On the Hochschild homology of $\ell^1$-rapid decay group algebras

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Abstract

We show that for many semi-hyperbolic groups the decomposition into conjugacy classes of the Hochschild homology of the $\ell^1$-rapid decay group algebra is injective.

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1 Introduction

Let $G$ be a finitely generated group and denote by $\mathbb{C}G$ its complex group ring. Choosing any finite, symmetric generating set for $G$, we get a word-length norm on it and can then define the $\ell^1$-rapid decay group algebra $\ell^1_\infty G$, which is the closure of $\mathbb{C}G$ under a certain family of norms (see Definition 3.1 for details).
The Fréchet algebra $\ell^1 G$ plays an important role in the isomorphism conjectures and related questions: its $K$-theory is the target of the Bost assembly map and is also an intermediate step in the Baum–Connes conjecture [Laf02], and its Hochschild and cyclic homology are related to the Bass, Burghelea and idempotent conjectures [JOR10].

In this paper we are interested in the Hochschild homology of $\ell^1 G$. In the case of the group ring $\mathbb{C} G$ Burghelea [Bur85] computed the Hochschild homology completely: $HH_n(\mathbb{C} G) \cong \bigoplus_{[g] \in (G)} H_n(Z_g; \mathbb{C})$, where on the right hand side the sum runs over all the conjugacy classes of $G$ and $Z_g$ is the centralizer of $g$. In the case of $\ell^1 G$ we have a map

$$HH^\text{cont}_n(\ell^1 G) \to \prod_{x \in (G)} HH^\text{cont}_n(\ell^1 G)_x \quad (1.1)$$

into the product over the conjugacy classes, but injectivity might be lost now. This was already noticed in [JOR10], where a computation of the single factors was carried out.

**Main Theorem.** Let $G$ be a finitely generated group from one of the following classes:

1. hyperbolic groups,
2. central extensions of hyperbolic groups,
3. Artin groups of extra-large type,
4. right-angled Artin groups, or
5. hyperbolic relative to a finite collection of groups from the previous four classes.

Then the map (1.1) is injective.

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## 2 Review of the homology of complex group rings

**Definition 2.1.** Let $A$ be an algebra over $\mathbb{C}$.

The *Hochschild homology* $HH_\ast(A)$ is defined as the homology of the complex

$$\cdots \to C_n(A) \xrightarrow{b} C_{n-1}(A) \xrightarrow{b} \cdots \xrightarrow{b} C_0(A) \to 0,$$

where $C_n(A) = A \otimes^{(n+1)}$ and $b$ is the Hochschild boundary operator

$$b(a_0 \otimes \cdots \otimes a_n)$$

$$= \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \cdots \otimes a_{n-1}.$$

\[\text{\footnote{We do not have surjectivity since on the right hand side we give up any control over the norms across the different conjugacy classes.}}\]
The complete computation of the Hochschild and cyclic homology of group rings is due to Burghelea [Bur85]. One can also consult Khalkhali [Kha13] Example 3.10.3 or Loday [Lod92] Chapter 7.4 or Connes [Con94] Example 3.2.7. We will state below only the result for Hochschild homology, since this is the one we need.

We write $\langle G \rangle$ for the conjugacy classes of $G$, and for an element $g \in G$ we write $Z_g < G$ for its centralizer.

**Theorem 2.2** (Burghelea [Bur85]). For all $n \in \mathbb{N}_0$ we have

$$HH_n(CG) \cong \bigoplus_{[g] \in \langle G \rangle} H_n(Z_g; \mathbb{C}).$$

(2.1)

### 2.1 Details of the computation I: reduction to the centralizers

We denote by $C_\ast(CG)$ the Hochschild complex of the group ring $\mathbb{C}G$. For a conjugacy class $x \in \langle G \rangle$ we denote by $C_\ast(CG)_x$ the $\mathbb{C}$-linear span of the set \{$(g_0, \ldots, g_n) : g_0 \cdots g_n \in x$\}.

Then $C_\ast(CG)_x$ is a subcomplex of $C_\ast(CG)$, and we have a splitting

$$C_\ast(CG) \cong \bigoplus_{x \in \langle G \rangle} C_\ast(CG)_x$$

(2.2)

of the Hochschild complex which is responsible for the corresponding splitting in (2.1).

The proof of the following lemma is based on Ji [Ji93, Ji95] and Nistor [Nis90], and the basic idea for their arguments can be traced back to Burghelea [Bur85].

**Lemma 2.3.** For all $h \in x$ the inclusion $C_\ast(CZ_h)[h] \to C_\ast(CG)_x$ induces an isomorphism on Hochschild homology groups.

