min}}) \) in \( SpX \).

**Corollary 4.11** (Corollary to the Proposition 4.13). If \( G \) and \( H \) are finitely generated, torsion-free and coarse motivically equivalent, then there exists an equivalence \( D_G \simeq D_H \) in \( Sp \).

If the groups \( G \) and \( H \) admit finite classifying spaces, then they are finitely generated and torsion-free. Moreover, if \( G \) and \( H \) are quasi-isometric, then they are coarse motivically equivalent. Therefore the above corollary solves Klein’s conjecture.
**Example 4.12.** The following example shows that coarse motivic equivalence is a strictly weaker relation than quasi-isometry.

We consider torsion-free and cocompact lattices $G$ in $SO(2n,1)$ and $H$ in $SU(n,1)$. Such lattices exist by a result of Borel [Bor63]. Then $G$ is quasi-isometric to the hyperbolic space $\mathbb{H}^{2n}$ and $H$ is quasi-isometric to the complex hyperbolic space $\mathbb{H}^n$ (of real dimension $2n$). By Mostow rigidity (Mostow [Mos73], Kleiner–Leeb [KL97, Cor. 1.1.4]) $\mathbb{H}^{2n}$ and $\mathbb{H}^n$ are not quasi-isometric, and hence $G$ and $H$ are not quasi-isometric.

The boundaries of the negatively curved spaces $\mathbb{H}^{2n}$ and $\mathbb{H}^n$ are both homeomorphic to $S^{2n-1}$. Hence, $\mathbb{H}^{2n}$ and $\mathbb{H}^n$ are both coarsely homotopy equivalent to the Euclidean cone over $S^{2n-1}$ (Higson–Roe [HR95, Sec. 8]). It follows that $G$ and $H$ are coarsely homotopy equivalent and therefore coarse motivically equivalent.

In order to prove our result we express the dualizing spectrum $D_G$ of $G$ as the value of a coarse cohomology theory applied to $G_{can,\min}$. In the following we explain this in greater detail. Referring to Section 4.1, we consider the case $C = \text{Sp}$ and $C := S$ (i.e., the sphere spectrum). By Definition 4.1 we get a coarse cohomology theory $Q_S : \text{BornCoarse}^{op} \to \text{Sp}$, which is a coarse version of stable cohomotopy. By $G_{can,\min}$ (see [BEKW17, Ex. 2.4]) we denote the group $G$ equipped with the bornological coarse structure given by the minimal bornology (i.e., consisting of finite subsets) and the canonical coarse structure (induced by the word metric associated to any choice of finite generating set). Our main technical result is now:

**Proposition 4.13.** If $G$ is finitely generated and torsion-free, then we have an equivalence $D_G \simeq Q_S(G_{can,\min})$.

### 4.3 Proof of Proposition 4.13

Let $\iota : \text{Top} \to \text{Spc}$ denote the canonical functor from topological spaces to the $\infty$-category of spaces. For a group $G$ we denote by $BG$ the category consisting of one object whose monoid of endomorphisms is given by the group $G$. Furthermore, by $\text{Orb}(G)$ we denote the orbit category of $G$ which is the category of transitive $G$-sets and equivariant maps. We form the categories

$$G\text{Top} := \text{Fun}(BG, \text{Top})$$

of $G$-topological spaces (i.e., objects are topological spaces with an action of $G$, and morphisms are equivariant continuous maps) and

$$G\text{Spc} := \text{Fun}(BG, \text{Spc}), \quad G[\text{Spc}] := \text{Fun(Orb}(G)^{op}, \text{Spc})$$

of spaces with a $G$-action and $G$-spaces. The category $G[\text{Spc}]$ models the $G$-equivariant homotopy theory and is the natural home for classifying spaces $E_F G$ for families $F$ of
subgroups of \( G \). We have a functor \( \iota_G : G\text{Top} \to G[\text{Spc}] \) which sends the \( G \)-topological space \( X \) to the functor

\[
\text{Orb}(G)^{\text{op}} \ni O \mapsto \iota_G(X)(O) := \text{Map}_G(O, X) \in \text{Spc} ,
\]

(4.7)

where \( \text{Map}_G(Y, X) \) denotes the topological space of \( G \)-equivariant maps from \( Y \) to \( X \) with the compact-open topology, and \( O \) is considered as a discrete \( G \)-topological space.

The category \( G\text{Spc} \) models the homotopy of topological spaces with \( G \)-action and equivariant maps, where weak equivalences are maps which are weak equivalences after forgetting the \( G \)-action. We have an isomorphism of monoids

\[
\text{End}_{\text{Orb}(G)}(G) \cong G^{\text{op}} ,
\]

and therefore an inclusion

\[
BG^{\text{op}} \to \text{Orb}(G) .
\]

(4.8)

This inclusion induces an adjunction

\[
\text{Res} : G[\text{Spc}] \leftrightarrows G\text{Spc} : \text{Coind}
\]

(4.9)

relating the two categories.

Furthermore, we let

\[
G\text{Sp} := \text{Fun}(BG, \text{Sp}) , \quad G[\text{Sp}] := \text{Fun}(\text{Orb}(G)^{\text{op}}, \text{Sp})
\]

be the categories of spectra with a \( G \)-action and of naive \( G \)-spectra. The inclusion \([4.8]\) induces an adjunction

\[
\text{Res} : G[\text{Sp}] \leftrightarrows G\text{Sp} : \text{Coind} .
\]

(4.10)

An \( \Omega \) spectrum is a spectrum \((E_n, \sigma_n)_{n\in\mathbb{N}}\) in topological spaces such that \( \sigma_n : E_n \to \Omega E_{n+1} \) is a weak equivalence. A weak equivalence between \( \Omega \)-spectra is a morphism which is a level-wise weak equivalence. We denote by \( \text{Sp}^\Omega \) the ordinary category of \( \Omega \)-spectra. The relative category \((\text{Sp}^\Omega, W)\), where \( W \) denotes the class of weak equivalences, is a presentation of the category of spectra. In particular, we have a functor

\[
\kappa : \text{Sp}^\Omega \to \text{Sp}^\Omega[W^{-1}] \cong \text{Sp} .
\]

We furthermore consider the category

\[
G\text{Sp}^\Omega := \text{Fun}(BG, \text{Sp}^\Omega)
\]

of \( \Omega \)-spectra with a \( G \)-action and use symbol \( \kappa \) also for the induced functor

\[
\kappa : G\text{Sp}^\Omega \to G\text{Sp} .
\]

We consider the \( G \)-spectrum \( S[G] \) in \( G\text{Sp} \).
Remark 4.14. In greater detail, $S[G]$ is given by $\coprod_{g \in G} S$, where the $G$-action is given by the action of $G$ on the index set by left multiplication. The technical description is

$$S[G] := \text{Ind}_G^G(S),$$

where $\text{Ind}_G^G : \text{Sp} \to G\text{Sp}$ is the left-adjoint of the forgetful functor $G\text{Sp} \to \text{Sp}$. Equivalently, we can choose an $\Omega$-spectrum $QS$ in $\text{Sp}_\Omega$ with $\kappa(QS) \simeq S$. Then we form the $G$-$\Omega$-spectrum $\text{Map}_c(G, QS)$ of compactly supported maps from $G$ to $QS$ (see below for details), where $G$ is considered as a discrete $G$-space with the left action. Then we have an equivalence

$$S[G] \simeq \kappa\text{Map}_c(G, QS).$$

We have a functor

$$\lim_{BG} : G\text{Sp} \to \text{Sp},$$

and, by definition, an equivalence

$$D_G \simeq \lim_{BG} S[G].$$

Let $X, Y$ be $G$-topological spaces and $Z$ be a pointed topological space. For a subset $K$ of $X$ we let $\text{Map}_K(X, Z)$ denote the subspace of $\text{Map}(X, Z)$ of maps which send $X \setminus K$ to the base point. We define the $G$-subset of $\text{Map}(X, Z)$

$$\text{Map}_c(X, Z) := \bigcup_K \text{Map}_K(X, Z)$$

of compactly supported maps, where $K$ runs over all compact subsets of $X$. We equip $\text{Map}_c(X, Z)$ with the inductive limit topology. Note that this topology is in general finer than the induced topology from $\text{Map}(X, Z)$.

Let $X, Y$ be $G$-topological spaces and $Z$ be a pointed topological space.

Lemma 4.15. If $G$ acts properly and cocompactly on $X$ and $Y$, then we have a homeomorphism

$$\text{Map}_G(X, \text{Map}_c(Y, Z)) \cong \text{Map}_G(Y, \text{Map}_c(X, Z)). \quad (4.11)$$

Proof. We define the $G$-space

$$\text{Map}_d(X \times Y, Z) := \text{colim}_{(K,L)} \text{Map}_{G(K \times L)}(X \times Y, Z)$$

equipped with the inductive limit topology, where $K$ (or $L$) runs over the compact subsets of $X$ (resp. of $Y$) and $G$ acts diagonally on $X \times Y$. We compare both sides of (4.11) with $\text{Map}_d(X \times Y, Z)^G$. We carry out the arguments only for the case of $\text{Map}_G(X, \text{Map}_c(Y, Z))$ since the other case is completely analogous.

Assume that $f$ belongs to $\text{Map}_G(X, \text{Map}_c(Y, Z))$. By the exponential law for maps between sets it corresponds to a $G$-equivariant map $\tilde{f} : X \times Y \to Z$ which by $G$-equivariance is
determined by its restriction to $K \times Y$ for any compact subset $K$ of $X$ with $GK = X$. Since we equip $\text{Map}_c(Y, Z)$ with the inductive limit topology, there exists a compact subset $L$ of $Y$ with $f(k, y) = *$ for all $k \in K$ and $y \in Y \setminus L$. In other words, $\tilde{f} \in \text{Map}_{G(K \times L)}(X \times Y, Z)^G$. In this way we define a map

$$\text{Map}_G(X, \text{Map}_c(Y, Z)) \to \text{Map}_d(X \times Y, Z)^G.$$ 

Assume now that $\tilde{f}$ belongs to $\text{Map}_d(X \times Y, Z)^G$. Then there is a pair $(K, L)$ of compact subsets of $X$ and $Y$, respectively, such that $\tilde{f}$ is supported on $G(K \times L)$. Since $G$ acts properly on $X$, the set $F := \{g \in G \mid gK \cap K \neq \emptyset\}$ is finite. Then $L := FL$ is compact and $\tilde{f}(x, y) = *$ for $x$ in $K$ and $y \in Y \setminus L$. Let $f : X \to \text{Map}(Y, Z)$ be the adjoint of $\tilde{f}$. Then $f|_K$ takes values in $\text{Map}_L(Y, Z)$. This shows that $f \in \text{Map}_G(X, \text{Map}_c(Y, Z))$. In this way we have constructed the inverse map $\text{Map}_d(X \times Y, Z)^G \to \text{Map}_G(X, \text{Map}_c(Y, Z))$.

