The coarse index class with support

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Abstract

We construct the coarse index class with support condition (as an element of coarse $K$-homology) of an equivariant Dirac operator on a complete Riemannian manifold endowed with a proper, isometric action of a group.

We further show a coarse relative index theorem and discuss the compatibility of the index with the suspension isomorphism.

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1 Introduction

Coarse geometry studies the large-scale geometry of spaces, and such spaces often come with an additional action of a discrete group. An interesting class of examples is provided by the universal coverings of compact spaces with the action of the fundamental groups. Large-scale invariants can be obtained by applying equivariant coarse homology theories.

Coarse (co)homology theories were introduced by Roe [Roe88, Roe93]. Since then they experienced a broad development with many applications to index theory, geometric group theory, isomorphism conjectures like the Baum–Connes and Farrell–Jones conjectures, surgery theory or the classification of metrics of positive scalar curvature.

In order to provide a very general formal framework for coarse geometry we introduced in [BEKW] the category of $G$-bornological coarse spaces. These are $G$-sets equipped with a compatible bornology and coarse structure. The bornology is used in order to encode local finiteness conditions, while the coarse structure captures the large-scale geometry of the space. We further proposed a notion of an equivariant coarse homology theory and studied the universal example.

In the present paper we consider equivariant coarse $K$-homology $K^G_X$ which we construct in detail in [BE].

One of the original sources of interest in coarse geometry was the application to the index theory of Dirac operators. The index of a Dirac operator on a compact manifold is an integer, or equivalently, a class in the $K$-theory of the $C^*$-algebra of compact operators. Roe [Roe93] observed that one can define an index of a Dirac operator on a complete Riemannian manifold as a $K$-theory class of a certain algebra, now called Roe algebra, capturing large-scale properties of the bornological coarse space. This index is interesting since it encodes non-trivial information on the non-invertibility of the Dirac operator.

One possible reason for the invertibility of a Dirac operator and hence the triviality of the index is the positivity of the zero-order term in the Weizenboeck formula, e.g., the scalar curvature of the manifold in the case of the spin Dirac operator. If this term is positive on some subset of the manifold only, then the index information is supported on the complement of that subset. The precise formulation of this fact is due to Roe [Roe16] and leads to the notion of the coarse index class of a Dirac operator with support.

Let $M$ be a complete Riemannian manifold with a proper action of a discrete group $G$ by isometries and let $A$ be an invariant subset of $M$. In this note we work out the details of the construction of the index class

$$\text{Ind}(\mathcal{D}, on(A)) \in K^G_X(\{A\})$$
(Definition 9.5) of a $G$-equivariant (degree $n$)-Dirac operator $\mathcal{D}$ which is uniformly locally positive outside of $A$. We need the technical assumption that $A$ is a support. It requires essentially that invariant coarse thickenings of $A$ are coarsely equivalent to $A$. For a precise formulation we refer to Definition 8.2.

We further show the following basic properties of this index class:

1. locality: Coarse Relative Index Theorem 10.4
2. compatibility with suspension: Theorem 11.1

In [Bun] we show that the presence of this index class together with knowledge of the above two basic properties are sufficient to apply the theory of abstract boundary value problems to Dirac operators. We observe that one can reproduce the main results of Piazza–Schick [PS14] and Zeidler [Zei16] on secondary coarse index invariants by calculations in coarse homotopy theory without opening the black-box of analysis of Dirac operators anymore.

In the following we explain the motivation for this paper and its achievements in greater detail. Let $M$ be a complete Riemannian manifold with a proper action of a discrete group $G$ by isometries. The invariant Riemannian metric of $M$ induces a invariant distance on $M$. We obtain a $G$-bornological coarse space $M$ by equipping $M$ with the coarse structure associated to this distance and the bornology of bounded subsets. We refer to [BEKW] for an introduction to $G$-bornological coarse spaces.

We consider the spectrum-valued equivariant coarse $K$-homology theory $K\chi^G$ constructed in [BE] and recalled here in Definition 5.2. We write $K\chi^G_*$ for the corresponding $\mathbb{Z}$-graded abelian group-valued functor. By $\{A\}$ we denote the big family of invariant thickenings of an invariant subset $A$ of $M$. Then $K\chi^G_n(\{A\})$ is the colimit of the equivariant coarse $K$-homology groups of the members of this family $\{A\}$ in degree $n$.

Let $\mathcal{D}$ be an equivariant $\text{Cl}^n$-invariant Dirac operator on $M$. Assume that $\mathcal{D}$ is uniformly locally positive outside of an invariant subset $A$ of $M$, i.e., the endomorphism term in the Weizenboeck formula for the square of the Dirac operator has a uniform positive lower bound on the complement on $A$. In Definition 9.5 we introduce the index class

$$\text{Ind}(\mathcal{D}, \text{on}(A)) \text{ in } K\chi^G_n(\{A\})$$

(1.1)

of the Dirac operator $\mathcal{D}$ with support on $A$.

The first construction of such an index class in the non-equivariant case was given by Roe [Roe16]. In this reference the index is an element of the $K$-theory of a Roe algebra which in Roe’s notation is written as $C^*(A \subseteq X)$.

For manifolds with a free and proper $G$-action such an index class has been constructed by Piazza–Schick [PS14] and Zeidler [Zei16] Def. 4.5. Zeidler’s index class is an element of the $K$-theory of Yu’s localization algebra denoted in Zeidler’s notation by $C^*_{L,A}(M)^G$. His index class is actually a refinement of the index classes considered in the present paper and by Piazza–Schick. The latter correspond to the image of Zeidler’s index class under the natural evaluation homomorphism $C^*_{L,A}(M)^G \to C^*(A \subseteq X)^G$. We should point out that this notation for the algebras, though standard, is somewhat misleading. These
algebras consist of operators which can be approximated by $G$-invariant finite propagation operators and not, as the notation suggests, of the $G$-invariants in the non-equivariant versions of these algebras. It is not clear that one can interchange the order of taking the closure of the finite-propagation operators and the $G$-invariants.

It is one of the themes of [Bun] that the analog of Zeidler’s refined index class can be reconstructed from the index classes considered in the present paper. The assumption made by Zeidler that the action of the group is free leads to the simplification that the controlled Hilbert space obtained from the $L^2$-sections of the bundle on which the Dirac operator acts is already ample. For proper actions this is not the case and a stabilization is necessary. In the present paper we in particular also provide the generalization of the construction of the index class from free to proper actions.

We now come to the main point of the present paper. The index classes discussed above are $K$-theory classes of $C^*$-algebras. The $K$-theory groups of these $C^*$-algebras can be used to construct a $\mathbb{Z}$-graded group-valued functor on sufficiently nice metric spaces with $G$-action which sends coarse equivalences to isomorphisms and have a Mayer–Vietoris sequence for equivariant coarsely excisive decompositions. In [BE16] (the non-equivariant case) and [BEKW] (the equivariant case) we upgraded the properties of such functors to the notion of an equivariant coarse homology theory defined on the category of $G$-bornological coarse spaces. In particular, the equivariant coarse $K$-homology theory $K\mathcal{X}_G$ is such a coarse homology theory which takes values in the stable $\infty$-category of spectra. In order to ensure a spectrum-level functoriality, we define the equivariant coarse $K$-homology spectrum $K\mathcal{X}_G^G(X)$ in terms of a $C^*$-category of locally finite equivariant $X$-controlled Hilbert spaces. For a complete Riemannian manifold $M$ with a proper isometric $G$-action and an invariant subset $A$ the comparison of the equivariant coarse $K$-homology groups $K\mathcal{X}_G^G(\{A\})$ with the $K$-theory groups of the Roe-algebra $K_*(C^*(A \subseteq M))$ is non-trivial.

In the non-equivariant case [BE16] we have constructed a natural isomorphism between the classical group-valued coarse $K$-homology functor defined using ample $M$-controlled Hilbert spaces and the functor $K\mathcal{X}_G^G$. Using this we got a canonical isomorphism of abelian groups

$$K_n(C^*(A \subseteq X)) \cong K\mathcal{X}_n(\{A\})$$

which can be used to turn Roe’s index class into a coarse $K$-homology class (1.1) as desired. We refer to [BE16, Sec. 7.9] for details.

The main achievement of the present paper is the equivariant generalization of the above identification. As we shall see in Section [9] the constructions the index classes by Roe or Zeidler easily generalize to Riemannian manifolds with a proper isometric action by $G$. So the main task is to generalize the identification of the relevant $K$-theory groups to the equivariant case. Though the general ideas are very similar to the non-equivariant case, the details are considerably more difficult. In particular, we need quite strong properness assumptions on the action of $G$ which are fortunately satisfied in the case of a proper isometric action on a complete Riemannian manifold.

In order to deal with the degree $n$ in a simple and uniform manner, following Zeidler [Zei16], we use operators commuting with an action of the Clifford algebra $\mathbb{C}I^n$. We are
further following this reference by representing the index class by a homomorphism from the graded algebra \( C_0(\mathbb{R}) \) to the Roe algebra obtained by functional calculus applied to the Dirac operator. This idea avoids considering the exact sequences used in Roe’s approach which are usually written as
\[
0 \to C^*(M) \to D^*(M) \to Q^*(M) \to 0.
\]

The definition of the \( C^* \)-algebras \( D^*(X) \) and \( Q^*(X) \) involves continuously controlled Hilbert spaces whose functorial properties are considerably more involved. In contrast to Zeidler [Zei16] we only consider Roe algebras and do not consider localization algebras.

**Example 1.1.** Here is the typical example. Let \( M \) be an \( n \)-dimensional Riemannian spin manifold with a proper action of \( G \) by isometries and assume that the scalar curvature is bounded below by a positive constant on the subset \( M \setminus A \). Denote by \( P \to M \) the \( G \)-equivariant \( \text{Spin}(n) \)-principal bundle. We can consider \( \text{Spin}(n) \) as a subgroup of the units in \( \mathbb{C}l^n \) which acts on \( \mathbb{C}l^n \) by left-multiplication. Then we define the Dirac bundle \( E := P \times_{\text{Spin}(n)} \mathbb{C}l^n \) with the \( \mathbb{C}l^n \)-action given by right-multiplication. Let \( \mathcal{D}^\text{spin}_M \) be the associated \( G \)-invariant \( \mathbb{C}l^n \)-linear Dirac operator on \( E \). Then we get a class
\[
\text{Ind}(\mathcal{D}^\text{spin}_M, \text{on}(A)) \in K\mathcal{X}_G^n(\{A\}),
\]
which is the example which one usually encounters.