**Proof.** We define a map $\pi_h : C_\ast(CG)_x \to C_\ast(CZ_h)[h]$ which is an inverse on homology. Let us define a map (which is in general not a homomorphism) $p_h : G \to Z_h$ by picking for each coset $y \in Z_h \setminus G$ a representative $s(y)$, i.e., $Z_h \cdot s(y) = y$, and then mapping $g$ to $gs(y)^{-1}$ if $g \in y$. It has the property $p_h(a g) = a p_h(g)$ for all $a \in Z_h$.

Let $(g_0, \ldots, g_n) \in C_\ast(CG)_x$ be given. Then there is $r \in G$ such that $g_0 \cdots g_n = r^{-1} h r$, because $h \in x$. Then we set on generators

$$\pi_h(g_0, \ldots, g_n) := (p_h(r g_0 \cdots g_n)^{-1} p_h(r g_0), p_h(r g_0)^{-1} p_h(r g_0 g_1), \ldots, p_h(r g_0 \cdots g_{n-1})^{-1} p_h(r g_0 \cdots g_n)).$$

(2.3)

One quickly checks that $\pi_h$ maps indeed into $C_\ast(CZ_h)[h]$. It is also well-defined in the sense that if we have two representations $g_0 \cdots g_n = r^{-1} h l = l^{-1} h l$, then $\pi_h(g_0, \ldots, g_n)$ is independent of the choice of $r$ or $l$ for the formula. This follows from the fact that in this situation we have $r = a l$ for some element $a \in Z_h$ and we have already noted above that $p_h(a g) = a p_h(g)$ for all $a \in Z_h$. But the map $\pi_h$ does in general depend on the choice of $p_h$.

We are using here the isomorphism $\mathbb{C}G \otimes \cdots \otimes \mathbb{C}G \cong \mathbb{C}(G \times \cdots \times G)$ to make sense of this.
A computation\(^3\) shows that \(\pi_h\) is a chain map, i.e., commutes with boundary operators, and therefore induces a map on homology groups.

We denote the inclusion \(C_\bullet(\mathbb{C}Z_h)_{[\ell]} \to C_\bullet(\mathbb{C}G)_x\) by \(\iota_h\). The composition \(\pi_h \circ \iota_h\) is the identity map and the composition \(\iota_h \circ \pi_h\) is chain homotopic to the identity (the latter will be shown in Section 2.3), which finishes this proof.

\[\square\]

### 2.2 Details of the computation II: homology of the centralizers

To compute the group homology \(H_\ast(G; \mathbb{C})\) we can either use the chain complex given by

\[C'_n(G) := G^n\] with the boundary operator

\[\partial(g_1, \ldots, g_n) := (g_2, \ldots, g_n) + \sum_{k=1}^{n-1} (-1)^k(g_1, \ldots, g_kg_{k+1}, \ldots, g_n) + (-1)^n(g_1, \ldots, g_{n-1})\]

or we can use the chain complex given by (here \((-)^G\) denotes the \(G\)-invariants)

\[C_n(G) := (G^{n+1})^G\] with the boundary operator

\[\partial(1, g_1, \ldots, g_n) := (g_1, \ldots, g_n) + \sum_{k=1}^{n} (-1)^k(1, g_1, \ldots, g_k, \ldots, g_n)\]

An isomorphism between these two chain complexes is given by the chain map

\[\psi: C'_n(G) \to C_n(G), \quad \psi(g_1, \ldots, g_n) := (1, g_1, g_2, \ldots, g_1g_2 \cdots g_n)\]

and its inverse

\[\psi^{-1}: C_n(G) \to C'_n(G), \quad \psi^{-1}(1, g_1, \ldots, g_n) = (g_1, g_1^{-1}g_2, g_2^{-1}g_3, \ldots, g_{n-1}^{-1}g_n)\]

Recall the notation from Section 2.1. We have a chain map \(\phi_g: C'_\bullet(Z_g) \to C_\bullet(\mathbb{C}Z_g)_{[g]}\) which is in degree \(n\) on generators given by \((g_1, \ldots, g_n) \mapsto ((g_1 \cdots g_n)^{-1}g, g_1, \ldots, g_n)\). Its inverse is given by \((g_0, \ldots, g_n) \mapsto (g_0, \ldots, g_n)\). This shows on the nose that \(\phi_g\) induces an isomorphism on homology groups \(H_\ast(Z_g; \mathbb{C}) \cong HH_\ast(Z_g)_{[g]}\), where we write \(HH_\ast(Z_g)_{[g]}\) for the homology of the complex \(C_\bullet(\mathbb{C}Z_g)_{[g]}\). Combined with Lemma 2.3 and (2.2) we therefore deduce the isomorphism (2.1) for Hochschild homology.