If $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ is a $G$-$\Omega$-spectrum, then for a $G$-topological space $X$ we get a $G$-$\Omega$-spectrum

$$\text{Map}(X, E) := (\text{Map}(X, E_n), \sigma_n^X)_{n \in \mathbb{N}},$$

where

$$\sigma_n^X : \text{Map}(X, E_n) \xrightarrow{\sigma_n} \text{Map}(X, \Omega E_{n+1}) \cong \Omega \text{Map}(X, E_{n+1}).$$

If $X$ is a $CW$-complex and $E$ is an $\Omega$-spectrum, then we have the equivalence in $\text{Sp}$

$$\kappa \text{Map}(X, E) \cong (\kappa E)^{1_X}. \quad (4.12)$$

In order to see this note that both sides are cohomology theories and coincide for $X = *$.

Furthermore, if $X$ is a free $G$-$CW$-complex and $E$ is a $G$-$\Omega$-spectrum, then we have the equivalence in $\text{Sp}$

$$\kappa \text{Map}_G(X, E) := \kappa \lim_{BG} \text{Map}(X, E) \cong \lim_{BG} [(\kappa E)^{\text{Res} (\iota_G X)}], \quad (4.13)$$

where $\text{Res}$ is as in (4.9) and $\iota_G$ as in (4.7). Again, both sides are cohomology theories on free $G$-spaces and coincide on $X = G$.

Similarly, we define the $G$-$\Omega$-spectrum $\text{Map}_c(X, E)$ by $(\text{Map}_c(X, E_n), \sigma_{c,n}^X)_{n \in \mathbb{N}}$, where

$$\sigma_{c,n}^X : \text{Map}_c(X, E_n) \xrightarrow{\sigma_n} \text{Map}_c(X, \Omega E_{n+1}) \cong \Omega \text{Map}_c(X, E_{n+1}).$$

For the last isomorphism we used that the circle $S^1$ is compact.

We now choose an $\Omega$-spectrum $QS$ representing the sphere spectrum. We consider $G$ as a discrete $G$-space and note that $G$ acts properly and cocompactly on $G$. Then we have (see Remark 4.14)

$$S[G] \cong \kappa \text{Map}_c(G, QS).$$

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The Rips complex of a $G$-coarse space $X$ with coarse structure $\mathcal{C}$ is defined by

$$\text{Rips}(X) := \colim_{U \in \mathcal{C}} P_U(X),$$

where the colimit is interpreted in the category $G\text{-Top}$ of $G$-topological spaces. Then by [BEKW17, Lem. 11.4] we have an equivalence

$$\iota_G \text{Rips}(G_{\text{can}}) \simeq E_{\text{Fin}} G,$$

in $G[\text{Spc}]$. Since we assume that $G$ is torsion-free we have an equivalence $E_{\text{Fin}} G \simeq EG$.

Since $G$ is finitely generated the coarse structure $\mathcal{C}$ of $G_{\text{can}}$ is generated by a single invariant entourage $U_{\text{gen}}$. Hence we have an equivariant homeomorphism

$$\text{Rips}(G_{\text{can}}) \simeq \colim_{n \in \mathbb{N}} P_{U_{\text{gen}}}(G).$$

We further observe that $P_{U_{\text{gen}}}(G)$ is a locally finite $G$-CW-complex, and that the morphisms $P_{U_{\text{gen}}}(G) \to P_{U_{\text{gen}}^{n+1}}(G)$ are inclusions of subcomplexes. It follows that the colimit over these inclusions is a homotopy colimit, i.e., that we have the equivalence

$$EG \simeq \iota_G \text{Rips}(G_{\text{can}}) \simeq \colim_{n \in \mathbb{N}} \iota_G P_{U_{\text{gen}}}(G)$$

in $G[\text{Spc}]$. Let

$$(\cdot)^G : G[\text{Sp}] \to \text{Sp}$$

be the evaluation functor at the one-point $G$-set. For a $G$-$\Omega$-spectrum $E$ we have the following chain of equivalences in $\text{Sp}$:

$$\lim_{BG} \kappa E \simeq (\text{Coind} \ \kappa E)^G \simeq ((\text{Coind} \ \kappa E)^{EG})^G \simeq ((\text{Coind} \ \kappa E)_{\colim_{n \in \mathbb{N}} \iota_G P_{U_{\text{gen}}}(G)})^G \simeq \lim_{n \in \mathbb{N}} ((\text{Coind} \ \kappa E)^G P_{U_{\text{gen}}}(G))^G \simeq \lim_{n \in \mathbb{N}} \lim_{BG} (\kappa E)^{\iota_G P_{U_{\text{gen}}}(G)})^G \overset{(4.13)}{=} \lim_{n \in \mathbb{N}} \kappa \text{Map}_G(P_{U_{\text{gen}}}(G), E),$$

where for the last equivalence we use the fact that $P_{U_{\text{gen}}}(G)$ is a free $G$-CW-complex.

We obtain the equivalence

$$D_G \simeq \lim_{n \in \mathbb{N}} \kappa \text{Map}_G(P_{U_{\text{gen}}}(G), \text{Map}_c(G, QS)).$$

Since $G$ acts properly and cocompactly on both $P_{U_{\text{gen}}}(G)$ and $G$, we get by Lemma 4.15

$$D_G \simeq \lim_{n \in \mathbb{N}} \kappa \text{Map}_G(G, \text{Map}_c(P_{U_{\text{gen}}}(G), QS)) \simeq \lim_{n \in \mathbb{N}} \kappa \text{Map}_c(P_{U_{\text{gen}}}(G), QS).$$
where the second isomorphism is induced by the evaluation at the identity of $G$. We now observe that the subsets of the form $\overline{P_{\text{ngen}}(G)} \setminus \overline{P_{\text{ngen}}(G \setminus B)}$ for all bounded subsets $B$ of $G_{\text{can,min}}$ (i.e., finite subsets of $G$) are cofinal in the set of compact subsets of $P_{\text{ngen}}(G)$. It follows that

$$\text{Map}_{c}(P_{\text{ngen}}(G), QS) \cong \colim_{B \in \mathcal{B}} \text{Fib}(\text{Map}(P_{\text{ngen}}(G), QS) \to \text{Map}(P_{\text{ngen}}(G \setminus B), QS)),$$

where $\mathcal{B}$ denotes the bornology of $G_{\text{can,min}}$. Hence we get

$$D_{G} \cong \lim_{n \in \mathbb{N}} \kappa \colim_{B \in \mathcal{B}} \text{Fib}(\kappa \text{Map}(P_{\text{ngen}}(G), QS) \to \kappa \text{Map}(P_{\text{ngen}}(G \setminus B), QS)) .$$

Since a filtered colimit of $\Omega$-spectra is again an $\Omega$-spectrum and filtered colimits preserve equivalences, we can switch the order of $\kappa$ and taking the colimit. Furthermore, because $P_{\text{ngen}}(G \setminus B) \to P_{\text{ngen}}(G)$ is an inclusion of a locally finite subcomplex, the induced map between $\Omega$-spectra is a fibration between $\Omega$-spectra. Therefore $\text{Fib}$ in the formula above can be commuted with $\kappa$. Therefore

$$D_{G} \cong \lim_{n \in \mathbb{N}} \kappa \colim_{B \in \mathcal{B}} \text{Fib}(\kappa \text{Map}(P_{\text{ngen}}(G), QS) \to \kappa \text{Map}(P_{\text{ngen}}(G \setminus B), QS)) .$$

We now use the relation (4.12) in order to get the equivalence

$$D_{G} \cong \lim_{n \in \mathbb{N}} \kappa \colim_{B \in \mathcal{B}} \text{Fib}(\kappa QS \iota_{P_{\text{ngen}}(G)} \to \kappa QS \iota_{P_{\text{ngen}}(G \setminus B)}).$$

Finally, by cofinality we replace the limit over $\mathbb{N}$ by the limit over $\mathcal{C}$. In view of (4.2) we get the desired equivalence

$$D_{G} \cong QS(G_{\text{can,min}}) ,$$

which finishes the proof of Proposition 4.13.

**Remark 4.16.** Let $\text{Ord}_{F}(G)$ denote the full subcategory of the orbit category of $G$ of transitive $G$-sets with stabilizers in the family of subgroups $F$. We set

$$G_{F}[Sp] := \text{Fun}(\text{Ord}_{F}(G)^{op}, Sp) .$$

Then for every two families $F$ and $F'$ with $F \subseteq F'$ we have a corresponding pair of adjoint functors $(\text{Ind}_{F}^{F'}, \text{Res}_{F}^{F'})$, see [BEKW17, Sec. 10.3]. If $E \in G_{\text{All}}[Sp]$, then we define

$$E^{(h_{F}G)} := \lim_{\text{Ord}_{F}(G)} \text{Res}_{F}^{All} E .$$

If $H$ is a subgroup of $G$, then we have an induction functor

$$\text{Ind}_{H,F}^{G} : H_{F \cap H}[Sp] \to G_{F}[Sp] .$$

We could consider $S[G]$ as an object $\text{Ind}_{\{1\}, \text{All}}^{G}(S)$ of $G_{\text{All}}[Sp]$. Then by construction

$$D_{G} \cong S[G]^{(h_{\{1\}}G)} .$$
An appropriate modification (with $BG \simeq \text{Orb}_{\{1\}}(G)$ replaced by $\text{Orb}_{\text{Fin}}(G)$) of the proof of Proposition 4.13 actually shows:

**Proposition 4.17.** For every finitely generated group $G$ we have an equivalence

$$S[G]^{(h_{\text{Fin}}G)} \simeq Q_S(G_{\text{can,min}}).$$

If $G$ is torsion-free, then we have the equality of families $\text{Fin} = \{1\}$ and Proposition 4.13 follows from Proposition 4.17.

5 Coarse cohomological $K$-theory functors

5.1 Dualizing coarse $K$-homology

In this section we use the theory from Section 2.2 to dualize coarse $K$-homology.