In the present note we verify the following two basic properties of the coarse index classes.

The first is locality which is technically expressed by the Coarse Relative Index Theorem 10.4 stated in detail and shown in Section 10.

The second is Theorem 11.1 on the compatibility of the suspension isomorphism in \( K\mathcal{X}_G^n \) with the suspension of Dirac operators, i.e., the transition from the Dirac operator \( \mathcal{D} \) of degree \( n \) on \( M \) to its canonical extension \( \tilde{\mathcal{D}} \) of degree \( n + 1 \) on \( \mathbb{R} \times M \). We show that
\[
\partial(\text{Ind}(\tilde{\mathcal{D}}, \text{on}(\mathbb{R} \times A))) = \text{Ind}(\mathcal{D}, \text{on}(A)),
\tag{1.2}
\]
where
\[
\partial : K\mathcal{X}_G^n(\{\mathbb{R} \times A\}) \to K\mathcal{X}_G^n(\{A\})
\]
is the boundary operator of the Mayer–Vietoris sequence associated to the equivariant decomposition \( ((-\infty, 0] \times M, [0, \infty) \times M) \) of \( \mathbb{R} \times M \).

The proofs of these two properties are non-formal and based on analytic facts.

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2 X-controlled Hilbert spaces

In this section we introduce the notion of an equivariant X-controlled Hilbert space and the notions of controlled and of locally compact operators. To keep the paper self-contained we reproduce some definitions from [BE16] and [BE].

Let $G$ be a group. A $G$-Hilbert space is a Hilbert space with a unitary action of $G$. Similarly, a $G$-$\ast$-algebra is a $\ast$-algebra with an action of $G$ by automorphisms of $\ast$-algebras. If the underlying algebra is a $C^\ast$-algebra, then we say that $A$ is a $G$-$C^\ast$-algebra. For example, the algebra of bounded operators $B(H)$ on a $G$-Hilbert space $H$ is a $G$-$C^\ast$-algebra.

If $A$ is a $G$-$\ast$-algebra, then the subset of $G$-invariants $A^G$ is a $\ast$-algebra. Similarly, if $A$ is a $G$-$C^\ast$-algebra, then $A^G$ is a $C^\ast$-algebra.

We consider a $G$-bornological coarse space $X$.

**Definition 2.1.** An equivariant $X$-controlled Hilbert space is a pair $(H,\phi)$ of an Hilbert space with a unitary action of $G$ and a finitely-additive projection-valued measure $\phi$ defined on all subsets of $X$ such that $H(B)$ is separable for every bounded subset of $X$ and $\phi$ is equivariant.

In other words, for every $g$ in $G$ and subset $Y$ of $X$ we have the equality of projections

$$g\phi(Y)g^{-1} = \phi(g^{-1}Y).$$

In Definition 2.1, we use the notation $H(Y) := \phi(Y)(H)$ for the subspaces of $H$ corresponding to subsets $Y$ of $X$.

Furthermore, let $(H, \phi)$ and $(H', \phi')$ be two equivariant $X$-controlled Hilbert spaces and $A : H \to H'$ be a bounded operator.

**Definition 2.2.**

1. $A$ is locally compact if for every bounded subset $B$ of $X$ the operators $\phi'(B)A$ and $A\phi(B)$ are compact.

2. $A$ is controlled if there exists an entourage $U$ of $X$ such that $\phi'(Y')A\phi(Y) = 0$ for every two subsets $Y$ and $Y'$ of $X$ such that $U[Y] \cap Y' = \emptyset$.

3. $A$ is invariant, if $gAg^{-1} = A$ for every $g$ in $G$.

Assume now that $X$ is a $G$-bornological coarse space. Let furthermore $(H, \phi)$ and $(H', \phi')$ be two equivariant $X$-controlled Hilbert spaces.

**Definition 2.3.** We let $C_{lc}(X, (H', \phi'), (H, \phi))$ denote closure (in operator norm) of the set of invariant bounded operators from $H'$ to $H$ which are locally compact and controlled.

The collection of spaces $C_{lc}(X, (H', \phi'), (H, \phi))$ for all pairs of equivariant $X$-controlled Hilbert spaces is closed under taking adjoints or forming linear combinations or compositions. In particular

$$C_{lc}(X, H, \phi) := C_{lc}(X, (H, \phi), (H, \phi))$$

is a $C^\ast$-algebra.
Remark 2.4. Note that $C_{lc}(X, (H', \phi'), (H, \phi))$ consists of $G$-invariant operators. We will refrain from using the notation $C_{lc}(X, (H', \phi'), (H, \phi))^G$ since this is misleading. We would use this notation for the $G$-invariants in the Banach space $C_{lc}({\text{Res}}^G_1(X, (H', \phi'), (H, \phi)))$ (this means that we first forget the $G$-action on the data, then form the Banach space of operators in the non-equivariant case, and finally take the invariants under the $G$-action). The space $C_{lc}(X, (H', \phi'), (H, \phi))$ might be strictly contained in $C_{lc}(X, (H', \phi'), (H, \phi))^G$, i.e., they might be actually different spaces.

3 Locally finite $X$-controlled Hilbert spaces

In order to define equivariant coarse $K$-homology $KX^G(X)$ we consider the $C^*$-category of equivariant $X$-controlled Hilbert spaces with the morphism spaces defined in Definition 2.3. In order to ensure that this $C^*$-category is unital we must restrict to locally finite equivariant $X$-controlled Hilbert spaces. In this section we recall this notion and prove some technical results which will be crucial later.

Let $X$ be a $G$-bornological coarse space.

**Definition 3.1.** An equivariant $X$-controlled Hilbert space is determined on points if the natural map $\bigoplus_{x \in X} H(\{x\}) \to H$ is an isomorphism.

The sum above is understood in the sense of Hilbert spaces, i.e., it involves a completion. Let $(H, \phi)$ be an equivariant $X$-controlled Hilbert space.

**Definition 3.2.** The support $\text{supp}(H, \phi)$ of $(H, \phi)$ is the subset $\{x \in X \mid H(\{x\}) \neq 0\}$ of the space $X$.

Note that the support of an equivariant $X$-controlled Hilbert space is an invariant subset of $X$.

Recall that a subset $S$ of $X$ is called locally finite, if the intersection $S \cap B$ is finite for every bounded subset $B$ of $X$.

**Definition 3.3.** An equivariant $X$-controlled Hilbert space is locally finite if the following conditions are satisfied:

1. $(H, \phi)$ is determined on points.
2. The support of $(H, \phi)$ is locally finite.
3. For every $x$ in $X$ the space $H(\{x\})$ is finite-dimensional.

If $(H, \phi)$ is locally finite, then for every bounded subset $B$ of $X$ the space $H(B)$ is finite-dimensional.

Let $(H, \phi)$ be an equivariant $X$-controlled Hilbert space, not necessarily locally finite.
Definition 3.4. A closed invariant subspace $\tilde{H}$ of $H$ is called locally finite if there exists a locally finite equivariant $X$-controlled Hilbert space $(\tilde{H}', \phi')$ and a controlled equivariant unitary embedding $U : \tilde{H}' \to H$ with image $\tilde{H}$.

Let $A$ be a bounded operator on $H$.

Definition 3.5. We say that $A$ is locally finite, if there exist locally finite subspaces $\tilde{H}_0$ and $\tilde{H}_1$ of $H$ and an operator $\tilde{A} : \tilde{H}_0 \to \tilde{H}_1$ such that $A = j_1 \circ \tilde{A} \circ p_0$, where $p_0$ is the orthogonal projection from $H$ to $\tilde{H}_0$ and $j_1$ is the inclusion of $\tilde{H}_1$ into $H$.

Note that a locally finite operator is automatically locally compact.

Definition 3.6. We let $C(X, H, \phi)$ be the closure of the subset of locally finite operators in $C_{lc}(X, H, \phi)$.

With the structures induced from $C_{lc}(X, H, \phi)$ the subset $C(X, H, \phi)$ is a $C^*$-algebra.

We now consider the possibility of approximating locally compact operators by locally finite ones. Our approach uses the existence of sufficiently nice equivariant partitions of the space $X$.

Let $X$ be a $G$-set. An equivariant family of subsets $(Y_i)_{i \in I}$ of $X$ is a family of subsets where $I$ is a $G$-set and we have the equality $gY_i = Y_{gi}$ for every $i$ in $I$ and $g$ in $G$.

Let $X$ be a $G$-bornological coarse space. Recall that the action of $G$ on $X$ is called proper if for every bounded subset $B$ the set $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite. Properness only involves the bornology of $X$. For our comparison results we need a stronger condition which we call very proper.

Let $X$ be a $G$-bornological coarse space.

Definition 3.7. We say that $X$ is very proper if for every entourage $U$ of $X$ there exists an equivariant partition $(B_i)_{i \in I}$ of $X$ such that

1. The partition is uniformly bounded, i.e., there exists an entourage $V$ of $X$ such that every subset $B_i$ is $V$-bounded,

2. It has finite stabilizers,

3. for every $i$ in $I$ the set $\{j \in I \mid U[B_i] \cap B_j \neq \emptyset\}$ is finite,

4. for every $i$ in $I$ there exists a point in $B_i$ which is fixed by the stabilizer of $i$,

5. for every bounded subset $B$ of $X$ the set $\{i \in I \mid B_i \cap B \neq \emptyset\}$ is finite, and

6. the set of orbits $I/G$ is countable.

If $X$ also has the structure of a $G$-measurable space, we say that $X$ is measurably very proper if we in addition can assume that the members of the partition are measurable.
The following proposition ensures that under suitable conditions the usual definition of the Roe algebra using locally compact operators coincides with the definition that we use which employs locally finite operators.

Let $X$ be a $G$-bornological coarse space and $(H, \phi)$ an equivariant $X$-controlled Hilbert space.

**Proposition 3.8.** If $X$ is very proper and $(H, \phi)$ is determined on points, then the natural inclusion is an equality \( C(X, H, \phi) = C_{lc}(X, H, \phi) \).

**Proof.** We consider an operator $A$ in $C_{lc}(X, H, \phi)$. We will show that for every given $\varepsilon$ in $(0, \infty)$ there exists a locally finite subspace $H'$ of $H$ and an operator $A': H' \to H'$ in $C_{lc}(X, H, \phi)$ such that $\|A - A'\| \leq \varepsilon$. Here we omit to write the projection from $H$ to $H'$ in front of $A'$ and the inclusion of $H'$ into $H$ after $A'$.