For every \(x \in \langle G \rangle\) and \(h \in x\) the formula for the composition

\[C_\bullet(\mathbb{C}G)_x \xrightarrow{\pi_h} C_\bullet(\mathbb{C}Z_h)_{[\ell]} \xrightarrow{\phi_h^{-1}} C'_n(Z_h) \xrightarrow{\psi} C_n(Z_h)\] (2.5)

is as follows: given a generator \((g_0, \ldots, g_n)\) with \(g_0 \cdots g_n = r^{-1}g_r\), its image under the above composition is the equivariant chain having the value 1 on the orbit of

\[(p_h(rg_0), p_h(rg_0g_1), \ldots, p_h(rg_0 \cdots g_n))\] (2.6)

\(^3\)When doing this keep in mind that at some point one has to apply (2.3) to \((g_n, g_0, g_1, \ldots, g_{n-1})\) and we have now \(g_ng_0g_1 \cdots g_{n-1} = g_{n}r^{-1}hrg_{n}^{-1}\), i.e., one has to apply in this case (2.3) with a different choice of \(r\), namely \(rg_{n}^{-1}\).
2.3 Details of the computation III: the chain homotopies

In this section we are following the presentation of Ji \[\text{[Ji95]}\], which is itself based on work of Nistor \[\text{[Nis90]}\].

We consider the chain complex \(E_\bullet(G)\) with the chain groups \(E_n(G) := G^{n+1}\) and the boundary operator

\[
\partial(g_0, \ldots, g_n) := (g_1, \ldots, g_n) + \sum_{k=1}^{n} (-1)^k (g_0, g_1, \ldots, \hat{g}_k, \ldots, g_n),
\]

i.e., the non-equivariant version of \(\text{(2.4)}\).

Let \(x\) be a conjugacy class of \(G\) and let \(h \in x\). Let us denote by \(i^E: E_\bullet(Z_h) \rightarrow E_\bullet(G)\) the inclusion map. To define a chain homotopy inverse \(h\) to it, recall first the definition of the map \(p_h: G \rightarrow Z_h\) from the beginning of the proof of Lemma \[2.3\]. Then \(p^E\) is the extension to \(E_\bullet(G)\) of it, i.e., on generators we have

\[
p_h(g_0, \ldots, g_n) := (p_h(g_0), \ldots, p_h(g_n)).
\]

To define the chain homotopy from \(i^E_h p^E_h\) to the identity on \(E_\bullet(G)\), we first define

\[
D_0: E_0(G) \rightarrow E_1(G), \quad D_0(g_0) := (g_0 s(Z_h \cdot g_0)^{-1}, g_0)
\]

and check that \((\text{id} - i^E_h p^E_h)(g_0) = (\partial D_0)(g_0)\). Then we define inductively

\[
D_n: E_n(G) \rightarrow E_{n+1}(G), \quad D_n(g_0, \ldots, g_n) := (g_0, (\text{id} - i^E_h p^E_h - D_{n-1} \partial_h)(g_0, \ldots, g_n))
\]

and this satisfies \(\text{id} - i^E_h p^E_h = D_{n-1} \partial_h + \partial_h D_n\).

We have a chain map

\[
\partial_h: E_n(G) \rightarrow C_n(\mathbb{C}G)_x, \quad \partial_h(g_0, \ldots, g_n) := (g_n^{-1} h g_0, g_0^{-1} g_1, \ldots, g_{n-1}^{-1} g_n).
\]

The kernel of \(\partial_h\) is spanned by \(\{g \cdot (g_0, \ldots, g_n) - (g_0, \ldots, g_n) : \forall i(g_i \in G) \text{ and } g \in Z_h\}\), and \(\partial_h\) is surjective. It follows that it induces an isomorphism \(E_\bullet(G) \otimes_{\mathbb{C}Z_h} \mathbb{C} \cong C_\bullet(\mathbb{C}G)_x\).

The maps \(i^E_h, p^E_h\) and the chain homotopies \(D_n\) are all \(\mathbb{C}Z_h\)-linear, i.e., they induce maps between the quotient complexes \(E_\bullet(G) \otimes_{\mathbb{C}Z_h} \mathbb{C}\) and \(E_\bullet(Z_h) \otimes_{\mathbb{C}Z_h} \mathbb{C}\). The main result of this subsection is now that we have a commuting diagram

\[
\begin{array}{ccc}
E_\bullet(G) \otimes_{\mathbb{C}Z_h} \mathbb{C} & \xrightarrow{\partial_h} & C_\bullet(\mathbb{C}G)_x \\
\downarrow i^E_h & & \downarrow \pi_h \\
E_\bullet(Z_h) \otimes_{\mathbb{C}Z_h} \mathbb{C} & \xrightarrow{\partial_h} & C_\bullet(\mathbb{C}Z_h)[p]
\end{array}
\]

This provides us the chain homotopies between the identity and \(\iota_h \circ \pi_h\).
3 Homology of \(\ell^1\)-rapid-decay group algebras

Let \(G\) be a finitely generated group. We fix a finite, symmetric generating set and get a word-length norm \(\ell^1\) on \(G\). Note that any two choices of finite, generating sets result in quasi-isometric word-length norms.