Coarse $K$-homology $K\chi^{hlg}$ is an important example of a coarse homology theory with many applications in index theory, group theory and topology. Classically, for a proper metric space $X$, one defines the coarse $K$-homology groups $K\chi^{hlg}(X)$ as the $K$-theory groups of the Roe algebra associated to $X$. In [BE16, Sec. 7] we gave a construction of a spectrum-valued version

$$K\chi^{hlg} : \text{BornCoarse} \to \text{Sp}$$

of coarse $K$-homology. According to [BE16, Def. 7.52] the functor $K\chi^{hlg}$ is defined as the composition

$$\text{BornCoarse} \xrightarrow{C^*} \text{C}^*-\text{Cat} \xrightarrow{A_f} \text{C}^*-\text{Alg} \xrightarrow{K} \text{Sp},$$

(5.1)

where $C^*$ sends a bornological coarse space to its Roe category, $A_f$ sends a $C^*$-category to the $C^*$-algebra freely generated by the morphisms of the category, and $K$ is a $K$-theory functor for (non-unital) $C^*$-algebras.

Since $KU$ is a commutative algebra object in $\text{Sp}$ we can form the presentable stable $\infty$-category $\text{Mod}(KU)$ of $KU$-modules. In order to define a coarse $K$-cohomology theory by dualization it is useful to refine $K\chi^{hlg}$ to a $\text{Mod}(KU)$-valued coarse homology theory.

**Remark 5.1.** Observe that $\text{map}_{\text{Sp}}(KU, KU)$ is a complicated spectrum, while

$$\text{map}_{\text{Mod}(KU)}(KU, C) \simeq F(C)$$

for any $KU$-module $C$, where $F : \text{Mod}(KU) \to \text{Sp}$ forgets the $KU$-module structure.

One could even further refine the construction below and consider $\text{map}_{\text{Mod}(KU)}(-,-)$ as a bifunctor having values in $\text{Mod}(KU)$, see Remark 2.11. In the present paper we will not do this to keep things simple.

So dualizing in $KU$-modules with target $KU$ will have the effect that the value of the dual cohomology theory on a point is $KU$ and not something big.

\[\uparrow\]

1Note that in contrast to [BE16] we added the superscript $-^{hlg}$ to the notation in order to indicate that this is the coarse $K$-homology theory.
In order to get a $\text{Mod}(KU)$-valued refinement of $K\mathcal{A}$ we observe that the $K$-theory functor for $C^*$-algebras has a factorization

$$K : C^* \text{-Alg} \xrightarrow{K} \text{Mod}(KU) \xrightarrow{F} \text{Sp}.$$  

In the verification that $K\mathcal{A}$ is a coarse homology theory ([BE16 Thm. 7.53]) we have only used the following properties of the $K$-theory functor:

1. $K$ sends exact sequences of $C^*$-algebras to fibre sequences.
2. $K$ preserves filtered colimits.
3. $K$ is homotopy invariant.

Since the functor $K$ has the same properties we have a strong $\text{Mod}(KU)$-valued coarse homology theory

$$K\mathcal{A} : \text{BornCoarse} \to \text{Mod}(KU)$$  

defined as the composition (compare with (5.1))

$$\text{BornCoarse} \xrightarrow{C^*} C^* \text{-Cat} \xrightarrow{A_f} C^* \text{-Alg} \xrightarrow{K} \text{Mod}(KU).$$

Let $C$ in $\text{Mod}(KU)$ by any $KU$-module. Then by Theorem 2.13 and Lemma 2.14 we get a strong coarse cohomology theory

$$D_C(K\mathcal{A}) : \text{BornCoarse}^{op} \to \text{Sp}.$$  

By Example 2.17 we furthermore have a natural $C$-valued pairing between $D_C(K\mathcal{A})$ and $K\mathcal{A}$. 

**Example 5.2.** A natural choice for $C$ is $KU = K(\mathbb{C})$ itself. If we identify $\pi_0(KU)$ with $\mathbb{Z}$, then (2.1) specializes to a pairing on the level of groups

$$P_X : D_{KU}(K\mathcal{A})(X) \otimes K\mathcal{A}_i(X) \to \mathbb{Z}.$$  

**Remark 5.3.** Let $A$ be a $C^*$-algebra and set $C := K(A)$. Then it is a natural question if the coarse cohomology theory $D_{K(A)}(K\mathcal{A})$ has a geometric or analytic description. In Section 5.2 we discuss a functor $K\mathcal{A} : \text{BornCoarse}^{op} \to \text{Sp}$ which could serve as a first approximation to the a solution of the problem, but does not solve it (we have a comparison map between the two functors for certain spaces $X$, see (5.2) further below).

**5.2 The functor $K\mathcal{A}$**

In this section, for a $C^*$-algebra $A$, we construct a functor

$$K\mathcal{A} : \text{BornCoarse}^{op} \to \text{Sp}.$$  

Our construction is an adaption of Emerson’s and Meyer’s construction [EM06] to our context. This functor is coarsely invariant, vanishes on flasques and is $u$-continuous. But
we were only able to prove that it satisfies excision under certain restrictive conditions on the space and the complementary pair (see Lemma 5.23). We do not know whether it satisfies excision in general, i.e., whether $KAX$ is a coarse cohomology theory. Our interest in this functor stems from the observation that for bornological coarse spaces $X$ of strongly bounded geometry (see Definition 5.26) we have a natural morphism

$$KAX(X) 	o \Sigma D_{K(A)} KX^{hbg}(X)$$

(5.2)
defined in (5.13).

Remark 5.4. The coarse spaces considered in [EM06] are in addition topological spaces. In order to compare our setup with their setup one should consider our bornological coarse spaces as discrete topological spaces. On the other hand, the bornologies considered in [EM06] are always the minimal bornologies compatible with the coarse structure, whereas in the present paper we consider more general bornological structures and only require compatibility with the coarse structure. The straightforward adaption of the definition of Emerson–Meyer to our context yields a functor $KAX^{em}$ which we describe in detail in Remark 5.11. There we further explain and motivate (by $u$-continuity) the difference between the definitions of $KAX$ and $KAX^{em}$. On bornological coarse spaces whose coarse structure is generated by a single entourage both definitions are equivalent (Lemma 5.13).

Emerson–Meyer discuss the coarse invariance and the vanishing on certain flasque spaces (of the form $[0, \infty) \otimes X$) for their functor $KAX^{em}$. Willett [Wil13] shows that the functor $KAX^{em}$ satisfies excision for coarse decompositions of proper metric spaces. Our proof of coarse excision (Lemma 5.23) essentially uses Lemma 5.19 whose proof is modeled on the argument of Willett. ♦

We now define the functors $KAX$ and $KAX^{em}$ and compare them in Lemma 5.13.

Let $(V, \| - \|)$ be a normed vector space and $X$ be a bornological coarse space. Let $U$ be an entourage of $X$ and let $Y$ be a subset of $X$. Let $f : X \to V$ be a function.

Definition 5.5. We define the $U$-variation of $f$ on $Y$ by

$$\text{Var}_U(f, Y) := \sup \{ \| f(x) - f(y) \| \mid (x, y) \in U \cap (Y \times Y) \} .$$

It is a consequence of the triangle inequality that

$$\text{Var}_{U^k}(f, X \setminus U^k[Y]) \leq k \cdot \text{Var}_U(f, X \setminus Y)$$

(5.3)

for every $k$ in $\mathbb{N}$.

We let $C_b(X, V)$ be the space of all bounded functions from $X$ to $V$.

Definition 5.6. We define the subspace $C_U(X, V)$ of $C_b(X, V)$ of functions with vanishing $U$-variation at $\infty$ by

$$C_U(X, V) := \{ f \in C_b(X, V) \mid \lim_{B \in B} \text{Var}_U(f, X \setminus B) = 0 \} .$$

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If $V$ is a Banach space, then so is $C_b(X, V)$ with the norm $\|f\| := \sup_{x \in X} \|f(x)\|$. The proof of the following lemma is elementary:

**Lemma 5.7.** If $V$ is a Banach space, then $C_U(X, V)$ is a closed subspace of $C_b(X, V)$.

If $V$ is a $C^*$-algebra, the $C_U(X, V)$ is a $C^*$-algebra with respect to the pointwise multiplication and the norm $\|\cdot\|$ defined above.

We let $C_0(X, V)$ denote the closure in $C_b(X, V)$ of the subspace of functions which are supported on some bounded subset of $X$. If $V$ is a $C^*$-algebra, then $C_0(X, V)$ is an ideal in $C_U(X, V)$.

We fix a $C^*$-algebra $A$. By $M^s(A) := M(A \otimes K)$ we denote the stable multiplier algebra of $A$. By definition, it contains $A \otimes K$ as an essential ideal.

**Definition 5.8.** We define the $C^*$-algebra $C_U(X; A) := \{ f \in C_U(X, M^s(A)) \mid (\forall (x, y) \in U \mid f(x) - f(y) \in A \otimes K) \}$. We observe that $C_0(X, A \otimes K)$ is a closed ideal in $C_U(X; A)$.

**Definition 5.9.** We define the $C^*$-algebra $\bar{C}_U(X; A) := C_U(X; A)/C_0(X, A \otimes K)$.

If $f : X' \to X$ is a morphism of bornological coarse spaces and $U'$ is an entourage of $X'$ such that $(f \times f)(U') \subseteq U$, then the pull-back along $f$ functorially induces a morphism of exact sequences

$$
0 \longrightarrow C_0(X', A \otimes K) \longrightarrow C_U(X; A) \longrightarrow \bar{C}_U(X; A) \longrightarrow 0
$$

$$
0 \longrightarrow C_0(X', A \otimes K) \longrightarrow C_U'(X'; A) \longrightarrow \bar{C}_U'(X'; A) \longrightarrow 0
$$

In particular, we have defined a functor

$$
\bar{C} : (\text{BornCoarse}^C)^{op} \to C^*-\text{Alg} , \quad (X, U) \mapsto \bar{C}_U(X; A).
$$

Recall the functor

$$
p : \text{BornCoarse}^C \to \text{BornCoarse} , \quad (X, U) \mapsto X
$$

from (3.4). We let

$$
K : C^*-\text{Alg} \to \text{Sp}
$$

be the topological $K$-theory functor for $C^*$-algebras (see [BE16, Sec. 7.4] for a discussion of its desired properties and of possible constructions).
Definition 5.10. We define the functor $KA_X$ as the right Kan extension of $K \circ \bar{C}$ along $p$:

\[ \begin{array}{ccc}
\text{(BornCoarse}^C \text{)}^{\text{op}} & \xrightarrow{\bar{C}} & \text{C}^\ast\text{-Alg} \xrightarrow{K} \text{Sp} \\
p \downarrow & & \downarrow \text{KA}_X \\
\text{BornCoarse}^{\text{op}} & & \\
\end{array} \]

Remark 5.11. The pointwise formula for the right Kan extension yields the formula

\[ KA_X(X) \simeq \lim_{U \in \mathcal{C}} K(\bar{C}_U(X; A)) . \tag{5.4} \]

for the value of the functor $KA_X$ on the bornological coarse space $X$, where $\mathcal{C}$ denotes the coarse structure of $X$.