As a first step we choose an invariant locally compact operator $A_1$ of controlled propagation such that $\|A - A_1\| \leq \varepsilon/2$. This is possible by the definition of $C_{lc}(X, H, \phi)$.

Since $X$ is very proper we can choose an equivariant partition $(B_i)_{i \in I}$ of $X$ with the properties listed in Definition 3.7 for the invariant entourage $\text{supp}(A_1)$ in place of $U$.

We assume that $I/G$ is infinite (the argument in the finite case is similar, but simpler). Then by Assumption 3.7.6 we can assume that $I = \bigsqcup_{n \in \mathbb{N}} I_n$ for transitive $G$-sets $I_n$. For every integer $n$ we choose a base point $i_n$ in $I_n$.

For every integer $n$ let $P_{i_n}$ be a finite-dimensional $G_{i_n}$-invariant projection on $H(B_{i_n})$. We will fix this projection later. Since $B_{i_n}$ is $G_{i_n}$-invariant, where $G_{i_n}$ is the stabilizer of $i_n$, we can approximate the identity of $H(B_{i_n})$ strongly by such projections. Here we use the Assumption 3.7.2 which implies that $G_{i_n}$ is finite. For $i$ in $I_n$ we define $P_i := gP_{i_n}g^{-1}$, where $g$ in $G$ is such that $g_{i_n} = i$. This projection on $H(B_i)$ is well-defined, and the family of projections $(P_i)_{i \in I}$ is $G$-invariant in the sense that $gP_ig^{-1} = P_{gi}$ for every $i$ in $I$ and $g$ in $G$.

We now define the $G$-invariant operator

$$A' := \sum_{i,j \in I} P_i A_1 P_j .$$

Then

$$A_1 - A' = \sum_{i,j \in I} \phi(B_i)A_1\phi(B_j) - \sum_{i,j \in I} P_i A_1 P_j = \sum_{i,j \in I} (\phi(B_i) - P_i A_1\phi(B_j)) + \sum_{i,j \in I} P_i A_1 (\phi(B_j) - P_j) .$$

For every $j$ in $I$ we define the set

$$I_j := \{i \in I \mid B_i \cap \text{supp}(A_1)[B_j] \neq \emptyset\} .$$
This set is finite by Assumption 3.7.3. Then we have

\[ A_1 - A' = \sum_j \sum_{i \in I_j} (\phi(B_i) - P_i)A_1\phi(B_j) + \sum_j \sum_{i \in I_j} P_iA_1(\phi(B_j) - P_j). \]

Using the orthogonality of the terms for different indices \( i \) we get the estimates

\[ \|A_1 - A'\| \leq \max_{i \in I} \sum_{\{j \in I | i \in I_j\}} \|\phi(B_i) - P_i\|A_1 + \max_{i \in I} \sum_{\{j \in I | i \in I_j\}} \|A_1(\phi(B_j) - P_j)\|. \]

The index sets of these sums are finite by Assumption 3.7.3. For \( i \) in \( I \) we set

\[ \ell_i := |\{j \in I | i \in I_j\}|. \]

Since \( A_1 \) is locally compact we can make \( \|A_1(\phi(B_i) - P_i)\| \) and \( \|\phi(B_i) - P_i\|A_1\| \) as small as we want by choosing \( P_i \) sufficiently big.

If we choose for every integer \( n \) the projection \( P_{in} \) so large that

\[ \|\phi(B_{in}) - P_{in}\|A_1\| \leq \frac{\varepsilon}{4\ell_{in}}, \quad (3.1) \]

then we get (using \( G \)-invariance)

\[ \max_{i \in I} \sum_{\{j \in I | i \in I_j\}} \|\phi(B_i) - P_i\|A_1 \leq \frac{\varepsilon}{4}. \]

In order to deal with the second term we must increase the projections further. We proceed by induction. Assume that we have chosen the projections such that

\[ \max_{i \in G\{i_0,i_1,...,i_n\}} \sum_{\{j \in I | i \in I_j\}} \|A_1(\phi(B_j) - P_j)\| \leq \frac{\varepsilon}{4}. \]

In the next step we further increase the projections \( P_{im} \) for all integers \( m \) with \( i_{n+1} \in GI_{im} \) (these are finitely many by Assumption 3.7.3) such that

\[ \max_{i \in G\{i_0,i_1,...,i_{n+1}\}} \sum_{\{j \in I | i \in I_j\}} \|A_1(\phi(B_j) - P_j)\| \leq \frac{\varepsilon}{4} \]

and (3.1) is satisfied. We now observe that with this procedure we increase every projection at most a finite number of times (by Assumption 3.7.3).

Let \( H' \) be the sub-Hilbert space of \( H \) spanned by the images of \( P_i \) for all \( i \) in \( I \). Then \( A' \) factorizes over \( H' \) and

\[ \|A - A'\| \leq \varepsilon. \]

By 3.7.1 can choose an entourage \( V \) which bounds the members of the partition. Then the propagation of \( A' \) is bounded by \( V \circ \text{supp}(A_1) \circ V \).

We now construct a control \( \phi' \) for \( H' \) exhibiting this subspace as a locally finite subspace. For every integer \( n \) we choose a point \( b_{in} \) in \( B_{in} \) which is fixed by \( G_{in} \). This is possible by...
Assumption 3.7.4. For \( i \) in \( I_n \) we then define \( b_i := g b_{i_n} \) where \( g \) in \( G \) is such that \( g i_n = i \). Note that \( b_i \) is well-defined. The collection of points \((b_i)_{i \in I}\) thus defined is equivariant in the sense that \( g b_i = b_{g i} \) for every \( i \) in \( I \) and \( g \) in \( G \).

We now define the equivariant projection-valued measure

\[
\phi' := \sum_{i \in I} \delta_{b_i} P_i
\]

on \( H' \). It turns \((H', \phi')\) into an equivariant \( X \)-controlled Hilbert space which is determined on points. It is locally finite by Assumption 3.7.5 and the fact that \( P_i \) is finite-dimensional for every \( i \) in \( I \).

Finally we see that the inclusion \( H' \to H \) is \( V \)-controlled. Hence \( H' \) is a locally finite subspace of \( H \).

Let \( X \) be a \( G \)-bornological coarse space and \((H, \phi)\) be an equivariant \( X \)-controlled Hilbert space. Let \( H' \) and \( H'' \) be two subspaces of \( H \). Then we define

\[
H' + H'' := \overline{H' + H''}.
\]

The following rather technical result will be needed in the proof of Theorem 6.1.

**Proposition 3.9.** Assume that

1. \( X \) is very proper,
2. \((H, \phi)\) is determined on points, and
3. \( H' \) and \( H'' \) are locally finite.

Then \( H' + H'' \) is locally finite.

**Proof.** By assumption we can find an symmetric invariant entourage \( V \) of \( X \) and equivariant projection-valued measures \( \phi' \) and \( \phi'' \) on \( H' \) and \( H'' \) such that \((H', \phi')\) and \((H'', \phi'')\) are locally finite equivariant \( X \)-controlled Hilbert spaces and the inclusions of \( H' \) and \( H'' \) into \( H \) have propagation controlled by \( V \).

Since \( X \) is very proper we can find an equivariant partition \((B_i)_{i \in I} \) of \( X \) with the properties listed in Definition 3.7 for the entourage \( V \).

We assume that \( I / G \) is infinite (the argument in the finite case is similar, but simpler). Then by 3.7.6 we can assume that \( I = \bigsqcup_{n \in \mathbb{N}} I_n \) for transitive \( G \)-sets \( I_n \). For every integer \( n \) we choose a base point \( i_n \) in \( I_n \).

For every integer \( n \) we choose a point \( b_{i_n} \) in \( B_{i_n} \) which is fixed by the stabilizer \( G_{i_n} \) of \( i_n \). This is possible by Assumption 3.7.4. For \( i \) in \( I_n \) we then define \( b_i := g b_{i_n} \) where \( g \) in \( G \) is such that \( g i_n = i \). Note that \( b_i \) is well-defined. The collection of points \((b_i)_{i \in I}\) thus defined is equivariant.

We will construct the equivariant projection-valued measure \( \psi \) on \( H' + H'' \) by induction.
Because of the inclusion 
\[ \phi(B_{i_0})H' \subseteq \phi'(V[B_{i_0}])H' \]
the subspace \( \phi(B_{i_0})H' \) of \( H \) is finite-dimensional. Analogously we conclude that \( \phi(B_{i_0})H'' \) is finite-dimensional. Consequently, \( \phi(B_{i_0})(H' + H'') = \phi(B_{i_0})(H' + H'') \) is finite-dimensional. We define the subspace 
\[ H_{i_0} := \phi(B_{i_0})(H' + H'') \]
of \( H \) and let \( Q_{i_0} \) be the orthogonal projection onto \( H_{i_0} \). Note that \( H_{i_0} \) is preserved by \( G_{i_0} \) and that \( Q_{i_0} \) is \( G_{i_0} \)-invariant. We then define, for all \( i \) in \( I_0 \), the subspaces \( H_i := gH_{i_0} \) and \( Q_i := gQ_{i_0}g^{-1} \), where \( g \) is such that \( gi_0 = i \). Note that these objects are well-defined. Furthermore, the spaces \( H_i \) for all \( i \) in \( I_0 \) are mutually orthogonal.

We let \( H_0 \) be the Hilbert subspace generated by the spaces \( H_i \) for all \( i \) in \( I_0 \). We further define the equivariant projection-valued measure \( \psi_0 \) on \( H_0 \) by
\[ \psi_0 := \sum_{i \in I_0} \delta_i Q_i . \]

Using \( \psi_0 \) we recognize \( H_0 \subseteq H \) as a locally finite subspace.

Let now \( n \) be an integer and assume that we have constructed an invariant subspace \( H_n \) and an equivariant projection-valued measure \( \psi_n \) recognizing the subspace \( H_n \) of \( H \) as a locally finite subspace.

As above we observe that \( \phi(B_{i_{n+1}})(H' + H'') \) is finite-dimensional. We define the closed subspace 
\[ \tilde{H}_{i_{n+1}} := H_n + \phi(B_{i_{n+1}})(H' + H'') \]
of \( H \) and we let \( Q_{i_{n+1}} \) be the orthogonal projection onto \( H_{i_{n+1}} := \tilde{H}_{i_{n+1}} \ominus H_n \). We note that \( H_{i_{n+1}} \) and \( Q_{i_{n+1}} \) are \( G_{i_{n+1}} \)-invariant. For every \( i \) in \( I_{n+1} \) we now define \( H_i := gH_{i_{n+1}} \) and \( Q_i := gQ_{i_{n+1}}g^{-1} \), where \( g \) in \( G \) is such that \( gi_{n+1} = i \). Note that these objects are well-defined. Furthermore, the spaces \( H_i \) for \( i \) in \( I_{n+1} \) are mutually orthogonal and orthogonal to \( H_n \).