**Definition 3.1.** For every \(k \in \mathbb{N}\) we define a norm \(\|\cdot\|_{k,1}\) on \(C^*G\) by

\[
\|f\|_{k,1} := \|(1 + |\cdot|)^k \cdot f(-)\|_{\ell^1 G}.
\]

We denote by \(\ell^1G\) the closure of \(C^*G\) under the family of norms \((\|\cdot\|_{k,1})_{k \in \mathbb{N}}\).

For any Fréchet algebra\(^4\) \(A\) we define \(HH^*_\text{cont}(A)\) analogously as its algebraic counterpart, but we use the completed projective tensor product to form the chain groups.

For \(x \in \langle G \rangle\) and \(n \in \mathbb{N}_0\) we equip \(C_n(G)\) with the induced subspace norm and denote by \(C_n(\ell^1G)_x\) the completion of \(C_n(\ell^1G)_x\) to shorten notation.

### 3.1 Rapid decay group homology

We will compute the single factors of \((3.2)\) by comparing them with a rapid decay version of group homology:

**Definition 3.2.** On the chain group \(C_n(G)\) we define for each \(k \in \mathbb{N}\) a norm \(\|\cdot\|_{k,1}\) by

\[
\|c\|_{k,1} := \sum_{(g_1, \ldots, g_n) \in G^n} |c(1, g_1, \ldots, g_n)| \cdot \text{diam}(1, g_1, \ldots, g_n)^k.
\]

We equip \(C_n(G)\) with the family of norms \((\|\cdot\|_{k,1} + \|d\|_{k,1})_{k \in \mathbb{N}}\) and denote its completion by \(C^*_\text{RD}(G)\). The resulting homology is denoted by \(H^*_\text{RD}(G)\).

\(^4\)A Fréchet algebra is a topological vector space whose topology is Hausdorff and induced by a countable family of semi-norms such that it is complete with respect to this family of semi-norms, and such that multiplication is jointly continuous.

\(^5\)Analogously to Footnote 2 on Page 3 we use here the isomorphism \(\ell^1(G) \hat{\otimes} \cdots \hat{\otimes} \ell^1(G) \simeq \ell^1(G \times \cdots \times G)\).

\(^6\)Ji–Ogle–Ramsey [JOR10] already carried out these computations (in greater generality). We write them down again since our later results depend on how these computations concretely look like.
To get a hold on the rapid decay group homology we have to impose a polynomial control on the higher-order Dehn functions of the group $G$. Let us define that now.  

**Definition 3.3** (Higher-order Dehn functions, Ji–Ramsey [JR09]). Let $X$ be a simplicial complex. For a simplicial $N$-chain $b$ which is a boundary we denote by $l_f(b)$ the least $\ell^1$-norm of an $(N+1)$-chain $a$ with $\partial a = b$. We denote the $\ell^1$-norm of $b$ by $|b|$. The $N$-th Dehn function $d^N(\cdot): \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ of $X$ is defined as

$$d^N(k) := \sup_{|b| \leq k} l_f(b),$$

where the supremum runs over all $N$-boundaries $b$ of $X$ with $|b| \leq k$.

For a group $G$ we choose a simplicial model for $BG$. The higher-order Dehn functions of $G$ are then defined as the higher-order Dehn functions of $EG$. If $G$ is of type $F_{N+1}$, then all the higher-order Dehn functions $d^n(\cdot)$ up to $n \leq N$ have finite values and the growth type (e.g., being asymptotically a polynomial of a certain degree) does not depend on the chosen model for $BG$ with finite $(N+1)$-skeleton [JR09, Section 2].

**Proposition 3.4.** Let $G$ be of type $F_\infty$ and let it have polynomially bounded higher-order Dehn functions.

1. The inclusion $C_\bullet(G) \to C_{\text{RD}}\bullet(G)$ induces an isomorphism $H_\bullet(G) \cong H_\bullet^{\text{RD}}(G)$.

2. For every $n \in \mathbb{N}$ and every $k \in \mathbb{N}$ exists a constant $C_k > 0$ and $p_k \in \mathbb{N}$ such that if a chain $c \in C_n^{\text{RD}}(G)$ is a boundary, then there is a chain $b \in C_{n+1}^{\text{RD}}(G)$ with $d(b) = c$ and $\|b\|_{k,1} \leq C_k \cdot \|c\|_{k+p_k,1}$.