Remark 5.12. In this remark we introduce the functor $KA_X^{\text{em}}$ following [EM06, Def. 5.4] and discuss the difference to $KA_X$. In the notation of Emerson–Meyer the homotopy groups $KA_X^{\text{em}}(X)$ of the spectrum $KA_X^{\text{em}}(X)$ (to be defined below) are given by $K_\ast(\mathcal{C}_\text{red}(X, A))$ with the $C^\ast$-algebra

\[ \mathcal{C}_\text{red}(X, A) := \bigcap_{U \in \mathcal{C}} \bar{C}_U(X; A) . \tag{5.5} \]

So the main difference between our definition and the one of Emerson–Meyer is the order of applying the limit over all $U$ in $\mathcal{C}$ and the $K$-theory functor. Our choice to take the limit after application of the $K$-theory functor is dictated by $u$-continuity.

The spectrum-valued version $KA_X^{\text{em}}$ can be defined by

\[ \begin{array}{ccc}
\text{(BornCoarse}^C \text{)}^{\text{op}} & \xrightarrow{C} & \text{C}^\ast\text{-Alg} \xrightarrow{K} \text{Sp} \\
p \downarrow & & \downarrow \text{KA}_X^{\text{em}} \\
\text{BornCoarse}^{\text{op}} & & \\
\end{array} \]

where the functor $\mathcal{C}_\text{red}$ is defined by right Kan extension. By the pointwise formula for the right Kan extension and by interpreting the intersection in (5.5) as a limit we get the equivalence

\[ KA_X^{\text{em}}(X) \simeq K\left( \lim_{U \in \mathcal{C}} \bar{C}_U(X; A) \right) . \tag{5.4} \]

The natural transformation

\[ K \circ \lim \to \lim \circ K \]

induces a natural transformation

\[ KA_X^{\text{em}} \to KA_X . \]

Since $K$ does not commute with limits we do not expect that this transformation is an equivalence for general $X$. But it is in a special case:
Lemma 5.13. If the coarse structure of $X$ is generated by a single entourage, then the natural morphism
\[ KAX_{em}(X) \to KAX(X) \]
is an equivalence.

Proof. The Inequality (5.3) and the compatibility of the coarse and bornological structures imply that
\[ C_U(X; A) = C_{U^k}(X; A) \]
for all $k$ in $\mathbb{N}$.
Therefore, if the coarse structure of $X$ is generated by a single entourage $U$, then we have the equality
\[ c^{\text{er}}(X, A) = \bar{C}_U(X; A) \, . \]
In this case the limit in (5.4) can be restricted to the cofinal subset of powers of the generating entourage $U$. But then it is a limit over a constant system with value $K(\bar{C}_U(X; A))$. Hence we have the equivalences
\[ KAX(X) \simeq K(\bar{C}_U(X; A)) \simeq K(c^{\text{er}}(X, A)) \simeq KAX_{em}(X) \, . \]

We now prove that $KAX$ is $u$-continuous, coarsely invariant and vanishes on flasques. At the end we will also discuss additivity in the special case $A = \mathbb{C}$.

Lemma 5.14. $KAX$ is $u$-continuous.

Proof. Let $X$ be a bornological coarse space with coarse structure $\mathcal{C}$. Using (5.4) twice we get the equivalences
\[ KAX(X) \simeq \lim_{U \in \mathcal{C}} K(\bar{C}_U(X; A)) \simeq \lim_{U \in \mathcal{C}} \lim_{V \in \mathcal{C}(U)} K(\bar{C}_V(X; A)) \simeq \lim_{U \in \mathcal{C}} KAX(X_U) \, . \]

Lemma 5.15. $KAX$ is coarsely invariant.

Proof. Let $f$ and $g$ be two morphisms of bornological coarse spaces from $X'$ to $X$ which are close to each other. Then there exists an entourage $U$ of $X$ such that $(f, g)(\text{diag}(X')) \subseteq U$.

Let $U'$ be an entourage of $X'$ and assume additionally $(f \times f)(U') \subseteq U$ and $(g \times g)(U') \subseteq U$.

Let $\phi$ be an element of $C_U(X; A)$. We consider the difference $\delta := f^*\phi - g^*\phi$ in $C_{U'}(X'; A)$.

We first observe that $\delta$ has values in $A \otimes K$. Indeed, if $x'$ is in $X'$, then $(f(x'), g(x')) \in U$. Hence $\phi(f(x')) - \phi(g(x')) \in A \otimes K$.

Furthermore, given $\epsilon > 0$ we can find a bounded subset $B$ of $X$ such that $\text{Var}_U(\phi, X \setminus B) \leq \epsilon$. Then we have $\|\delta(x')\| \leq \epsilon$ for all $x' \in X' \setminus f^{-1}(B)$. Consequently, we can approximate $\delta$ by functions supported on bounded subsets of $X'$, i.e., $\delta \in C_0(X', A \otimes K)$. We conclude that the image of $\delta$ in $\bar{C}_{U'}(X'; A)$ vanishes.

Thus for every entourage $U'$ of $X'$ we can choose the entourage $U$ of $X$ sufficiently large such that $(f \times f)(U') \subseteq U$, $(g \times g)(U') \subseteq U$ and $(f, g)(\text{diag}(X')) \subseteq U$, and in this case we
have $\bar{f}^* = \bar{g}^*$. This implies the equivalence $K\mathcal{X}(f) \simeq K\mathcal{X}(g)$ as morphisms of spectra from $K\mathcal{X}(X)$ to $K\mathcal{X}(X')$. \hfill $\square$

**Lemma 5.16.** $K\mathcal{X}$ vanishes on flasques.

*Proof.* Let $X$ be a flasque bornological coarse space and $f : X \to X$ a morphism implementing flasqueness. We choose an identification of Hilbert spaces $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \cong \ell^2(\mathbb{N})$. It induces an isomorphism $K \otimes K \cong K$. For every $i$ in $\mathbb{N}$ we let $p_i$ be the projection onto the subspace generated by $\delta_i$ in $\ell^2(\mathbb{N})$. We then have a homomorphism $\iota_i : K \to K \otimes K \cong K$ given by $A \mapsto p_i \otimes A$. It extends to a homomorphism $s_i : M^*(A) \to M^*(A)$. Since the images of these homomorphisms are mutually orthogonal they can be summed up to a homomorphism

$$s : \bigoplus_{n \in \mathbb{N}} M^*(A) \to M^*(A).$$

We use this homomorphism in order to define the homomorphism

$$S : C_b(X, M^*(A)) \to C_b(X, M^*(A)) , \quad S(\phi) := s(\oplus_{n \in \mathbb{N}} f_n^* \phi).$$

We note that $S$ is continuous.

If $\phi$ in $C_b(X, M^*(A))$ is supported on a bounded subset $B$ of $X$, then $f_n^* \phi = 0$ for all $n$ with $f^n(X) \cap B = \emptyset$. There exists an $n_0$ in $\mathbb{N}$ such that the latter condition is satisfied for all $n$ in $\mathbb{N}$ with $n \geq n_0 + 1$. Then $S(\phi)$ is supported on the bounded subset $\bigcup_{n=0}^{n_0} f^{n-1}(B)$ of $X$. By continuity, $S$ restricts to a map $C_b(X, A) \to C_b(X, A)$.

Let $U$ be a coarse entourage of $X$. Then there exists a coarse entourage $V$ of $X$ such that $\bigcup_{n \in \mathbb{N}} f^n(U) \subseteq V$. Assume that $x, y$ are points in $X$ such that $(x, y) \in U$. If $\phi \in C_V(X; A)$, then for every $\epsilon$ in $(0, \infty)$ there exists a bounded subset $B$ such that $\|\phi(x) - \phi(y]\| \leq \epsilon$ for all $(x, y) \in V|_{\mathbb{N}} \cap B$. Let $n_0$ be in $\mathbb{N}$ such that $f^{n_0}(X) \cap B = \emptyset$. Then $\|\phi(f^n(x)) - \phi(f^n(y))\| \leq \epsilon$ for all $n$ in $\mathbb{N}$ with $n \geq n_0 + 1$. For all $n \in \mathbb{N}$ we have $\phi(f^n(x)) - \phi(f^n(y)) \in A \otimes K$. Because $A \otimes K$ is closed in $M^*(A)$ we conclude that $S(\phi)(x) - S(\phi)(y) \in A \otimes K$. We further consider the bounded subset $B' := \bigcup_{n=0}^{n_0} f^{n-1}(B)$. If $(x, y) \in U|_{\mathbb{N}} \cap B'$, then $\|\phi(f^n(x)) - \phi(f^n(y))\| \leq \epsilon$ for all $n$ in $\mathbb{N}$, i.e., we have $\|S(\phi)(x) - S(\phi)(y)\| \leq \epsilon$. We conclude that $S$ restricts to a homomorphism $S : C_V(X; A) \to C_V(X; A)$.

Hence we get an endomorphism $\bar{S}$ of pro-systems $(\bar{C}_U(X; A))_{U \in \mathcal{U}}$ which induces an endomorphism of spectra $\sigma$ of $K\mathcal{X}(X)$. By construction we have $f^* S \oplus \text{id} \cong S$. Therefore this isomorphism induces an equivalence

$$K\mathcal{X}(f) \circ \sigma + \text{id}_{K\mathcal{X}(X)} \simeq \sigma . \tag{5.6}$$

Since $f$ is close to the identity and $K\mathcal{X}$ is coarsely invariant we conclude that

$$\sigma + \text{id}_{K\mathcal{X}(X)} \simeq \sigma \tag{5.7}$$

and hence $K\mathcal{X}(X) \simeq 0$. \hfill $\square$

Recall Definition 2.9 of additivity.
Lemma 5.17. $K\mathcal{C}X$ is additive.