We let \( H_{n+1} \) be the Hilbert space generated by \( H_n \) and all the spaces \( H_i \) for \( i \) in \( I_{n+1} \). This space is \( G \)-invariant and we can define the equivariant projection-valued measure \( \psi_{n+1} \) on \( H_{n+1} \) by
\[ \psi_{n+1} := \psi_n + \sum_{i \in I_{n+1}} \delta_i Q_i . \]

It recognizes \( H_{n+1} \) as a locally finite subspace of \( H \).

We now observe that
\[ \psi := \sum_{n \in \mathbb{N}} \sum_{i \in I_n} \delta_i Q_i \]
is an equivariant projection-valued measure on the sum \( H' + H'' \), and that the inclusion of this sum into \( H \) is \( V \)-controlled, if \( V \) is a bound for the size of the members of the partition \( (B_i)_{i \in I} \). Furthermore, \( (H' + H'', \psi) \) is locally finite. \( \square \)
4 Existence of ample $X$-controlled Hilbert spaces

Let $X$ be a $G$-bornological coarse space and $(H, \phi)$ an equivariant $X$-controlled Hilbert space.

**Definition 4.1.** $(H, \phi)$ is called ample if it is determined on points and for every equivariant $X$-controlled Hilbert space $(H', \phi')$ there exists a controlled unitary inclusion $H' \to H$.

Let $X$ be a $G$-bornological coarse space.

**Proposition 4.2.** If $X$ is very proper, then $X$ admits an ample equivariant $X$-controlled Hilbert space.

**Proof.** Since $X$ is very proper we can find an equivariant partition $(B_i)_{i \in I}$ of $X$ with the properties listed in Definition 3.7 for some entourage $U$.

We assume that $I/G$ is infinite (the argument in the finite case is similar, but simpler). By Assumption 3.7.6 we can then assume that $I = \bigsqcup_{n \in \mathbb{N}} I_n$ for a family $(I_n)_{n \in \mathbb{N}}$ of transitive $G$-sets. For every $n \in \mathbb{N}$ we choose a base point $i_n \in I_n$.

For every $n \in \mathbb{N}$ we consider the $G$-Hilbert space $H_n := (L^2(G) \otimes L^2(G_{i_n}))^{G_{i_n}} \otimes \ell^2$, where the $G_{i_n}$-invariants are taken with respect to the action of $G_{i_n}$ on $L^2(G)$ by right-multiplication on $G$ and on $L^2(G_{i_n})$ by the left-multiplication on $G_{i_n}$. The $G$-action on $H_n$ is induced from the left-multiplication of $G$ on itself. We then define the $G$-Hilbert space

$$H := \bigoplus_{n \in \mathbb{N}} H_n.$$ 

We interpret the elements of $H_n$ as functions on $G$ with values in $L^2(G_{i_n}) \otimes \ell^2$. For every $n \in \mathbb{N}$ and $i$ in $I_n$ we let $Q_i$ be the projection onto the subspace $H_i$ of functions in $H_n$ supported on the subset $\{g \in G \mid gi_n = i\}$ of $G$. Since this subset is preserved by the right-action $G_{i_n}$ the support condition is compatible with the $G_{i_n}$-invariance condition, and $H$ is spanned by these subspaces $H_i$ for all $i$ in $I$. We further note that $gQ_ig^{-1} = Q_{gi}$.

We define the measure

$$\phi := \sum_{i \in I} \delta_{b_i} Q_i$$

which turns out to be invariant. In this way we get an equivariant $X$-controlled Hilbert space $(H, \phi)$. It is determined on points.
We now show that \((H, \phi)\) is ample. Let \((H', \phi')\) be any equivariant \(X\)-controlled Hilbert space. Then we must construct a controlled unitary embedding \(H' \to H\).

For every \(i\) in \(I\) we consider the subspace \(H_i' := H'_i(B_i)\). Since the subset \(B_i\) is bounded and \(G_i\)-invariant, this is a separable representation of the finite (by Assumption 3.7.2) group \(G_i\).

The action of \(G\) on \(H\) restricts to an action of \(G_i\) on \(H_i\) which is of the form \(L^2(G_i) \otimes \ell^2\). By the Peter–Weyl theorem it contains every irreducible representation of \(G_i\) with countably infinite multiplicity. Hence for every \(n\) we can choose a unitary embedding \(u_{i_n} : H_{i_n}' \to H_{i_n}\) which is \(G_{i_n}\)-invariant. We extend this \(G\)-equivariantly by

\[ u_i := gu_{i_n}g^{-1} : H_i' \to H_i, \]

where \(g\) is any element of \(G\) such that \(g_{i_n} = i\). The sum \(u := \oplus_{i \in I} u_i\) is then an equivariant embedding of \(H'\) into \(H\). By Assumption 3.7.2 there exists an entourage \(V\) of \(X\) which bounds the size of the members of the partition. The embedding \(u\) is then \(V\)-controlled.

**Remark 4.3.** The above proof would also work without Assumption 3.7.6 that \(I/G\) is countable. Moreover we have neither used Assumption 3.7.3 nor Assumption 3.7.5.

### 5 \(C^*\)-categories and the construction of \(K\mathcal{X}^G\)

In this section we introduce the \(C^*\)-categories of locally finite equivariant controlled Hilbert spaces and define the equivariant coarse \(K\)-homology functor. This material is taken from [BE] and reproduced here for the sake of self-containedness.

**Definition 5.1.** We let \(\mathbf{C}(X)\) be the \(\mathbb{C}\)-linear \(*\)-category given by the data that

1. the objects of \(\mathbf{C}(X)\) are equivariant locally finite \(X\)-controlled Hilbert spaces, and
2. the \(\mathbb{C}\)-vector space of morphisms between the objects \((H', \phi')\) and \((H, \phi)\) is defined as \(\mathbf{C}_{lc}(X, (H', \phi'), (H, \phi))\).

Note that any operator between locally finite \(X\)-controlled Hilbert spaces is locally compact. One can check that \(\mathbf{C}(X)\) is a \(C^*\)-category. As explained in [Bun16], being a \(C^*\)-category is a property of a \(\mathbb{C}\)-linear \(*\)-category.

If \(f : X \to X'\) is a morphism of \(G\)-bornological coarse spaces, then we get a functor between \(C^*\)-categories

\[ f_* : \mathbf{C}(X) \to \mathbf{C}(X') \]

given as follows:

1. \(f_*\) sends the object \((H, \phi)\) to \((H, f_* \phi)\), and
2. \(f_*\) is given by the identity on morphisms.
If $g: X' \to X''$ is a second morphism, then we have the equality $(g \circ f)_* = g_* \circ f_*$. We thus have defined a functor

$$C: GBornCoarse \to C^*\text{-Cat},$$

where $C^*\text{-Cat}$ denotes the category of $C^*$-categories and functors. We let

$$K: C^*\text{-Cat} \to \text{Sp}$$

be a topological $K$-theory functor from $C^*$-categories to spectra (a discussion of constructions of such a functor may be found in [BE16, Sec. 7.4 & 7.5]). Note that a $C^*$-algebra can be considered as a $C^*$-category with one object. A basic requirement for $K$ is that it sends a $C^*$-algebra to a spectrum whose homotopy groups are isomorphic to the $K$-theory groups of the $C^*$-algebra.

**Definition 5.2.** We define the equivariant coarse $K$-homology functor

$$K\chi^G: GBornCoarse \to \text{Sp}$$

by

$$K\chi^G := K \circ C.$$ 

**Theorem 5.3 ([BE]).** $K\chi^G$ is an equivariant coarse homology theory.

In this paper we will not recall the notion of an equivariant coarse homology theory but rather refer to [BEKW] for definitions. But we remark that $K\chi^G$ in particular sends coarse equivalences to equivalences of spectra and satisfies excision for invariant, coarsely excisive decompositions.

### 6 Comparison with the $K$-theory of Roe algebras

Let $X$ be a $G$-bornological coarse space and let $(H, \phi)$ be an equivariant $X$-controlled Hilbert space. Let $X'$ and $(H', \phi')$ be similar data.

**Theorem 6.1.**

1. Assume that $X$ is very proper and that $(H, \phi)$ is ample. Then there exists a canonical (up to equivalence) equivalence of spectra

$$\kappa_{(X; H, \phi)}: K(C(X, H, \phi)) \to K\chi^G(X).$$

2. Assume that $X$ and $X'$ are very proper and that $(H, \phi)$ and $(H', \phi')$ are ample. For a morphism $f: X' \to X$ of $G$-bornological coarse spaces and an equivariant unitary
inclusion \( V : H' \to H \) which induces a controlled morphism \( f_*(H', \phi') \to (H, \phi) \) we have a commuting diagram

\[
K(C(X', H', \phi')) \xrightarrow{K_f} KX^G(X') \quad (6.1)
\]

\[
K(C(X, H, \phi)) \xrightarrow{K_0} KX^G(X)
\]

**Proof.** Given Proposition [3.9] the proof is analogous to the proof of [BE16, Thm. 7.70]. Since we need some of the details later we will recall the construction of the equivalence \( \kappa(X, H, \phi) \).

We consider the category \( C(X, H, \phi) \) whose objects are triples \( (H', \phi', U) \), where \( (H', \phi') \) is an object of \( C(X) \), and \( U \) is an equivariant controlled inclusion \( U : H' \to H \) as a locally finite subspace. We consider the Roe algebra \( C(X, H, \phi) \) as a non-unital \( C^* \)-category \( C(X) \) with a single object. Then we have functors between non-unital \( C^* \)-categories

\[
C(X) \xrightarrow{F} C(X, H, \phi) \xrightarrow{I} C(X, H, \phi)
\]

(6.2)

The functor \( F \) forgets the inclusions. The action of \( I \) on objects is clear. Furthermore it maps the morphism \( A : (H', \phi', U') \to (H'', \phi'', U'') \) in \( C(X, H, \phi) \) to the morphism \( I(A) := U''AU'^{\ast} \) of \( C(X, H, \phi) \).