Here $p_k$ is the degree of the $n$-th Dehn function of $G$ and the constant $C_k$ depends on its coefficients.

**Proof.** Proofs of Point 1 are given by Ogle [Ogl05], Meyer [Mey06] and Ji–Ramsey [JR09]. A geometric proof of it was provided by the first named author in [Eng18, Section 4] (though in this reference the version of Riley is used for the higher-order Dehn functions, the proof also works with the version of Ji–Ramsey).

The proof of Point 2 is contained in the first part of the proof of Theorem 2.6(3 $\Rightarrow$ 1) of Ji–Ramsey [JR09].

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### 3.2 Computation of the homology localized at a conjugacy class

The results in this section are already known [JOR10]. But we need their proofs in the next section, hence we have to write everything down again.

Let $G$ be a finitely generated group, $x \in \langle G \rangle$ a conjugacy class, and $h \in x$. Recall that we denote by $Z_h G$ the centralizer of $h$ in $G$. We fix a word-length norm on $G$.

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7Note that there several different competing definitions for higher-order Dehn functions. We have chosen the version of Ji–Ramsey [JR09]. Another version is due to Riley [Ril03] and yet another due to Alonso–Pride–Wang [APW99].
In the proof of Lemma 2.3 we defined a map \( p_h : G \to Z_h \). It was based on picking for each coset \( y \in Z_h \setminus G \) a representative \( s(y) \) and then mapping \( g \) to \( gs(y)^{-1} \) if \( g \in y \). We pick now the representative \( s(y) \in y \) such that it minimizes the length in its class, i.e., \( |s(y)| \leq |g| \) for all \( g \in y \). This does not necessarily uniquely determine the element \( s(y) \), but this is of no problem to us. If we equip \( Z_h \subset G \) with the induced subspace norm\(^8\) then the map \( p_h \) is 2-Lipschitz, i.e., \( |p_h(g)| \leq 2|g| \) for all \( g \in G \).

**Definition 3.5 ([JOR10 Page 99]).** We say that \( G \) has a polynomialsolvably conjugacy bound at \( g \in G \), if there exists a polynomial \( P \) such that we have the following: for each \( h \in [g] \) exists an \( r \in G \) with \( h = r^{-1}gr \) and \( |r| \leq P(|h|) \).

We let \( h \in G \) and equip \( Z_h \subset G \) with the induced subspace norm. Then the inclusion of complexes \( \iota_h : C_\bullet(\ell_\infty^1 Z_h)[h] \to C_\bullet(\ell_\infty^1 G)[h] \) is continuous. If \( G \) has a polynomialsolvably conjugacy bound at \( h \) and if we pick the elements \( s(y) \) as above, then the map \( \pi_h \) from (2.3) is a continuous map\(^7\) and therefore extends continuously to a map

\[
\pi_h : C_\bullet(\ell_\infty^1 G)[h] \to C_\bullet(\ell_\infty^1 Z_h)[h].
\]

(3.3)

Since the chain homotopies from \( \iota_h \circ \pi_h \) to the identity are also continuous\(^9\) we conclude that \( \pi_h \) induces isomorphisms on homology groups.

Let us consider the composition \( \psi \circ \phi_h^{-1} \) of the last two maps from (2.5) and the inverse of this composition. If we equip \( Z_h \) with a word-length norm, then this composition and its inverse are continuous and hence we have an isomorphism

\[
HH^\text{cont}_\bullet(\ell_\infty^1 Z_h)[h] \cong H^\text{RD}_\bullet(Z_h).
\]

Putting it all together, we have proved the following:

**Proposition 3.6 ([JOR10 Corollary 1.4.6]).** Let \( G \) be a finitely generated group. For an \( h \in G \) assume that the centralizer \( Z_h \) is quasi-isometrically embedded in \( G \), and let \( G \) have a polynomialsolvably conjugacy bound at \( h \). Then

\[
HH^\text{cont}_\bullet(\ell_\infty^1 G)[h] \cong H^\text{RD}_\bullet(Z_h).
\]

Note that the result of the above proposition was already obtained by Ji–Ogle–Ramsey. They do not have the assumption of \( Z_h \) being quasi-isometrically embedded in \( G \) because they use the induced metric on \( Z_h \) (which might not be quasi-isometric to a word-length metric on it), whereas we equip \( Z_h \) with a word-length metric to make the homology groups \( H^\text{RD}_\bullet(Z_h) \) independent of \( G \). Note also that Ji–Ogle–Ramsey were able to remove the assumption on the conjugacy bound in [JOR14].