Proof. We must show that for every set $I$ we have the equivalence

$$K\mathcal{C}X\left(\bigsqcup_{I} \star\right) \simeq \bigoplus_{I} K\mathcal{C}X(\star).$$

We have the chain of equivalences

$$K\mathcal{C}X\left(\bigsqcup_{I} \star\right) \simeq K\left(\prod_{I} \mathbb{B}/\bigoplus_{I} \mathbb{K}\right) \simeq \Sigma K\left(\bigoplus_{I} \mathbb{K}\right) \simeq \bigoplus_{I} \Sigma K(\mathbb{K}) \simeq \bigoplus_{I} K\mathcal{C}X(\star),$$

because $K(\prod_{I} \mathbb{B}) \simeq 0$ due to an Eilenberg swindle: we can perform the Eilenberg swindle showing $K(\mathbb{B}) \simeq 0$ (see e.g., Higson–Roe [HR00, Ex. 4.6.3]) simultaneously in each factor of $\prod_{I} \mathbb{B}$. \qed

5.3 Excision

In this section we show a restricted version of excision for the functor $K\mathcal{A}X$.

Let $Y$ be a subset of a bornological coarse space $X$ and let $U$ be an entourage of $X$ which contains the diagonal of $X$. Let $U_{Y} := (Y \times Y) \cap U$ be the induced entourage on $Y$. Recall that $Y_{X_{U}}$ is the set $Y$ with the bornological coarse structure induced from the bornological coarse space $X_{U}$, and that $Y_{U_{Y}}$ is the set $Y$ with the coarse structure generated by the entourage $U_{Y}$ and bornology induced from $X$. In general the identity of the underlying sets is a morphism $Y_{U_{Y}} \to Y_{X_{U}}$ of bornological coarse spaces.

Definition 5.18. We say that $Y$ is $U$-convex if the morphism $Y_{U_{Y}} \to Y_{X_{U}}$ is an isomorphism.

The following lemma is the main technical lemma of this section. Its proof follows the arguments of Willett [Wil13, Lem. 3.4].

Lemma 5.19. Assume the following:

1. $Y$ is $U$-convex.
2. $X_{U}$ has at most finitely many coarse components intersecting $Y$ non-trivially.

Then the restriction

$$C_{U}(X; A) \to C_{U_{Y}}(Y; A)$$

is surjective.
Proof. We first assume that $X_U$ has a single coarse component, that $Y$ is not bounded, and that the inclusion $Y \to X$ is not a coarse equivalence. Because $Y$ is $U$-convex we conclude that $Y_U$ also has a single coarse component. Let $\hat{f}$ be in $C_{U_Y}(Y; A)$. We choose a base-point $y_0$ in $Y$ and set $f := \hat{f} - \hat{f}(y_0)$. Then $f : Y \to A \otimes K$ and the $U_Y$-variation of $f$ vanishes at $\infty$ of $Y$. It suffices to find for every given $\epsilon$ in $(0, \infty)$ a function $\tilde{f} : X \to A \otimes K$ with $U$-variation vanishing at $\infty$ of $X$ such that $\|\tilde{f}_Y - f\| \leq \epsilon$.

In the following we fix some notation and terminology.

The coarse structure on $X_U$ is generated by a metric $d$ given by

$$d(x, y) := \min\{n \in \mathbb{N} \mid (x, y) \in U^n\},$$

where we set $U^0 := \text{diag}(X)$. We can define a similar metric $d_Y$ on $Y$ using the entourage $U_Y$. Because $U^n_Y \subseteq (U^n)_Y$ for all $n$, it is clear that $d_Y \leq d_Y$. Because $Y$ is $U$-convex, there exist a function $\kappa : \mathbb{N} \to \mathbb{N}$ such that $(U^k)_Y \subseteq U^{\kappa(k)}_Y$ for all $k$. This implies that $d_Y \leq \kappa \circ d_Y$. Consequently, $d_Y$ and $d_Y$ induce the same coarse structure on $Y$, namely the one generated by $U_Y$.

For $x$ in $X$ and $R$ in $[0, \infty)$ we let $B(x, R) := \{y \in X \mid d(y, x) \leq R\}$ denote the ball of radius $R$ with center $x$. Let $\phi : X \to V$ be a function with values in some Banach space $V$.

For $R$ in $(0, \infty)$ we define the variation function

$$(\nabla_R \phi)(x) := \sup\{\|\phi(x) - \phi(y)\| \mid y \in B(x, R)\}.$$

Note that for $R'$ in $(0, \infty)$ with $R \leq R'$ we have

$$\nabla_R \phi \leq \nabla_R \phi,$$

and that the triangle inequality implies

$$(\nabla_{2R} \phi)(x) \leq 2 \sup_{y \in B(x, R)} (\nabla_R \phi)(y). \quad (5.8)$$

Let $p : X \to M$ be some function to a metric space $(M, m)$ and $a$ be a point in $M$. Then we say that

$$\lim_{x \to \infty} p(x) = a$$

if and only if

$$\lim_{B \in B} \sup_{x \in X \setminus B} m(p(x), a) = 0.$$

It immediately follows from the definitions that a function $\phi$ on $X$ has vanishing $U$-variation at $\infty$ if

$$\lim_{x \to \infty} \nabla_1 \phi(x) = 0.$$

It follows from the compatibility of the bornology with the coarse structure and (5.8) that this condition is equivalent to the condition that

$$\lim_{x \to \infty} \nabla_R \phi(x) = 0.$$
for all $R$ in $(0, \infty)$.

We now start the construction of the extension $\tilde{f}$ of $f$.

We choose inductively an exhaustion of $Y$ by (bornologically) bounded subsets $(Y_n)_{n \in \mathbb{N}}$ such that

$$(\nabla_{2n+1} f)(y) \leq 2^{-n} \epsilon$$

for all $y$ in $Y \setminus Y_n$. We then define the function

$$v : Y \to (0, \infty), \quad v(y) := \min\{n \in \mathbb{N} \mid y \in Y_n\}.$$

Then (using that $Y$ is unbounded for the first equality and [5,8] for the third)

$$\lim_{y \to \infty} v(y) = \infty, \quad \lim_{y \to \infty} (\nabla v(y) f)(y) = 0 \quad \text{and} \quad (\nabla v(y) f)(y) \leq \epsilon$$

for all $y$ in $Y$.

For $y$ in $Y$ we define the subset

$$U_y := B(y, \sqrt{v(y)})$$

of $X$.

We choose a point $y_0 \in Y_0$. We can choose a function $\rho : [0, \infty) \to [0, \infty)$ with the following properties:

1. $\rho(0) = 0$.
2. $\rho(t) > 0$ for $t > 0$.
3. $\lim_{t \to \infty} \rho(t) = \infty$.
4. $\rho$ is monotone.
5. For $x$ in $X$ the condition $d(x, Y) \leq \rho(d(x, y_0))$ implies $x \in \bigcup_{y \in Y} U_y$.
6. $\rho$ is subadditive.

In order to construct $\rho$ we start with the function

$$\tilde{\rho}(t) := \sup \{d(x, Y) \mid x \in B(y_0, t) \cap \bigcup_{y \in Y} U_y\}.$$

Then

1. $\tilde{\rho}(0) = 0$.
2. $\tilde{\rho}(t) \leq t$.
3. $\lim_{t \to \infty} \tilde{\rho}(t) = \infty$.
4. $\tilde{\rho}$ is monotone.
5. For $x$ in $X$ the condition $d(x, Y) \leq \tilde{\rho}(d(x, y_0))$ implies $x \in \bigcup_{y \in Y} U_y$.
For we use the assumption that \( Y \to X \) is not a coarse equivalence. Then it is possible to find a subadditive function \( \rho \) as desired with \( \rho(t) \leq \frac{1}{2} \hat{\rho}(t) \) for all \( t \in (0, \infty) \).

We consider the subset

\[
F := \{ x \in X \mid d(x, Y) \leq \rho(d(x, y_0)) \}
\]

of \( X \). Then \( Y \subseteq F \) and \( F \) is covered by the family \( (U_y)_{y \in Y} \) of subsets of \( X \). We can choose a partition \( (V_y)_{y \in Y} \) of \( F \) such that \( V_y \subseteq U_y \). Finally we set

\[
\psi : X \to [0, \infty) , \quad \psi(x) := \max \left\{ 0, \frac{\rho(d(x, y_0)) - d(x, Y)}{\rho(d(x, y_0))} \right\} .
\]

We now define the function

\[
\tilde{f} : X \to A \otimes K , \quad \tilde{f}(x) := \begin{cases} \psi(x) \cdot \sum_{y \in Y} \chi_{V_y}(x) f(y) & \text{if } x \in F , \\ 0 & \text{if } x \in X \setminus F . \end{cases}
\]

For \( z \) in \( Y \) we have \( \psi(y) = 1 \) and hence

\[
\| \tilde{f}(z) - f(z) \| = \| f(y) - f(z) \| \leq (\nabla_{\psi(y)} f)(y) \leq \epsilon ,
\]

where \( y \) in \( Y \) is the unique point such that \( z \in V_y \).

We now show that for every \( R \) in \( (0, \infty) \) the \( R \)-variation of \( \tilde{f} \) vanishes at \( \infty \). We fix \( R \) in \( (0, \infty) \) and consider \( x, z \) in \( X \) with \( d(x, z) \leq R \). We must consider three cases:

1. If \( \psi(x) = 0 \) and \( \psi(z) = 0 \), then \( \| \tilde{f}(x) - \tilde{f}(z) \| = 0 \).

2. If \( \psi(x) \neq 0 \) and \( \psi(z) = 0 \), then

\[
\| \tilde{f}(x) - \tilde{f}(z) \| = \psi(x) \cdot \| f(y_x) \| \leq \psi(x) \cdot \| f \| ,
\]

where the point \( y_x \) in \( Y \) is uniquely determined by the condition \( x \in V_{y_x} \).