We get an induced diagram of \( K \)-theory spectra

\[
K(C(X)) \xrightarrow{K_F} K(C(X, H, \phi)) \xrightarrow{K_I} K(C(X, H, \phi)) \quad (6.3)
\]

Using Proposition [3.9] as in the proof of [BE16, Thm. 7.70] we now see that \( K(F) \) and \( K(I) \) are equivalences. We set \( \kappa(X, H, \phi) := K(F) \circ K(I)^{-1} \).

The proof of the second assertion is the same as in [BE16, Thm. 7.70].

Combining Theorem 6.1 with Proposition 3.8 we immediately get the following corollary.

Let \( X \) be a \( G \)-bornological coarse space and let \( (H, \phi) \) be an equivariant \( X \)-controlled Hilbert space.

**Corollary 6.2.** If \( X \) is very proper and \((H, \phi)\) is ample, then we have a canonical (up to equivalence) equivalence of spectra \( K(C_{lc}(X, H, \phi)) \simeq KX^G(X) \).
7 Incorporating the degree

In this section graded means $\mathbb{Z}/2\mathbb{Z}$-graded. Let $\mathbb{C}^n$ denote the graded complex Clifford algebra of the Euclidean space $\mathbb{R}^n$.

Let $X$ be a $G$-bornological coarse space and let $n$ be in $\mathbb{N}$.

**Definition 7.1.** An equivariant $X$-controlled Hilbert space of degree $n$ is an equivariant $X$-controlled Hilbert space $(H, \phi)$ with an additional grading and a graded action of $\mathbb{C}^n$. We require that the measure $\phi$ commutes with the action of $\mathbb{C}^n$ and preserves the grading.

We use the notation $(H, \phi, n)$ in order to denote an equivariant $X$-controlled Hilbert space of degree $n$.

Let $(H, \phi, n)$ be an equivariant $X$-controlled Hilbert space of degree $n$.

**Definition 7.2.** We let $C_{\text{lc}}(X, H, \phi, n)$ be the graded $C^*$-algebra of operators in $C_{\text{lc}}(X, H, \phi)$ which commute with the action of $\mathbb{C}^n$.

**Definition 7.3.** An equivariant $X$-controlled Hilbert space $(H, \phi, n)$ of degree $n$ is ample if it is determined on points and for every equivariant $X$-controlled Hilbert space $(H', \phi', n)$ of degree $n$ there exists a controlled unitary embedding $H' \rightarrow H$ which preserves the grading and is $\mathbb{C}^n$-linear.

If $(H, \phi, n)$ is an ample equivariant $X$-controlled Hilbert space of degree $n$ and $(H', \phi', n)$ is an arbitrary equivariant $X$-controlled Hilbert space of degree $n$, then we can find a $\mathbb{C}^n$-linear controlled unitary embedding $H' \oplus H \rightarrow H$. In this case we say that the induced embedding $H' \rightarrow H$ has ample complement.

We consider an equivariant $X$-controlled Hilbert space $(H, \phi, n)$ of degree $n$ and any other equivariant $X$-controlled Hilbert space $(H', \phi', n)$ of degree $n$.

**Lemma 7.4.** Any two unitary embeddings $(H', \phi', n) \rightarrow (H, \phi, n)$ with the property that the complement of the sum of their images is ample are homotopic to each other.

**Proof.** Let $U$ and $V$ by two such embeddings. Assume first that the images are orthogonal. Then $\cos(t)U + \sin(t)V$ for $t$ in $[0, \pi/2]$ is a homotopy between $U$ and $V$.

We now consider the general case. We can choose an embedding $W$ of $H \oplus H$ into the complement of the sum of the images of the embeddings $U$ and $V$. Composing $W$ with the embedding $H \rightarrow H \oplus H$, $x \mapsto x \oplus 0$ and $V$ we get an embedding $V'$. Similarly, composing $W$ with the embedding $H \rightarrow H \oplus H$, $x \mapsto 0 \oplus x$ and $U$ we get an embedding $U'$. By the above $U$ and $U'$, $U'$ and $V'$, and $V'$ and $V$ are homotopic to each other. \qed
Let \((H, \phi)\) be an ample equivariant \(X\)-controlled Hilbert space. Then we form the graded Hilbert space
\[
\hat{H} := (H \oplus H^{op}) \otimes \mathbb{C}^n
\]
with the projection valued measure
\[
\hat{\phi} := (\phi \oplus \phi) \otimes \text{id}_{\mathbb{C}^n}.
\]
It has an \(\mathbb{C}^n\)-action induced from the right-multiplication on the \(\mathbb{C}^n\)-factor. We get an equivariant \(X\)-controlled Hilbert space \((\hat{H}, \hat{\phi}, n)\) of degree \(n\).

We note that the construction of \(\hat{H}\) implicitly depends on the degree though this is not reflected by the notation.

**Lemma 7.5.** \((\hat{H}, \hat{\phi}, n)\) is ample.

**Proof.** Let \((H', \phi', n')\) be a second equivariant \(X\)-controlled Hilbert space of degree \(n\). Then we can choose an even equivariant controlled unitary embedding \(i : H' \to H \oplus H^{op}\). We then define, using a standard basis \((e_i)_{i=1}^n\) of \(\mathbb{R}^n\),
\[
U : H' \to (H \oplus H^{op}) \otimes \mathbb{C}^n, \quad U(h) := \sum_{(k,i_1<\ldots<i_k)} (-1)^k i(h e_{i_k} \cdots e_{i_1}) \otimes e_{i_1} \cdots e_{i_k}.
\]
This map is a controlled, equivariant and \(\mathbb{C}^n\)-equivariant unitary embedding (for a suitably normalized scalar product on \(\mathbb{C}^n\)). \(\square\)

We now write \(H \oplus H^{op} \cong H \otimes (\mathbb{C} \oplus \mathbb{C}^{op})\) and identify the graded algebra \(\text{End}(\mathbb{C} \oplus \mathbb{C}^{op})\) with \(\mathbb{C}^{1,1}\). This gives an isomorphism of graded \(C^*\)-algebras
\[
C(X, \hat{H}, \hat{\phi}, n) \cong C(X, H, \phi) \otimes \mathbb{C}^{1,1} \otimes \mathbb{C}^n.
\]

**Remark 7.6.** Of course, we have \(\mathbb{C}^{1,1} \cong \mathbb{C}^2\) in the complex case, but it is better to write \(\mathbb{C}^{1,1}\) instead of \(\mathbb{C}^2\) in order to see later what happens in the real situation. We have \(K_*(A \otimes \mathbb{C}^{1,1}) \cong K_*(A)\) in the real and the complex case, but \(K_*(A \otimes \mathbb{C}^2) \cong K_*(A)\) only in the complex case. \(\square\)

**Remark 7.7.** In this remark we discuss the exterior tensor product. We consider two \(G\)-bornological coarse spaces \(X\) and \(X'\). Furthermore, let \((H, \phi, n)\) and \((H', \phi', n')\) be equivariant \(X\)- and \(X'\)-controlled Hilbert spaces of degrees \(n\) and \(n'\) which are determined on points. We define the measure \(\phi \otimes \phi'\) on \(X \times X'\) on \(H \otimes H'\) by
\[
(\phi \otimes \phi')(W) := \sum_{(x,x') \in W} \phi(\{x\}) \otimes \phi'(\{x'\}).
\]
Here we take advantage of the assumption that the factors are determined on points. In the general case, since we must define a measure on all subsets of \(X \times X'\), the construction would be more complicated and would involve choices of suitable partitions of \(X\) and \(X'\).
The graded Hilbert space $H \otimes H'$ carries an action of the graded algebra $Cl^n \otimes Cl^{n'}$ which is isomorphic to $Cl^{n+n'}$. Hence we get an equivariant $(X \times X')$-controlled Hilbert space $(H \otimes H', \phi \otimes \phi', n + n')$ of degree $n + n'$.

We furthermore get a homomorphism of $C^*$-algebras

$$C_{lc}(X, H, \phi, n) \otimes C_{lc}(X', H', \phi', n') \to C_{lc}(H \otimes H', \phi \otimes \phi', n + n')$$

given by $A \otimes A' \mapsto A \otimes A'$ (here the domain is understood as an element of the abstract tensor product of $C^*$-algebras, and the image is an operator on $H \otimes H'$). This map induces a homomorphism in $K$-theory groups

$$\boxtimes : K_\ell(C_{lc}(X, H, \phi, n)) \otimes K_{\ell'}(C_{lc}(X', H', \phi', n')) \to K_{\ell+\ell'}(C_{lc}(H \otimes H', \phi \otimes \phi', n+n')) \quad (7.1)$$

which will be used in Section 9.

\[ \square \]

8 Proper complete $G$-manifolds are very proper

Let $M$ be a complete Riemannian manifold with a proper and isometric action of $G$. We consider $M$ as a $G$-bornological coarse space with the bornological and coarse structures induced by the Riemannian distance.

**Proposition 8.1.** $M$ is measurably very proper.

**Proof.** We fix an open invariant entourage $V$ of $M$. For a point $x$ in $M$ let $G_x$ denote the stabilizer subgroup of $x$. There exists a linear action of $G_x$ on $\mathbb{R}^n$ and a $G_x$-invariant open neighbourhood $D_x$ of 0 in $\mathbb{R}^n$ such that a tubular neighbourhood of the orbit $Gx$ in $M$ is equivariantly diffeomorphic to $G \times_{G_x} D_x$. We let $U_x$ be the image of $\{1\} \times D_x$ under this diffeomorphism. We can assume that the sets $U_x$ are $V$-bounded for all $x$ in $M$.

We consider the quotient $\bar{M} := M/G$ (as a topological space) and write $\bar{U}_x$ for the image of $U_x$ under the natural projection $\pi : M \to \bar{M}$. The sets $(\bar{U}_x)_{x \in M}$ form an open covering of $\bar{M}$. We now use that $\bar{M}$ is $\sigma$-compact. Let $(\bar{K}_n)_{n \in \mathbb{N}}$ be an increasing exhaustion of $\bar{M}$ by compact subsets with $\bar{K}_n \subseteq \text{int}(\bar{K}_{n+1})$ for all $n$. For every integer $n$ we can choose a finite subset $\bar{I}_n$ of $\pi^{-1}(\bar{K}_n \setminus \text{int}(\bar{K}_{n-1}))$ such that $\bar{K}_n \setminus \text{int}(\bar{K}_{n-1}) \subseteq \bigcup_{x \in I_n} \bar{U}_x$. For simplicity we can in addition assume that the $G$-orbits of the points in $\bar{I}_n$ are disjoint. For $x$ in $\bar{I}_n$ we define the $G_x$-invariant open subset

$$V_x := U_x \cap \pi^{-1}(\text{int}(\bar{K}_{n+1}) \setminus \bar{K}_{n-2}) .$$

We then define the subset $\bar{I} := \bigcup_{n \in \mathbb{N}} \bar{I}_n$ of $M$ and the $G$-closure $I := G\bar{I}$. Furthermore, for every $x$ in $\bar{I}$ and $g$ in $G$ define the open subset $V_{gx} := gV_x$ (this is independent of the choice of $g$) of $M$.