Since for the identity \( e \in G \) the centralizer is the whole group and since the conjugacy class at \( e \) is trivial, we immediately get from the above proposition the following:

\(^{10}\)One has to redo Section 2.3 in the setting of \( \ell^1 \)-rapid decay algebras here. Especially, one has to convince oneself that under the assumptions here we have an isomorphism \( \ell^1 \text{RD}(G) \otimes_{\ell^1} \mathbb{C} \cong C_\bullet(\ell_\infty^1 Z_h)[h] \).
Corollary 3.7. Let $G$ be a finitely generated group. Then

$$HH^\ast_\ast(\ell^1_\infty G)[e] \cong H^\ast_\ast(RD)(G).$$

Corollary 3.8. Let $G$ be of type $F_\infty$ and let it have polynomially bounded higher-order Dehn functions.

Then the map $HH_\ast_\ast(CG)[e] \to HH^\ast_\ast(\ell^1_\infty G)[e]$ is an isomorphism.

Proof. From Section 2, we know that $HH_\ast_\ast(CG)[e] \cong H_\ast_\ast(G)$ and from the Corollary 3.7 that $HH^\ast_\ast(\ell^1_\infty G)[e] \cong H^\ast_\ast(RD)(G)$. The claimed result follows with Proposition 3.4.1.

3.3 Injectivity of the product decomposition

The discussion in the previous two sections can be used to prove the following results:

Lemma 3.9. Let $G$ be a countable group and pick for each conjugacy class $x \in \langle G \rangle$ an element $h_x \in x$ such that the following holds:

1. $G$ has a polynomially solvable conjugacy bound at each $h_x$.\(^{11}\)

2. The centralizers $Z_{h_x} \subset G$ are all of type $F_\infty$ and satisfy the following two conditions:
   a) Every centralizer admits a word-length norm such that the inclusion $Z_{h_x} \subset G$ is a quasi-isometric embedding.
   b) Each centralizer has polynomially bounded higher-order Dehn functions.

Then the map $HH_\ast_\ast(CG) \to HH^\ast_\ast(\ell^1_\infty G)$ is injective.

Proof. We have the following diagram, where the left vertical isomorphism is due to the Theorem 2.2 and the right vertical map exists due to our assumptions:

$$
\begin{array}{c}
HH_\ast_\ast(CG) \cong \\
\oplus_{[g] \in \langle G \rangle} H_\ast_\ast(Z_g; \mathbb{C}) \longrightarrow \prod_{[g] \in \langle G \rangle} H^\ast_\ast(RD)(Z_g; \mathbb{C})
\end{array}
$$

By Proposition 3.4.1, the lower horizontal map is injective, hence the lemma follows. \(\square\)

Lemma 3.10. Let $G$ be a countable group. For every conjugacy class $x \in \langle G \rangle$ we pick an element $h_x \in x$ minimizing the word-length norm in its conjugacy class. We assume that the following holds:

1. There is a polynomial $P(-, -)$ in two variables and $G$ has a polynomially solvable conjugacy bound at $h_x$ with governing polynomial $P(-, |h_x|)$.

2. The centralizers $Z_{h_x} \subset G$ are all of type $F_\infty$ and satisfy the following two conditions:

\(^{11}\)see Definition 3.5
Then the map \((3.2)\), i.e., \(HH^*_n(\ell_\infty^1 G) \to \prod_{x \in \langle G \rangle} HH^*_n(\ell_\infty^1 G)_x\) is injective.

**Proof.** Let \([c] \in HH^*_n(\ell_\infty^1 G)\). We first apply the chain map \((3.1)\) to map \(c\) to a cycle in the space \(\prod_{x \in \langle G \rangle} C_n(\ell_\infty^1 G)_x\). Note that each \(C_n(\ell_\infty^1 G)_x\) has the induced subspace norms from \(C_n(\ell_\infty^1 G)\). We fix one norm on \(C_n(\ell_\infty^1 G)\) for the rest of this proof, and consider the corresponding induced norms on each \(C_n(\ell_\infty^1 G)_x\). The norm of every factor of the image of \(c\) under \((3.1)\) is bounded from above by the norm of \(c\).

For each conjugacy class \(x \in \langle G \rangle\) we let \(h_x \in x\) be as in the assumptions of this lemma, i.e., \(h_x\) minimizes the word-length norm in its conjugacy class \(x\). To each of the factors \(C_n(\ell_\infty^1 G)_x\) we now apply the map \((3.3)\), i.e., we apply \(\pi_{h_x}: C_n(\ell_\infty^1 G)_x \to C_n(\ell_\infty^1 Z_{h_x})\|h_x\|\). Because of the Assumptions 1 and 2a of this lemma we can conclude now that the norms of the resulting chains are bounded from above by a factor times the norm of \(c\) and such that the factors for the conjugacy classes grow at most polynomially in \(|h_x|\).