Recall that \( d(x, z) \leq R \). By the triangle inequality for distances and the monotonicity and subadditivity of \( \rho \) we get the inequality

\[
\psi(x) = \max \left\{ 0, \frac{\rho(d(x, y_0)) - d(x, Y)}{\rho(d(x, y_0))} \right\} 
\begin{align*}
&\leq \max \left\{ 0, \frac{\rho(d(z, y_0)) + \rho(R) - d(z, Y) + R}{\rho(d(x, y_0))} \right\} \\
&\leq \max \left\{ 0, \frac{\rho(R) + R}{\rho(d(x, y_0))} \right\}
\end{align*}
\]

where for the last inequality we use the assumption \( \psi(z) = 0 \). In order to proceed in this case we use that \( \lim_{x \to \infty} \rho(d(x, y_0)) = \infty \).
3. If $\psi(x) \neq 0$ and $\psi(z) \neq 0$, then we have

$$
\|\tilde{f}(x) - \tilde{f}(z)\| \leq \psi(x) \cdot \|f(y_x) - f(y_z)\| + |\psi(x) - \psi(z)| \cdot \|f(y_z)\|, 
$$

(5.9)

where the points $y_x, y_z$ in $Y$ are uniquely determined by the conditions $x \in V_{y_x}$ and $z \in V_{y_z}$, respectively. The second term in (5.9) satisfies

$$
|\psi(x) - \psi(z)| \cdot \|f(y_z)\| \leq |\psi(x) - \psi(z)| \cdot \|f\|.
$$

We have the estimate

$$
|\psi(x) - \psi(z)| \\
\leq \left| \frac{\rho(d(z, y_0)) \rho(d(x, y_0)) - d(x, Y)}{\rho(d(x, y_0)) \rho(d(z, y_0))} \right| \\
\leq \left| \frac{\rho(d(z, y_0)) d(x, Y) - \rho(d(x, y_0)) d(z, Y)}{\rho(d(x, y_0)) \rho(d(z, y_0))} \right| \\
\leq \left| \frac{\rho(d(z, y_0)) d(z, Y) - \rho(d(x, y_0)) d(z, Y)}{\rho(d(x, y_0)) \rho(d(z, y_0))} + \frac{\rho(d(z, y_0)) R}{\rho(d(x, y_0)) \rho(d(z, y_0))} \right| \\
\leq \frac{\rho(R) d(z, Y) + \rho(d(z, y_0)) R}{\rho(d(x, y_0)) \rho(d(z, y_0))} \\
\leq \frac{\rho(R) + R}{\rho(d(x, y_0))} \\
\leq \frac{\rho(R)}{\rho(d(x, y_0))} + \frac{R}{\rho(d(x, y_0))}
$$

To proceed with the second term of (5.9) we again use that $\lim_{x \to \infty} \rho(d(x, y_0)) = \infty$.

The first term in (5.9) can be estimated as follows. We have

$$
\psi(x) \cdot \|f(y_x) - f(y_z)\| \leq \|f(y_x) - f(y_z)\|.
$$

Furthermore, $d(x, y_x) \leq \sqrt{v(y_x)}$ and $d(z, y_z) \leq \sqrt{v(y_z)}$. It follows that

$$
d(y_x, y_z) \leq R + \sqrt{v(y_x)} + \sqrt{v(y_z)}.
$$

Note that we have $\lim_{y \to \infty} v(y_x) = \infty$. Consequently, for sufficiently large bounded subsets $B$ of $X$ we have $d(y_x, y_z) < \max(v(y_x), v(y_z))$ for all $x \in X \setminus B$ (here we use $d(x, z) \leq R$). For these points $x$ we then have the estimate

$$
\|f(y_x) - f(y_z)\| \leq \max\{\nabla v(y_x) f(y_x), (\nabla v(y_z) f)(y_z)\}.
$$

For every bounded subset $B$ of $X$ there exists a bounded subset $B'$ of $X$ such that $x \in X \setminus B'$ together with $d(x, z) \leq R$ implies that $y_x \in X \setminus B$ and $y_z \in X \setminus B$. We therefore get

$$
\lim_{x \to \infty} \max\{\nabla v(y_x) f(y_x), (\nabla v(y_z) f)(y_z)\} = 0,
$$

finishing the estimate for the first term in (5.9).
The above three estimates together show that
\[ \lim_{x \to \infty} (\nabla_R \tilde{f})(x) = 0. \]

This finishes this proof in the case that \( X_U \) has a single coarse component, that \( Y \) is unbounded, and that \( Y \to X \) is not a coarse equivalence.

If \( Y \) is unbounded but \( i : Y \to X \) is a coarse equivalence, then we choose an inverse equivalence \( j : X \to Y \) such that \( j \circ i = \text{id}_Y \) and define the extension \( f \) in \( C_U(X; A) \) of \( \hat{f} \) in \( C_{U_Y}(Y; A) \) by \( f := j^* \hat{f} \).

If \( Y \) is non-empty and bounded, then we choose a base point \( y_0 \) in \( Y \) and define the extension \( f \) in \( C_U(X; A) \) of \( \hat{f} \) in \( C_{U_Y}(Y; A) \) by
\[
f(x) := \begin{cases} 
\hat{f}(x) & \text{if } x \in Y, \\
\hat{f}(y_0) & \text{if } x \notin Y.
\end{cases}
\]

If \( X_U \) has finitely many coarse components intersecting \( Y \) non-trivially, then we can construct the extension separately on every of these component.

We extend \( f \) by zero to coarse components of \( X_U \) which do not intersect \( Y \) at all. \( \Box \)

From the previous lemma we immediately get the following corollary. Let \( Y \) be a subset of a bornological coarse space \( X \) and \( U \) be an entourage of \( X \).

**Corollary 5.20.** Assume:

1. \( Y \) is \( U \)-convex.
2. \( X_U \) has at most finitely many coarse components intersecting \( Y \) non-trivially.

Then the restriction \( \tilde{C}_U(X; A) \to \tilde{C}_{U_Y}(Y; A) \) is surjective.

Let \( X \) be a bornological coarse space with coarse structure \( C \).

**Definition 5.21.** \( X \) has eventually finitely many coarse components if there exists an entourage \( U \) of \( X \) such that \( X_U \) has at most finitely many coarse components.

Let \((Z, Y)\) be a complementary pair in \( X \).

**Definition 5.22.** We say that \((Z, Y)\) is convex if there exists a cofinal set \( C' \subseteq C \) such that for every \( U \) in \( C' \) there exists a cofinal subset \( I' \subseteq I \) such that for every \( i \) in \( I' \)

1. \( Y_i \) is \( U \)-convex in \( X \), and
2. \( Z \cap Y_i \) is \( U_Z \)-convex in \( Z \).

Let \( X \) be a bornological coarse space and \((Z, Y)\) be a complementary pair on \( X \).

**Lemma 5.23.** Assume:
1. $X$ has eventually finitely many coarse components.

2. $(Z, Y)$ is convex.

Then $\mathcal{K}AX$ satisfies excision for $(Z, Y)$.

Proof. Let $C'$ be as in Definition 5.22. In the following $U$ belongs to $C'$ and is so large that $X_U$ has at most finitely many coarse components.

If $Y$ is a $U$-convex subset of $X$, then by Corollary 5.20 we have an exact sequence of $C^*$-algebras

$$0 \to \tilde{C}_U(X, Y; A) \to \tilde{C}_U(X; A) \xrightarrow{r} \tilde{C}_{U_Y}(Y; A) \to 0,$$

where the ideal $\tilde{C}_U(X, Y; A)$ is defined as the kernel of the restriction map $r$.

Let $I'$ be as in Definition 5.22. For every $i$ in $I'$ we get a map of exact sequences

$$0 \to \tilde{C}_{U_i}(X, Y_i; A) \to \tilde{C}_{U_i}(X; A) \xrightarrow{\bar{r}_i} \tilde{C}_{U_{Y_i}}(Y_i; A) \to 0 \quad (5.10)$$

We claim that the family of maps $(\bar{r}_i)_{i \in I'}$ is an isomorphism of pro-systems. To show this we define an inverse map. For every $i$ in $I'$ with $Z \cup Y_i = X$ we can choose $j$ in $I'$ such that $U[Y_i] \subseteq Y_j$. We let

$$s : C_b(Z; M^s(A)) \to C_b(X; M^s(A))$$

denote the homomorphism given by extension by zero.

We claim that $s$ restricts to a homomorphism

$$s : C_0(Z, A \otimes K) \to C_0(X, A \otimes K). \quad (5.11)$$

Indeed, if $f$ in $C_0(Z, A \otimes K)$ is supported on a bounded subset $B$ of $Z$, then $s(f)$ is also supported on $B$. The claim now follows because $s$ is continuous. It follows that $s$ descends to a homomorphism

$$\bar{s} : \tilde{C}_{U_Z}(Z; A) \to C_b(X; M^s(A))/C_0(X, A \otimes K).$$

We now claim that $\bar{s}$ restricts to a homomorphism

$$s_i : \tilde{C}_{U_Z}(Z, Z \cap Y_j; A) \to \tilde{C}_U(X, Y_i; A)$$

(recall that $j$ depends on $i$). Let $[f]$ be an element of $\tilde{C}_{U_Z}(Z, Z \cap Y_j; A)$. Because of (5.11) we can assume that $f|_{Z \cap Y_i} = 0$. A priori we only have $s(f) \in C_b(X, M^s(A))$ and we must check that $[s(f)]$ really belongs to $\tilde{C}_U(X, Y_i; A)$.

First of all, $s(f)$ vanishes on $Y_j$ and hence on $Y_i$. 

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Let now \((x, y)\) be in \(U\). We assert that \(s(f)(x) - s(f)(y) \in A \otimes K\). This is clear if \(x, y\) both belong to \(Z\) or both belong to \(X \setminus Z\). Assume now that \(x \in Z\) and \(y \in X \setminus Z\). Then \(s(f)(y) = 0\). Since \(y \in Y_i\) we have \(x \in Y_j\). Hence \(f(x) = 0\) and thus \(s(f)(x) = 0\) as well. Hence we have shown the assertion.

We now fix \(\epsilon\) in \((0, \infty)\). Then we can choose a bounded subset \(B\) of \(Z\) such that for all \((x, y)\) in \(U_{Z \setminus B}\) we have \(\|f(x) - f(y)\| \leq \epsilon\). We shall see that \(\|s(f)(x) - s(f)(y)\| \leq \epsilon\) for all \((x, y)\) in \(U_{X \setminus B}\). Indeed, either \((x, y)\) \(\in U_{Z \setminus B}\), or \(s(f)(x) = 0\) and \(s(f)(y) = 0\) by the same argument as above.

We thus have verified the claim.