We get an equivariant open covering $(V_x)_{x \in I}$ such that

1. the family of subsets is uniformly $V$-bounded,
2. $I$ has finite stabilizers,

3. for every entourage $W$ of $M$ and $x$ in $I$ the set $\{y \in I \mid W[V_x] \cap V_y \neq \emptyset\}$ is finite (indeed, $\pi(W[V_x])$ is contained in $\overline{K}_n$ for some sufficiently large integer $n$),

4. for every $x$ in $I$ the point $x$ belongs to $V_x$ and is stabilized by $G_x$,

5. for every bounded subset $B$ the set $\{y \in I \mid V_y \cap B \neq \emptyset\}$ is finite (indeed, since we assume that $M$ is complete, the closure of $B$ is compact and hence $\pi(B)$ is contained in $\overline{K}_n$ for some sufficiently large integer $n$), and

6. the set $I/G$ is countable.

It remains to turn this covering into a partition. To this end we order the set $\tilde{I} = I/G$ and choose an identification with $\mathbb{N}$. For every integer $n$ we thus have a base point $x_n$ in the $n$’th orbit in $I$, namely the unique point of the orbit also belonging to $\tilde{I}$. Using local finiteness of the covering, for every integer $n$ we can choose a $G_{x_n}$-invariant neighbourhood $W_{x_n}$ of $x_n$ in $V_{x_n}$ which does not contain any other point of $I$. In a first step, for every integer $n$ we set

$$V'_{x_n} := V_{x_n} \setminus \bigcup_{g \in G} \bigcup_{m \in \mathbb{N}, m < n} gW_{x_m}.$$ 

Then we extend this to an equivariant family of subsets $(V'_x)_{x \in J}$ by setting $V'_{gx} := gV'_{x_n}$ for all integers $n$ and $g \in G$ (note that this is well-defined). Next we turn this family into a partition $(B_x)_{x \in J}$ by setting

$$B_{gx_n} := V'_{gx_n} \setminus \bigcup_{m \in \mathbb{N}, m < n} \bigcup_{h \in G} V'_{hx_m}.$$ 

Since we have replaced the sets $V_x$ by the sets $V'_x$ in the preceding step we have ensured that for every $x$ in $I$ we still have $x \in B_x$.

We finally note that the sets $B_x$ for all $x$ in $I$ are measurable. \hfill \qed

Note that the proof of Proposition 8.1 yields in addition that we can prescribe the bound $V$ on the partition. This will be used below.

Let $M$ be a complete Riemannian manifold with a proper and isometric action of $G$. We consider $M$ as a $G$-bornological coarse space with the bornological and coarse structures induced by the Riemannian distance. In the following we show that certain invariant open subsets $Z$ with the induced bornological coarse structures of $M$ are also very proper. The argument in 5 above does not apply directly since bounded subsets of $Z$ need not be relatively compact. A simple idea is to intersect the partition constructed for $M$ with $Z$. But then in general the condition 3.7.4 is violated. In the following we introduce an assumption on $Z$ which ensures 3.7.4 in this procedure.

Recall from [BEKW] that an invariant subset $A$ of a $G$-bornological coarse space $X$ is called nice if the inclusion $A \to V[A]$ is an equivalence of $G$-bornological coarse spaces for every invariant entourage $V$ containing the diagonal. The following notion introduces a similar property.
Let $V$ be an invariant entourage of $X$.

**Definition 8.2.**

1. We say that $A$ is nice for $V$ if for every $x$ in $X$ and $a$ in $A \cap V[x]$ we have $G_x \subseteq G_a$.

2. We say that $A$ is a support if there exists a cofinal set of invariant entourages $U$ of $X$ such that the $U$-thickening $U[A]$ of $A$ is nice for some entourage (which may depend on $U$).

Let $M$ be as above and let $Z$ be an invariant open subset of $M$. We consider $Z$ as a $G$-bornological coarse space with the induced structures from $M$.

**Proposition 8.3.** If $Z$ is nice for some invariant open entourage $V$ of $M$, then $Z$ is measurably very proper.

**Proof.** By Proposition 8.1 we can find a $V$-bounded equivariant measurable partition $(B_i)_{i \in I}$ satisfying the conditions listed in Definition 3.7. We let $I' := \{i \in I \mid B_i \cap Z \neq \emptyset\}$ and define the equivariant partition $(Z \cap B_i)_{i \in I'}$ of $Z$. The only non-trivial condition to check is 3.7.4. By assumption, for every $i$ in $I'$ there exists a point $x$ in $B_i$ such that $G_x \subseteq G_i$. Let $x'$ be in $Z \cap B_i$. Then $x' \in V[x] \cap Z$. Since $Z$ is nice for $V$ we have $G_x \subseteq G_{x'}$. This implies $G_i \subseteq G_{x'}$ as required.

**Example 8.4.** For $Z$ to be nice for some open entourage $V$ of $M$ is not necessary for $Z$ being very proper. We consider the action of the group $G := \mathbb{Z}/3\mathbb{Z}$ by rotations on $M := \mathbb{R}^2$ with the standard Euclidean metric. We consider the open invariant subset $Z := \mathbb{R}^2 \setminus \{0\}$. If the unit ball $B(0,1)$ was a member of our partition for $\mathbb{R}^2$ fixed by $G$, then the problem is that $Z \cap B(0,1)$ does not contain any $G$-fixed point. One can check that $Z$ is still very proper, but it is not nice for any open invariant entourage of $\mathbb{R}^2$. 

9 Construction of the index class

Let $G$ be a group and $M$ be a complete Riemannian manifold with proper action of $G$ by isometries. The Riemannian distance induces a $G$-bornological coarse structure on $M$. We use the symbol $M$ also in order to denote the resulting $G$-bornological coarse space.

We consider a graded Hermitian vector bundle $E \to M$ with a right action of $\mathbb{C}^n$. We form the $G$-Hilbert space $H_0 := L^2(M, E)$. It is graded and carries a right action by $\mathbb{C}^n$. In order to turn this Hilbert space into an equivariant $M$-controlled Hilbert space of degree $n$ we must construct an equivariant projection-valued measure $\phi_0$.

By Proposition 8.1 the $G$-bornological coarse and $G$-measurable space $M$ is measurably very proper. We can thus choose an equivariant measurable partition of unity $(B_i)_{i \in I}$ with the properties listed in Definition 3.7. In particular, by Assumption 3.7.4 we can choose an equivariant family $(b_i)_{i \in I}$ of base points such that $b_i \in B_i$. We define

$$\phi_0 := \sum_{i \in I} \delta_{b_i} \chi(B_i),$$

where $\delta_{b_i}$ denotes the Dirac measure at $b_i$. The measure $\phi_0$ is the required projection-valued measure.
where $\chi(B_i)$ is the multiplication operator on $H_0$ with the characteristic function of $B_i$, and $\delta_{b_i}$ is the Dirac measure at $b_i$. We thus get an equivariant $M$-controlled Hilbert space $(H_0, \phi_0, n)$ of degree $n$. Note that the equivariant $M$-controlled Hilbert space $(H_0, \phi_0, n)$ is determined on points.

Note that the Hilbert space $H_0$ has a natural continuous control $\rho_0$ (see [BE16, Rem. 7.37] or [BE17, Sec. 15]), where compactly supported functions act by multiplication operators.

Lemma 9.1. We have an equality $C_{lc}(M, H_0, \rho_0, n) = C_{lc}(M, H_0, \phi_0, n)$.

This uses that the members of the partition $(B_i)_{i \in I}$ used to construct $\phi_0$ are uniformly bounded, and a proof of the lemma (in the non-equivariant case) may be found in [BE16, Rem. 7.37]. This comparison is relevant below in order to apply the results of J. Roe [Roe16] which are formulated for the continuous controlled case.

We now consider an invariant $\Cl^n$-linear Dirac operator $D$ acting on sections of $E$. Such an operator is associated to a Dirac bundle structure on $E$ which consists, in addition to the grading, of a Hermitian metric, of a right $\Cl^n$-action of a Clifford multiplication by tangent vectors, and of a connection satisfying some natural compatibility conditions. The Dirac operator $D$ is an essentially selfadjoint unbounded operator on $H_0$ with domain the smooth and compactly supported sections of $E$.

We consider the $\ast$-algebra $C_0(\mathbb{R})$ with grading $z : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ which is defined by $z(\chi)(x) := \chi(-x)$. For $\chi$ in $C_0(\mathbb{R})$ we define $\chi(D)$ in $B(H)$ using the functional calculus for essentially selfadjoint unbounded operators.

Proposition 9.2. The map $C_0(\mathbb{R}) \to B(H)$ given by $\chi \mapsto \chi(D)$ is a grading-preserving homomorphism of $C^\ast$-algebras $I(D) : C_0(\mathbb{R}) \to C_{lc}(M, H_0, \phi_0, n)$.

Proof. This is a basic and well-known fact in coarse index theory. If $\chi$ in $C_0(\mathbb{R})$ is such that the Fourier transform $\hat{\chi}$ is smooth and has compact support, then $\chi(D)$ is $G$-invariant, locally compact and controlled. One can approximate the elements of $C_0(\mathbb{R})$ uniformly by such functions. The homomorphism preserves the grading since $D$ is odd. \qed

Let $B$ be a subset of $M$. The Weizenboeck formula

$$D^2 = \nabla^\ast \nabla + R$$

determines a selfadjoint bundle endomorphism $R$ in $C^\infty(M, \text{End}(E)^{sa})$, where $\nabla$ denotes the connection on the Dirac bundle $E$.

Definition 9.3. $D$ is uniformly locally positive on $B$ if there exists a positive real number $c$ such that for every $x$ in $B$ we have $c \cdot \text{id}_{E_x} \leq R(x)$.

We call $c$ a lower bound for $D$ on $B$.