We use now the composition of the last two maps from \((2.5)\) on each factor and arrive in the space \(\prod_{x \in \langle G \rangle} C^RD_n(Z_{h_x})\). Again we can conclude that the norm of each factor of the image of \(c\) in \(\prod_{x \in \langle G \rangle} C^RD_n(Z_{h_x})\) is bounded from above by a constant times the norm of \(c\) and the constant grows at most polynomially in \(|h_x|\).

Now assume that \([c] \in HH^*_n(\ell_\infty^1 G)\) is mapped to zero under the map \((3.2)\). Then we know that the image of \(c\) in \(\prod_{x \in \langle G \rangle} C^RD_n(Z_{h_x})\) is a boundary. By Point 2 of Proposition 3.4 in combination with Assumption 2b of this lemma this implies that there exist chains \(b_x \in C^R_{n+1}(G)\) with \(d(b_x) = c_x\), where \(c_x \in C^RD_n(Z_{h_x})\) is the \(x\)-component of the image of \(c\), and we have \(|b_x|_{k,1} \leq C_k(x) \cdot |c_x|_{k+p_{k,1}}\), where the constants \(C_k(x)\) depend polynomially on \(|h_x|\). Here \(k \in \mathbb{N}\) corresponds to the at the beginning of this proof chosen norm.

We now use the inverse of the composition of the last two maps from \((2.5)\) on each of the factors to map \(\prod_{x \in \langle G \rangle} b_x\) into \(\prod_{x \in \langle G \rangle} C_{n+1}(\ell_\infty^1 Z_{h_x})\|h_x\|\). Due to the above estimates on the norms of the chains \(b_x\) we conclude that the image of \(\prod_{x \in \langle G \rangle} b_x\) in \(\prod_{x \in \langle G \rangle} C_{n+1}(\ell_\infty^1 Z_{h_x})\|h_x\|\) has finite norms that grow at most polynomially in \(|h_x|\). Now we apply \(\prod_{x \in \langle G \rangle} t_{h_x}\), where the maps \(t_{h_x}: C_{n+1}(\ell_\infty^1 Z_{h_x})\|h_x\| \to C_{n+1}(\ell_\infty^1 G)_x\) are the inclusions, to arrive at a chain in the space \(\prod_{x \in \langle G \rangle} C_{n+1}(\ell_\infty^1 G)_x\) with an analogous estimate on the norms. Therefore it assembles to a well-defined chain \(b\) in \(C_{n+1}(\ell_\infty^1 G)\).

The claim is now that the Hochschild boundary of \(b\) is \(c\). To see this one applies the chain homotopies between the identities and the compositions \(t_{h_x} \circ \pi_{h_x}\) from Section 2.3. The important observation to do here is that under the assumptions of this lemma they assemble to a single well-defined chain homotopy between the identity and \(\prod_{x \in \langle G \rangle} t_{h_x} \circ \pi_{h_x}\) with the needed polynomial estimates in \(|h_x|\). 

\(\square\)
4 Examples of groups satisfying the assumptions

Semi-hyperbolic groups were introduced by Alonso–Bridson [AB95].

**Theorem 4.1.** Let $G$ be a semi-hyperbolic group satisfying Assumption 1 of Lemma 3.10. Then $G$ satisfies Assumption 2 of that lemma, and hence (3.2) is injective for $G$.

**Proof.** By [BH99, Proposition III.Γ.4.14] we get that each centralizer $Z_{h_x}$ is quasi-convex in $G$. Going into the proof of the cited result, we see that the quasi-convexity constant depends polynomially on $|h_x|$ (here we are already using that Assumption 1 of Lemma 3.10 is satisfied, because otherwise we would only get an exponential dependents).

By [BH99, Proposition III.Γ.4.12] we get that each centralizer $Z_{h_x}$ is quasi-isometrically embedded and semi-hyperbolic itself for a certain choice of finite generating set of it. Going again into the proof we see that the constants of the quasi-isometric embedding depend only on the constants of the semi-hyperbolicity of $G$, i.e., Assumption 2a of Lemma 3.10 is satisfied even with a constant instead of a polynomial. The semi-hyperbolicity constants will depend polynomially on $|h_x|$.

The centralizers $Z_{h_x}$ are of type $F_\infty$ since they are combable; Alonso [Al92, Theorem 2]. That the centralizers $Z_{h_x}$ will have polynomially bounded higher-order Dehn functions was already noticed by Ji–Ramsey [JR09, End of 2nd paragraph on p. 257] since they are polynomially combable (in our case they are even quasi-geodesically combable). Since the hyperbolicity constants of the centralizers depend polynomially on $|h_x|$, the polynomial bounds on the higher-order Dehn functions will be polynomial in $|h_x|$.