The family \((s_i)_{i \in I}\) thus constructed represents a map of pro-systems which is inverse to \((r_i)_{i \in I}\). We now apply \(\lim_{i \in I'}\lim_{i \in I} K\) to \((5.10)\). We get a morphism of fibre sequences

\[
\begin{array}{cccc}
\lim_{i \in I'}\lim_{i \in I} K(C_U(X, Y_i; A)) & \longrightarrow & KAX(X) & \longrightarrow & KAX(Y) \\
\downarrow \cong & & \downarrow & & \downarrow \\
\lim_{i \in I'}\lim_{i \in I} K(C_{U_Z}(Z, Z \cap Y_i; A)) & \longrightarrow & KAX(Z) & \longrightarrow & \Sigma \lim_{i \in I'}\lim_{i \in I} K(C_{U_Z}(Z, Z \cap Y_i; A)) \\
\end{array}
\]

In view of the outer equivalences the middle square is cartesian. \(\square\)

**Example 5.24.** We have

\[
KAX(*) \cong K(Q^s(A)) \equiv \Sigma K(A) \cong \Sigma D_{K(A)}(KAX^{hlb})(*)
\]

The complementary pair \([[0, \infty) \times \mathbb{R}^{n-1}, ((-\infty, n] \times \mathbb{R}^{n-1})_{n \in \mathbb{N}}\) is convex in \(\mathbb{R}^n\) with the standard metric structures. Consequently, we can use excision and vanishing on flasques in order to calculate inductively that

\[
KAX(\mathbb{R}^n) \cong \Sigma^{n+1}K(A) \cong \Sigma D_{K(A)}(KAX^{hlb})(\mathbb{R}^n).
\]

(5.12)

Note that the coarse structure of \(\mathbb{R}^n\) is generated by a single entourage. Therefore by Lemma 5.13 we have an equivalence \(KAX^{em}(\mathbb{R}^n) \cong KAX(\mathbb{R}^n)\), and the calculation above is the same as [Wil13, Ex. 3.8].

Therefore one could ask whether \(KAX(X)\) is equivalent to \(\Sigma D_{K(A)}(KAX^{hlb})(X)\) in general. This is not the case. To this end we consider the case \(A = \mathbb{C}\), an infinite set \(I\), and the bornological coarse space \(X := \bigsqcup^\alpha_{i \in I} s\). Then we have

\[
KAX(X) \cong \bigoplus_I \Sigma KU
\]

by additivity (Lemma 5.17). On the other hand, by additivity of the coarse homology theory \(KAX^{hlb}\) we have an equivalence \(KAX^{hlb}(X) \cong \prod_I KU\), and hence

\[
D_{KU}(KAX^{hlb})(X) \cong \operatorname{map}_{\operatorname{Mod}(KU)}\left(\prod_I KU, KU\right) \cong \bigoplus_I KU
\]

showing that \(KAX(X) \not\cong \Sigma D_{KU}(KAX^{hlb})(X)\) for \(X = \bigsqcup^\alpha_{i \in I} s\). \(\diamondsuit\)
5.4 The pairing

The goal of this section is to describe a natural (in $X$) pairing

$$P_X : K\mathcal{A}(X) \otimes_{KU} K\mathcal{A}^{hlg}(X) \to \Sigma K(A)$$

defined for bornological coarse spaces $X$ of strongly bounded geometry ([BE16 Def. 6.100] or Definition 5.26 below) with finitely many coarse components. By adjunction it gives rise to a natural (see Equation 5.21 and Remark 5.31) morphism

$$p_X : K\mathcal{A}(X) \to \text{map}_{\text{Mod}(KU)}(K\mathcal{A}^{hlg}(X), \Sigma K(A)) \simeq \Sigma D_{K(A)}(K\mathcal{A}^{hlg})(X) \ .$$

(5.13)

In order to simplify the notation we only consider the case $A = \mathbb{C}$.

Remark 5.25. We do not know if one can extend the pairing naturally to all bornological coarse spaces.

We consider a coarse space $X$. For a subset $S$ of $X$ and an entourage $U$ of $X$ we define the multiplicity of $S$ with respect to $U$ to be

$$\text{mult}_U(S) := \sup_{s \in S} |S \cap U[s]|$$

in $\mathbb{N} \cup \{\infty\}$. A subset $S$ of $X$ has uniformly bounded coarse multiplicity if $\text{mult}_U(S)$ is finite for every coarse entourage $U$ of $X$.

Let $X$ be a bornological coarse space.

Definition 5.26 ([BE16 Def. 6.100]). $X$ has strongly bounded geometry if $X$ has uniformly bounded coarse multiplicity and the minimal compatible bornology.

For the definitions of notions related to $X$-controlled Hilbert spaces and Roe algebras we refer to [BE16 Sec. 7]. Let $X$ be a bornological coarse space.

Let $V$ be an auxiliary $\infty$-dimensional separable Hilbert space, and let us write $\mathbb{B} := B(V)$ and $\mathbb{K} := K(V)$.

We consider an $X$-controlled Hilbert space $(H, \phi)$ which is determined on points. The projection-valued measure $\phi$ extends to an action $\phi_V$ of the $C^*$-algebra $C_b(X, \mathbb{B})$ of bounded $\mathbb{B}$-valued functions on $H \otimes V$ such that

$$\phi_V(f) := \sum_{x \in X} \phi(\{x\}) \otimes f(x) \ .$$

(5.14)

This sum converges in the strong topology on $B(H \otimes V)$.

Let now $(H, \phi)$ and $(H', \phi')$ be two locally finite $X$-controlled Hilbert spaces. For every coarse entourage $U$ of $X$ we let $C^*_U(X)$ be the $C^*$-subcategory of the Roe category $C^*(X)$
whose morphisms are generated by bounded operators with propagation controlled by \( U \).

By definition of \( C^*(X) \) have an isomorphism

\[
C^*(X) \cong \operatorname{colim}_{U \in \mathcal{C}} C^*_U(X) .
\]

For any two objects \((H, \phi)\) and \((H', \phi')\) of \( C^*(X) \) we let \( C^*_U(X, (H, \phi), (H', \phi')) \) denote the morphism space \( \operatorname{Hom}_{C^*_U(X)}((H, \phi), (H', \phi')) \). We consider the inclusion

\[
i : C^*_U(X, (H, \phi), (H', \phi')) \to B(H \otimes V, H' \otimes V) , \quad A \mapsto A \otimes \text{id}_V . \tag{5.15}
\]

Note that \( C^*_U(X; \mathbb{C}) \) (Definition 5.8) is a subalgebra of \( C_b(X, \mathbb{B}) \). The formula (5.14) defines an action of \( C^*_U(X; \mathbb{C}) \) on \( H \otimes V \). Note that \( C_0(X, \mathbb{K}) \) is a subalgebra of \( C^*_U(X; \mathbb{C}) \).

Let \( A \) be in \( C^*_U(X, (H, \phi), (H', \phi')) \) and \( f \) be in \( C^*_U(X; \mathbb{C}) \).

**Lemma 5.27.** The operators \( i(A) \phi_V(f) \) and \( \phi'_V(f)i(A) \) are compact.

**Proof.** We consider the case of \( i(A) \phi_V(f) \). The other case is similar.

It suffices to show that \( i(A) \phi_V(f) \) is compact for functions \( f \) with bounded support. Let \( B \) be a bounded subset of \( X \) and assume that \( f \in C_0(X, \mathbb{K}) \) is supported on \( B \). Since we assume that \( (H, \phi) \) is locally finite, the complex vector space \( \phi(B)H \) is finite-dimensional. We now have

\[
i(A) \phi_V(f) = \sum_{x \in \operatorname{supp}(H, \phi) \cap B} A\phi(B)\phi(\{x\}) \otimes f(x) . \tag{5.16}
\]

Since \( A\phi(B) \) is finite-dimensional, it is compact. Since \( \operatorname{supp}(H, \phi) \cap B \) is finite and \( f(x) \) is compact for all \( x \) in \( X \), the right-hand side of (5.16) is a finite sum of compact operators and hence compact.

Let \( U \) be a coarse entourage of \( X \), \( A \) be in \( C^*_U(X, (H, \phi), (H', \phi')) \), and \( f \) be in \( C^*_U(X; \mathbb{C}) \).

**Lemma 5.28.** We assume:

1. \( \pi_0(X_U) \) is finite.
2. \( X_U \) has strongly bounded geometry.

Then the difference

\[
i(A) \phi_V(f) - \phi'_V(f)i(A) : H \to H'
\]

is compact.

**Proof.** It suffices to show this for generators, i.e., for operators with propagation at most \( U \). So let \( A \) have propagation \( U \). Then it does not mix the components of \( X_U \). Since \( X_U \) has finitely many components its suffices to consider the case of a single component.
We fix some point $x$ in $X$ and define $b := f(x)$ in $\mathbb{B}$. Then we have the equality

\[ \phi_V(f) = \sum_{x \in X} \phi\{x\} \otimes g(x) + 1 \otimes b , \]

where $g(x) \in \mathbb{K}$ for all $x$ in $X$ and the sum converges strongly. Then

\[ i(A)\phi_V(f) - \phi'_V(f)i(A) = \sum_{x \in X} (A\phi\{x\}) - \phi'(\{x\})A \otimes g(x) . \]

We now calculate

\[
\sum_{x \in X} (A\phi\{x\}) - \phi'(\{x\})A \otimes g(x)
\]

\[
= \sum_{x \in X} A\phi\{x\} \otimes g(x) - \sum_{y \in X} \phi'(\{y\})A \otimes g(y)
\]

\[
= \sum_{x,y \in X} \phi'(\{y\})A\phi\{x\} \otimes g(x) - \sum_{y \in X} \phi'(\{y\})A \otimes g(y)
\]

\[
= \sum_{x,y \in X} \phi'(\{y\})A\phi\{x\} \otimes g(x) + \sum_{x,y \in X} \phi'(\{y\})A\phi\{x\} \otimes (g(x) - g(y))
\]

\[
- \sum_{y \in X} \phi'(\{y\})A \otimes g(y)
\]

\[
= \sum_{x,y \in X} \phi'(\{y\})A\phi\{x\} \otimes (g(x) - g(y))
\]

Fix a positive real number $\epsilon$. We can find a bounded subset $B$ of $X$ such that

\[ \text{Var}_U(g, X \setminus B) \leq \frac{\epsilon}{\text{mult}_U(X)(\|A\| + 1)} . \]

(5.17)

Since $(H, \phi)$ and $(H', \phi')$ are locally finite, the difference

\[
\sum_{x,y \in X} \phi'(\{y\})A\phi\{x\} \otimes (g(x) - g(y)) - \sum_{x,y \in X \setminus B} \phi'(\{y\})A\phi\{x\} \otimes (g(x) - g(y))
\]

is compact. Using orthogonality of the families of projections $(\phi\{x\})_{x \in X}$ and $(\phi'(\{x\}))_{x \in X}$, respectively, we can restrict the sum over $x, y$ in $X \setminus B$ to those pairs which in addition satisfy $y \in U\{x\}$ (since $A$ has propagation controlled by $U$). But then we have

\[
\left\| \sum_{x,y \in X \setminus B \atop y \in U\{x\}} \phi'(\{y\})A\phi\{x\} \otimes (g(x) - g(y)) \right\| \leq \epsilon
\]

due to (5.17). Since $\epsilon$ is arbitrary and the compact operators form a closed subspace, we conclude that $i(A)\phi_V(f) - \phi'_V(f)i(A)$ is compact.