We define the Roe algebra $C_{lc}(\mathcal{Y}, H_0, \phi_0, n)$ of a big family $\mathcal{Y}$ in $M$ as the $C^\ast$-subalgebra of the Roe algebra $C_{lc}(M, H_0, \phi_0, n)$ generated by operators which are supported on members
of the family $\mathcal{Y}$. Recall that $\{A\}$ denotes the big family of thickenings of the invariant subset $A$ of $M$. In the ungraded case we have the equality $C_{lc}(\{A\}, H_0, \varphi_0) = C^*(A \subseteq M)$, where the right-hand side is Roe’s notation $[Roe16]$.

We now assume that $A$ is a $G$-invariant subset of $M$ such that $\mathcal{D}$ is uniformly locally positive on the complement $M \setminus A$ with lower bound $c^2$. We furthermore assume that $A$ is a support (Definition 8.2).

Let $\chi$ be in $C_0(\mathbb{R})$.

**Proposition 9.4.** If $\text{supp}(\chi) \in (-c, c)$, then we have $\chi(\mathcal{D}) \in C_{lc}(\{A\}, H_0, \varphi_0, n)$.

**Proof.** This has been shown by J. Roe [Roe16] Lem. 2.3. 

By Proposition 9.4 the homomorphism $I(\mathcal{D})$ restricts to a homomorphism

$$I_c(\mathcal{D}) : C_0((-c, c)) \to C_{lc}(\{A\}, H_0, \varphi_0, n).$$

This homomorphism represents the primary index class (cf. Zeidler [Zei16] Def. 4.1]

$$i(\mathcal{D}, \text{on}(A)) \text{ in } K_0(C_{lc}(\{A\}, H_0, \varphi_0, n))$$

(9.1)

in the $K$-theory (for graded $C^*$-algebras). It is independent of the choice of $c$.

Let $(H, \phi)$ be an ample equivariant $M$-controlled Hilbert space. By Lemma 7.5 we can choose an equivariant controlled and $C^{1,n}$-linear unitary embedding

$$U : (H_0, \varphi_0, n) \to (\hat{H}, \hat{\phi}, n).$$

By Lemma 7.4 any two such embeddings become homotopic if we compose them further with the first summand embedding $(\hat{H}, \hat{\phi}, n) \to (\hat{H} \oplus \hat{H}, \hat{\phi} \oplus \hat{\phi}, n)$.

The embedding $U$ induces a homomorphism of $C^*$-algebras

$$C_{lc}(\{A\}, H_0, \varphi_0, n) \to C_{lc}(\{A\}, \hat{H}, \hat{\phi}, n) \cong C_{lc}(\{A\}, H, \phi) \otimes C^{1,1} \otimes C^{1,n}.$$

This homomorphism of $C^*$-algebras induces a homomorphism of $K$-theory groups

$$K_0(C_{lc}(\{A\}, H_0, \varphi_0, n)) \to K_0(C_{lc}(\{A\}, H, \phi) \otimes C^{1,1} \otimes C^{1,n})$$

which is independent of the choice of $U$. We have a natural isomorphism

$$K_0(C_{lc}(\{A\}, H, \phi) \otimes C^{1,1} \otimes C^{1,n}) \cong K_n(C_{lc}(\{A\}, H, \phi) \otimes C^{1,1}) \cong K_n(C_{lc}(\{A\}, H, \phi)).$$

Finally, since we assume that $A$ is a support, we have the isomorphisms

$$K_n(C_{lc}(\{A\}, H, \phi)) \cong K_n(C(\{A\}, H, \phi)) \cong K\mathcal{X}^G_n(\{A\})$$

given by Proposition 3.8 (applied to the members $U[A]$ of $\{A\}$ for a cofinal subfamily of invariant entourages $U$ of $X$ such that $U[A]$ is nice for some invariant entourage and hence very proper by Proposition 8.3) and Theorem 6.1. Putting these homomorphisms and isomorphisms together we get a well-defined homomorphism

$$\kappa : K_0(C_{lc}(\{A\}, H_0, \varphi_0, n)) \to K\mathcal{X}^G_n(\{A\}).$$
Definition 9.5 (Coarse index class with support). The index class of $\mathcal{D}$ with support on $A$ is defined by

$$\text{Ind}(\mathcal{D}, \text{on}(A)) := \kappa(i(\mathcal{D}, \text{on}(A))) \text{ in } K\chi^C_0(\{A\}) .$$

Let us discuss now the compatibility of the primary index classes of Dirac operators with products. Let $\mathcal{D}$ be an invariant Dirac operator of degree $n$ on a complete Riemannian manifold $M$ with isometric proper $G$-action and such that $\mathcal{D}$ is uniformly locally positive outside of an invariant subset $A$ of $M$ which is a support. Let $\mathcal{D}', M', A', n'$ be similar data. Then we have classes

$$i(\mathcal{D}, \text{on}(A)) \text{ in } K_0(C(\{A\}, H_0, \phi_0, n))$$

and

$$i(\mathcal{D}', \text{on}(A')) \text{ in } K_0(C(\{A'\}, H_0', \phi_0', n')) .$$

We can form the invariant Dirac operator $\mathcal{D} \otimes 1 + 1 \otimes \mathcal{D}'$ on $M \times M'$ of degree $n + n'$. It is uniformly locally positive outside $A \times A'$ which is again a support. We therefore get a class

$$i(\mathcal{D} \otimes 1 + 1 \otimes \mathcal{D}', \text{on}(A \times A')) \text{ in } K_0(C_{lc}(\{A \times A'\}, H_0'', \phi_0'', n + n')) .$$

Recall that $H_0 = L^2(M, E)$, $H_0' = L^2(M', E')$ and $H_0'' = L^2(M \times M', E \otimes E')$. There is a canonical isomorphism of Hilbert spaces

$$u : H_0 \otimes H_0' \xrightarrow{\cong} H_0'' .$$

It preserves the grading, is $C^{n+n'}$-linear, and it is controlled if we equip the domain with the measure $\phi_0 \otimes \phi_0'$ and the target with the measure $\phi_0''$ (see Remark 7.7), where $\phi_0$, $\phi_0'$ and $\phi_0''$ are constructed by the procedure explained above. In particular, we have an isomorphism of graded $C^*$-algebras

$$u_* : C_{lc}(\{A \times A'\}, H_0 \otimes H_0', \phi_0 \otimes \phi_0', n + n') \to C_{lc}(\{A \times A'\}, H_0'', \phi_0'', n + n') , \quad B \mapsto uBu^* .$$

Recall the exterior product (7.1).

**Proposition 9.6.** We have

$$u_* \left( i(\mathcal{D}, \text{on}(A)) \boxtimes i(\mathcal{D}', \text{on}(A')) \right) = i(\mathcal{D} \otimes 1 + 1 \otimes \mathcal{D}', \text{on}(A \times A')) .$$

**Proof.** The proof is the same as the proof of Zeidler [Zei16, Thm. 4.14]. Zeidler assumes free actions, but this not relevant for the argument.

**10 A coarse relative index theorem**

In this section we prove a relative index theorem for the coarse index with support. It is the technically precise expression that two Dirac operators on two manifolds which are isomorphic to each other on some subset have the same index if the index is considered as
an element in the relative coarse $K$-homology of the manifold relative to the complements of the respective subsets. In order to compare the indices we use the excision morphism between the relative coarse $K$-homology groups. We will actually show a slightly more general theorem which takes into account that an equivariant isometry between two subsets is not necessarily an isomorphism of $G$-bornological coarse spaces if their structures are induced from the respective ambient manifolds.

We start with the description data needed to state the coarse relative index theorem. Let $G$ be a discrete group. We consider a complete Riemannian manifold $M$ with a proper action of $G$ by isometries and with an invariant Dirac operator $\mathcal{D}_M$ of degree $n$. We assume that there is an invariant subset $A$ of $M$ which is a support and such that $\mathcal{D}_M$ is uniformly locally positive on the complement of $A$.

Let $Z$ be an open and very proper (e.g., nice for some invariant open entourage of $M$, see Proposition 8.3) invariant subset of $M$ and let $Z^c$ denote its complement. Note that since $A$ is a support, there is a cofinal set of invariant entourages $U$ such that $Z \cap U[A]$ is measurably very proper (cf. Proposition 8.3). We have the big family $\{Z^c\} \cap Z$.

We assume that for every entourage $U$ of $M$ there exists an entourage $V$ of $M$ such that $U[Z \setminus V[Z^c]] \cap Z^c = \emptyset$. In other words, for every prescribed distance $R$ in $(0, \infty)$ there exists a member of $Y$ of $\{Z^c\}$ such that the distance of $Z \setminus Y$ to the complement of $Z$ is bigger than than $R$.

Let $M'$, $A'$, $\mathcal{D}_{M'}$ and $Z'$ be similar data.

We assume that there is an equivariant diffeomorphism $i : Z \xrightarrow{\sim} Z'$ which preserves the Riemannian metric. We assume that $i$ also induces a morphism of $G$-bornological coarse spaces where the bornological coarse structures on $Z$ or $Z'$ are induced from $M$ or $M'$, respectively.

**Remark 10.1.** We note that the coarse structure on $Z$ does not only depend on the metric of $Z$. Two points which are very distant to each other in $Z$ might be actually close to each other in $M$ since they might be connected by a short path leaving $Z$. So our assumption is that for points $x, y$ in $Z$ the distance between the image points $i(x)$ and $i(y)$ in $Z'$ measured in the geometry of $M'$ can be bounded in terms of the distance between $x$ and $y$ measured in $M$.

In order to state our compatibility assumptions for the big families we use the following language. We consider sets $X$ and $X'$ and a map of sets $f : X \to X'$. Let $\mathcal{Y} = (Y_i)_{i \in I}$ and $\mathcal{Y}' = (Y'_i)_{i' \in I'}$ be filtered families of subsets of $X$ and $X'$, respectively. We say that $f$ induces a morphism $\mathcal{Y} \to \mathcal{Y}'$ if for every $i$ in $I$ there exists $i'$ in $I'$ such that $f(Y_i) \subseteq Y'_{i'}$.

We now continue with the assumptions for the coarse relative index theorem. We assume that the map $i : Z \to Z'$ induces a morphism between the big families $\{A\} \cap Z$ and $\{A'\} \cap Z'$, and a morphism between the big families $\{Z^c\} \cap Z$ and $\{Z'^c\} \cap Z'$.

We finally assume that the isometry $i$ is covered by an equivariant isomorphism of Dirac operators $(\mathcal{D}_M)|_Z \cong (\mathcal{D}_{M'})|_{Z'}$. 

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Definition 10.2. We call data satisfying the above assumptions a coarse relative index situation.