Hence we have checked Assumption 2 of Lemma 3.10, and the injectivity statement follows from it. \qed

**Example 4.2.** The following groups are known to be semi-hyperbolic:

1. hyperbolic groups (Alonso–Bridson [AB95]),
2. central extensions of hyperbolic groups (Neumann–Reeves [NR97]),\(^{12}\)
3. CAT(0) groups (Alonso–Bridson [AB95]),
4. systolic groups (Januszkiewicz–Świątkowski [JS06]),
5. Artin groups of finite type (Charney [Cha92]),
6. Artin groups of almost large type (Huang–Osajda [HO17]),
7. right-angled Artin groups (Charney–Davis [CD95] proved they are CAT(0) cube groups, which were shown to be bi-automatic by Niblo–Reeves [NR98], and hence semi-hyperbolicity follows),

\(^{12}\)Note that hyperbolicity is important here. The 3-dimensional integral Heisenberg group is a central extension of $\mathbb{Z} \times \mathbb{Z}$ by $\mathbb{Z}$, but it is not quasi-geodesically combable (since it has a cubic Dehn function, see e.g. [ECH+92, Section 8.1] or [Ger92, Remark 5.9]) and hence can not be semi-hyperbolic.
8. groups acting geometrically and in an order preserving way on Euclidean buildings of the type \(A_n, B_n\) or \(\tilde{C}_n\) (Noskov [Nos00] for the case of groups acting freely, and Świątkowski [Swi06] for the general case).

The above groups are known to be bi-automatic (in the case of CAT(0) groups we have to restrict to CAT(0) cube groups) and hence they have exponentially solvable conjugacy bounds. Let us compile now results from the literature about which of these groups even satisfy the much stronger Assumption 1 of Lemma 3.10.

**Lemma 4.3.** Let \(G\) be a group from one of the following classes of groups:

1. hyperbolic groups,
2. central extensions of hyperbolic groups,
3. Artin groups of extra-large type\(^{13}\) or
4. right-angled Artin groups.

Then \(G\) satisfies Assumption 1 of Lemma 3.10.

**Proof.** The case of hyperbolic groups follows immediately from [BH99, Lemma III.Γ.2.9]. Note that in this case the polynomial \(P(\cdot, \cdot)\) can be chosen to be linear in both variables and its coefficients also depend linearly on \(\delta\), where \(\delta\) is the hyperbolicity constant of \(G\). The extension of the results from hyperbolic groups to central extensions of them was achieved in [Sal16].

That Artin groups of extra-large type have a solvable conjugacy problem was shown in [AS83], and it follows from the proofs given in [HR15] that they even satisfy Assumption 1 of Lemma 3.10 (with a polynomial of degree one in both variables).\(^{14}\)

Right-angled Artin groups (and some of their CAT(0) subgroups) have linearly solvable conjugacy bounds by [LWZ90, CGW09].

**Lemma 4.4.** Let \(G\) be a group which is hyperbolic relative to a finite collection of groups from the four classes in the above Lemma 4.3.

Then \(G\) is semi-hyperbolic and satisfies Assumption 1 of Lemma 3.10.

**Proof.** Building on the thesis of Farb [Far94], Rebbechi [Reb01, Theorem 9.1] showed that if \(G\) is hyperbolic relative to a finite collection of bi-automatic subgroups with prefix-closed normal forms, then \(G\) itself is bi-automatic. Now all the groups stated in Lemma 4.3 are bi-automatic, and an inspection of the corresponding proofs reveals that all these bi-automatic structures can be chosen to have prefix-closed normal forms. Hence \(G\) is bi-automatic and therefore semi-hyperbolic.

Assumption 1 of Lemma 3.10 is proven in [JOR10, Theorem 2.2.10].\(^{15}\)

\(^{13}\)Artin groups of extra-large type are especially of almost large type and hence are bi-automatic. An earlier proof of bi-automaticity of Artin groups of extra-large type was given by Peifer [Pei96].

\(^{14}\)In their paper the algorithm to decide conjugacy runs in cubic time, but the final conjugating element will have a linear bound on its length.
Lemma 4.5. Let $G$ be a CAT(0) group. Then it satisfies Assumption 3 of Lemma 3.10 for its conjugacy classes of finite order elements.$^{15}$

Proof. Follows immediately from [BH99, Corollary III.Γ.1.14]. Note that as for hyperbolic groups the polynomial can be taken to be linear in both variables.

References


$^{15}$It seems that for CAT(0) groups a polynomial bound on the conjugacy problem for elements of infinite order is currently not known. But in the case of groups acting on special CAT(0) cube complexes there is some recent progress [AB18, Theorem B].