We define a $C^*$-category $\mathbb{Q}^*$ as follows:
1. The objects of $\mathbb{Q}^*$ are the Hilbert spaces $H$ (in the universe used to define all the categories $\mathbf{C}^*(X)$ we are using).

2. The morphisms $H \to H'$ of $\mathbb{Q}^*$ are the quotient Banach spaces

$$B(H \otimes V, H' \otimes V)/K(H \otimes V, H' \otimes V).$$

(5.18)

The $*$-operation is defined in the obvious way.

We consider the Calkin algebra $\mathbb{Q} := \mathbb{B}/\mathbb{K}$ as a $\mathbf{C}^*$-category with one object. The Hilbert space $C$ gives rise to a functor $\mathbb{Q} \to \mathbb{Q}^*$.

**Lemma 5.29.** The functor $\mathbb{Q} \to \mathbb{Q}^*$ induces an equivalence in $K$-theory. In particular we have an equivalence $K(A^f(\mathbb{Q}^*)) \simeq \Sigma \mathbb{K}U$.

**Proof.** The $K$-theory of a $\mathbf{C}^*$-category is equivalent to the filtered colimit of the $K$-theories of subcategories with finitely many objects. If $\mathbb{Q}'$ is a subcategory of $\mathbb{Q}^*$ with finitely many objects, then $K(A^f(\mathbb{Q}')) \simeq K(A(\mathbb{Q}'))$. We now observe that $A(\mathbb{Q}')$ is isomorphic to the Calkin algebra of a Hilbert space $H \otimes V$, where $H$ is the sum of the objects of $\mathbb{Q}'$.

We assume that $\mathbb{C}$ is an object of $\mathbb{Q}'$. The object $\mathbb{C}$ gives rise to an embedding $\mathbb{C} \to H$, hence an embedding $V \to H \otimes V$, and finally to an embedding $\mathbb{Q} \to A(\mathbb{Q}')$. This embedding induces an equivalence in $K$-theory. To this end we compare the fibre sequences associated to the exact sequences

$$0 \to \mathbb{K} \to \mathbb{B} \to \mathbb{Q} \to 0$$

and

$$0 \to \mathbb{K}(H \otimes V) \to \mathbb{B}(H \otimes V) \to A(\mathbb{Q}') \to 0.$$

We use that the $K$-theories of the algebras $\mathbb{B}$ and $B(H \otimes V)$ vanish, and that the inclusion $\mathbb{K} \to K(H \otimes V)$ induces an equivalence. This last assertion follows again from the fact that the $K$-theory of $K(H \otimes V)$ can be expressed as a filtered colimit of the $K$-theories of the subalgebras $K(W)$ for separable subspaces $W$ of $H \otimes V$. \qed

The maximal tensor product of $\mathbf{C}^*$-categories was defined in [Del12]. We regard $\tilde{\mathcal{C}}_U(X; \mathbb{C})$ as a $\mathbf{C}^*$-category with one object and morphisms $\tilde{\mathcal{C}}_U(X; \mathbb{C})$. In the following we describe a functor

$$R : \tilde{\mathcal{C}}_U(X; \mathbb{C}) \otimes \mathbf{C}^*(X_U) \to \mathbb{Q}^*.$$

This functor sends the object $(\ast, (H, \phi))$ of the tensor product to the object $H$ of $\mathbb{Q}^*$. It furthermore sends the morphism

$$[f] \otimes A : (\ast, (H, \phi)) \to (\ast, (H', \phi'))$$

to the morphism

$$[A \otimes \phi_V(f)] : H \to H'$$

in $\mathbb{Q}^*$, where the brackets indicate that we have to take the class of the bounded operator $A \otimes \phi_V(f) : H \otimes V \to H' \otimes V$, see (5.18). By Lemma 5.27 the operator $[A \otimes \phi_V(f)]$ is well-defined independently of the choice of the representative $f$ of the class $[f]$. If $\pi_0(X_U)$
is finite and $X$ has strongly bounded geometry, then $R$ is a functor by Lemma 5.28, i.e., it is compatible with the composition.

For any two $C^\ast$-categories $C$ and $D$ we have a homomorphism of $C^\ast$-algebras
\[ A^f(C) \otimes A^f(D) \to A^f(C \otimes D) . \] (5.19)

In the following we use that the functor $K : \text{CAlg} \to \text{Mod}(KU)$ admits a lax symmetric monoidal refinement which yields the first natural morphism in the composition
\[ K(A^f(C)) \otimes_{KU} K(A^f(D)) \to K(A^f(C) \otimes A^f(D)) \xrightarrow{\text{(5.19)}} K(A^f(C \otimes D)) . \]

We now apply this to $C := \bar{C}_U(X; C)$ and $D := C^\ast _U(X)$. Using that $A^f(\bar{C}_U(X; C)) \cong \bar{C}_U(X; C)$
we get a pairing
\[ K(\bar{C}_U(X; C)) \otimes_{KU} K(A^f(C^\ast_U(X))) \to K(A^f(\bar{C}_U(X; C) \otimes C^\ast_U(X))) \]
\[ \xrightarrow{\text{Lemma } 5.29} K(A^f(Q^\ast)) \]
\[ \to \Sigma KU . \]

This pairing is compatible with the maps induced by inclusions $U \subseteq U'$ of entourages. So we get a pairing
\[ \lim_{U' \in \mathcal{C}} K(\bar{C}_U(X; C)) \otimes_{KU} K(A^f(C^\ast_U(X))) \to \Sigma KU , \]
and furthermore
\[ \lim_{U' \in \mathcal{C}} K(\bar{C}_U(X; C)) \otimes_{KU} \operatorname{colim}_{U \in \mathcal{C}} K(A^f(C^\ast_U(X))) \to \Sigma KU . \]

Let $X$ be a bornological coarse space.

**Proposition 5.30.** Assume that the following conditions are satisfied for a cofinal set of entourages $U$ of $X$:

1. $X_U$ has finitely many coarse components.
2. $X_U$ has strongly bounded geometry.

Then we have constructed a pairing
\[ P_X : K\mathcal{C}_X(X) \otimes_{KU} K\mathcal{X}^{hlg}(X) \to \Sigma KU . \] (5.20)

Furthermore, if $f : X' \to X$ is a morphism between bornological coarse spaces satisfying the assumptions above, then we have the following functoriality of the pairing:
\[ P_{X'} \circ (f^* \otimes \text{id}_{K\mathcal{X}^{hlg}(X')}) \simeq P_X \circ (\text{id}_{K\mathcal{C}_X(X)} \otimes f_*) : K\mathcal{C}_X(X) \otimes_{KU} K\mathcal{X}^{hlg}(X') \to \Sigma KU . \]
By adjunction (5.20) gives rise to a morphism

\[ p_X : K\mathcal{C}(X) \to \Sigma D_{KU}(K\mathcal{X}^{hlg}(X)) \, . \]

We have further seen that for a morphism \( f : X' \to X \) as above the square

\[
\begin{array}{ccc}
K\mathcal{C}(X) & \xrightarrow{p_X} & \Sigma D_{KU}(K\mathcal{X}^{hlg}(X)) \\
\downarrow f^* & & \downarrow f^* \\
K\mathcal{C}(X') & \xrightarrow{p_X'} & \Sigma D_{KU}(K\mathcal{X}^{hlg}(X'))
\end{array}
\]

commutes in the homotopy category \( \text{Ho}(\text{Sp}) \).

**Remark 5.31.** It would require additional work (that we have not carried out) to refine the above functoriality of \( p \) so that it becomes a morphism (like in Definition 2.16) between the restrictions of the functors \( K\mathcal{C} \) and \( \Sigma D_{KU}(K\mathcal{X}^{hlg}) \) to a suitable subcategory of \( \text{BornCoarse}^{op} \).

After fixing an identification \( \pi_1(\Sigma KU) \cong \mathbb{Z} \) we get a pairing

\[ P_{X,*} : K\mathcal{C}^{*+1}(X) \otimes_{KU} K\mathcal{X}_*(X) \to \mathbb{Z} \]

on the level of groups. For \( X \) a single point, this is the canonical pairing \( \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z} \) in the degree \( * = 0 \) (in the other degrees we have trivial groups), see the following example.

**Example 5.32.** The morphism \( K\mathcal{C}(*) \to \Sigma D_{KU}(K\mathcal{X}^{hlg})(*) \) is an equivalence. To see this, let us analyse the pairing

\[ P_* : K\mathcal{C}(*) \otimes_{KU} K\mathcal{X}^{hlg}(*) \to \Sigma KU \, . \]

In this case the functor \( \tilde{\mathcal{C}}_{\text{diag}(*)}(X; \mathbb{C}) \otimes \mathbb{C}_{\text{diag}(*)} \to \mathbb{Q}^* \) is given on morphisms by

\[ [Q] \otimes A \mapsto [Q \otimes A] \, . \]

The map induced in \( K \)-theory is equivalent to the map

\[ K(\mathbb{Q}) \otimes_{KU} K(K(\ell^2)) \to K(Q(\ell^2 \otimes V)) \]

which in turn is equivalent to the product (given by the multiplicative structure on \( KU \))

\[ \Sigma KU \otimes_{KU} KU \to \Sigma KU \, . \]

Hence this product is equivalent to the pairing from (5.22).

**Example 5.33.** Let \( X = \mathbb{Z} \) with the bornological and coarse structures induced from the metric. We consider the object \( (H, \phi) = (L^2(\mathbb{Z}), \phi) \) of \( C^*(X) \), where \( \phi(A) \) denotes the projection onto the subspace \( L^2(A) \) for every subset \( A \) of \( \mathbb{Z} \). We let \( U \) be the shift on \( H \) in the positive direction. Then \( U \) can be considered as a class \([U]\) in \( K\mathcal{X}_1^{hlg}(X) \). We further consider the projection in \( \tilde{C}_{U_1}(\mathbb{Z}; \mathbb{C}) \) represented by the function

\[ Q = 1 - \chi_{\mathbb{Z}, e_1} \in \tilde{C}_{U_1}(\mathbb{Z}, \mathbb{B}) \, . \]

It gives rise to an element \([Q] \in K\mathcal{C}(0)(\mathbb{Z}) \). Then we have

\[ P_{*,1}([Q], [U]) = 1 \, . \]
References