We have a diagram of horizontal pair-fibre sequences of spectra

\[
\begin{array}{c}
\xymatrix{
K\mathcal{X}^G(\{Z^c\} \cap \{A\}) \ar[r] \ar[d] & K\mathcal{X}^G(\{A\}) \ar[r] \ar[d] & K\mathcal{X}^G(\{A\}, \{Z^c\} \cap \{A\}) \\
K\mathcal{X}^G(\{Z \cap \{Z^c\} \cap \{A\}\}) \ar[r] & K\mathcal{X}^G(\{Z \cap \{A\}\}, \{Z \cap \{Z^c\} \cap \{A\}\}) \\
K\mathcal{X}^G(\{Z' \cap \{Z'^c\} \cap \{A'\}\}) \ar[r] & K\mathcal{X}^G(\{Z' \cap \{A'\}\}, \{Z' \cap \{Z'^c\} \cap \{A'\}\}) \\
K\mathcal{X}^G(\{Z'^c\} \cap \{A'\}) \ar[r] & K\mathcal{X}^G(\{A'\}) \ar[u] \\
\end{array}
\]  

\[(10.1)\]

The lower and the upper left square are push-out squares by excision. This explains the lower and upper right vertical equivalences. The middle vertical morphisms are induced by the morphism \(i : Z \to Z'\).

Definition 10.3. The morphism induced by the right column in \((10.1)\)

\[e : K\mathcal{X}^G(\{A\}, \{Z^c\} \cap \{A\}) \to K\mathcal{X}^G(\{A'\}, \{Z'^c\} \cap \{A'\})\]

is called the excision morphism associated to the coarse relative index situation.

We let

\[\overline{\text{Ind}}(\mathcal{D}_M, \text{on}(A)) \in K\mathcal{X}^G_n(\{A\}, \{Z^c\} \cap \{A\})\]

denote the image of \(\text{Ind}(\mathcal{D}_M, \text{on}(A))\) under the natural map

\[K\mathcal{X}^G(\{A\}) \to K\mathcal{X}^G(\{A\}, \{Z^c\} \cap \{A\})\].

We similarly define the class

\[\overline{\text{Ind}}(\mathcal{D}_M', \text{on}(A')) \in K\mathcal{X}^G_n(\{A'\}, \{Z'^c\} \cap \{A'\})\].

Theorem 10.4 (Coarse relative index theorem). We have the equality

\[e(\overline{\text{Ind}}(\mathcal{D}_M, \text{on}(A))) = \overline{\text{Ind}}(\mathcal{D}_M', \text{on}(A'))\].

Proof. The proof has two parts. This first is the comparison of the index classes of the Dirac operators as \(K\)-theory classes of respective quotients of Roe algebras. This part is probably well-known to the experts in the field.

The second part is the transition from the \(K\)-theories of Roe algebras to the \(K\)-theory of Roe categories which are in the background of the construction of the equivariant coarse \(K\)-homology functor \(K\mathcal{X}^G\).
We start with the basic analytic facts. We consider the Hilbert spaces $H_0 := L^2(M, E)$ and $H'_0 := L^2(M', E')$, where $E$ and $E'$ denote the corresponding Dirac bundles underlying $\mathcal{D}_M$ and $\mathcal{D}_{M'}$. Also, we let $\phi_0$ and $\phi'_0$ be the projection-valued measure defined by multiplication with the characteristic functions of Borel measurable subsets. Note that the pairs $(H_0, \phi_0)$ and $(H'_0, \phi'_0)$ are not equivariant $X$-controlled Hilbert spaces since the measures are only defined on Borel subsets. But this measurable control suffices to define the Roe algebras appearing in the proof. We abuse notation and use the same symbols as in the case of equivariant $X$-controlled Hilbert spaces. All occurring subsets of $M$ or $M'$ in the following are assumed to be measurable.

The assumed isomorphism of the objects we consider over $Z$ and $Z'$, respectively, induces an isometry $u : H_0(Z) \to H'_0(Z')$ such that $i_*\phi_0|Z = u^*\phi'_0|Z, u$. We note that $u$ is controlled, but not necessarily $u^*$ (except if $i$ is also an isomorphism of $G$-bornological coarse spaces).

In the following we will usually suppress the degree from the notation.

It is known from proofs of excision for coarse $K$-homology (e.g., [BE16, Proof of Prop. 7.54]) that the inclusion

$$C_{lc}(Z \cap \{A\}, H_0, \phi_0) \to C_{lc}(\{A\}, H_0, \phi_0)$$

induces an isomorphism of $C^*$-algebras

$$\frac{C_{lc}(Z \cap \{A\}, H_0, \phi_0)}{C_{lc}(Z \cap \{Z^c\} \cap \{A\}, H_0, \phi_0)} \cong \frac{C_{lc}(\{A\}, H_0, \phi_0)}{C_{lc}(\{Z^c\} \cap \{A\}, H_0, \phi_0)}. \quad (10.2)$$

Here we define the Roe algebra of a big family as the $C^*$-subalgebra of the Roe algebra of the ambient space generated by operators which are supported on members of the family. The inverse of the isomorphism (10.2) is given by $[F] \mapsto [\phi_0(Z)F\phi_0(Z)]$. We have a similar isomorphism for the primed objects. The isometry $u$ induces a morphism of $C^*$-algebras

$$[u] : \frac{C_{lc}(Z \cap \{A\}, H_0, \phi_0)}{C_{lc}(Z \cap \{Z^c\} \cap \{A\}, H_0, \phi_0)} \to \frac{C_{lc}(Z' \cap \{A'\}, H'_0, \phi'_0)}{C_{lc}(Z' \cap \{Z'^c\} \cap \{A'\}, H'_0, \phi'_0)}$$

given by $[F] \mapsto [uFu^*]$. In general it is not an isomorphism, because the control conditions in the target might be weaker than in the source (this is related to the fact that $u^*$ is not necessarily controlled).

Let $c$ in $(0, \infty)$ be a lower bound for the uniform local positivity of $\mathcal{D}_M$ and $\mathcal{D}_{M'}$ on the complements of the sets $A$ and $A'$, respectively, see Definition 9.3. We consider a function $\chi$ in $C_0^\infty((-c,c))$.

**Lemma 10.5.** We have an equality

$$[u](\phi_0(Z)\chi(\mathcal{D}_M)\phi_0(Z)) = [\phi'_0(Z')\chi(\mathcal{D}_{M'})\phi'_0(Z')].$$

**Proof.** Let

$$\Delta := u\phi_0(Z)\chi(\mathcal{D}_M)\phi_0(Z)u^* - \phi'_0(Z')\chi(\mathcal{D}_{M'})\phi'_0(Z').$$

We must show that $\Delta \in C_{lc}(Z' \cap \{Z'^c\} \cap \{A'\}, H'_0, \phi'_0)$. 

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Note that $\Delta \in C_{\perp}(Z' \cap \{ A' \}, H_0, \phi_0)$ by Proposition 9.4. We fix $\varepsilon$ in $(0, \infty)$. Then we fix $R$ in $(0, \infty)$ so large such that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}[[-R,R[]} |\hat{\chi}(t)| \, dt \leq \frac{\varepsilon}{4}.$$ 

By our assumptions on $Z'$ we can choose an entourage $V'$ of $M'$ such that we will have $U'_R(Z' \setminus V'[Z'^c]) \cap Z'^c = \emptyset$, where $U'_R := \{(x', y') \in M' \times M' \mid \text{dist}_M(x', y') \leq R\}$. Then we have the equality

$$\chi(D_M) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itD_M^*} \hat{\chi}(t) \, dt.$$

We get

$$\phi'_0(Z' \setminus V'[Z'^c])\Delta = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}[[-R,R[]} \phi'_0(Z' \setminus V'[Z'^c])(ue^{itD_M}u^* - e^{itD_{M'}})\phi'_0(Z') \hat{\chi}(t) \, dt,$$

where we can omit the interval $[-R, R]$ from the domain of integration since the integrand vanishes there identically firstly by the unit propagation speed of the wave groups $t \mapsto e^{itD_M}$ and $t \mapsto e^{itD_{M'}}$, and secondly because by assumption we have $i_*\phi_0|_Z = u^*\phi_0|_{Z'}u$ and $i$ is covered by an equivariant isomorphism of Dirac operators $(D_M)|_Z \cong (D_{M'})|_{Z'}$. In view of the choice of $R$ and the unitarity of the wave operators this implies the first one of the following two estimates:

$$\|\phi'_0(Z' \setminus V'[Z'^c])\Delta\| \leq \frac{\varepsilon}{2}, \quad \|\Delta\phi'_0(Z' \setminus V'[Z'^c])\| \leq \frac{\varepsilon}{2}.$$

The second inequality follows by considering adjoints with $\chi$ replaced by the function given by $t \mapsto \hat{\chi}(-t)$.

We now write

$$\Delta-\phi'_0(Z' \cap V'[Z'^c])\Delta\phi'_0(Z' \cap V'[Z'^c]) = \Delta\phi'_0(Z' \cap V'[Z'^c]) + \phi'_0(Z' \setminus V'[Z'^c])\Delta\phi'_0(Z' \cap V'[Z'^c])$$

and conclude that

$$\|\Delta - \phi'_0(Z' \cap V'[Z'^c])\Delta\phi'_0(Z' \cap V'[Z'^c])\| \leq \varepsilon.$$ 

But $\phi'_0(Z' \cap V'[Z'^c])\Delta\phi'_0(Z' \cap V'[Z'^c])$ belongs to $C_{\perp}(Z' \cap \{ Z'^c \} \cap \{ A' \}, H_0, \phi_0)$. Since $\varepsilon$ can be taken arbitrarily small we conclude that $\Delta \in C_{\perp}(Z' \cap \{ Z'^c \} \cap \{ A' \}, H_0, \phi_0)$.

Let us continue with the proof of Theorem 10.4. We let $\bar{i}(D_M, on(A))$ be the image of the index class $[9.1]$ under the composition

$$K_0(C_{\perp}(\{ A \}, H_0, \phi_0, n)) \to K_0\left(\frac{C_{\perp}(\{ A \}, H_0, \phi_0, n)}{C_{\perp}(\{ Z^c \} \cap \{ A \}, H_0, \phi_0, n)}\right) \cong K_0\left(\frac{C_{\perp}(Z \cap \{ A \}, H_0, \phi_0, n)}{C_{\perp}(Z \cap \{ Z^c \} \cap \{ A \}, H_0, \phi_0, n)}\right).$$

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We further use the very properness of \( D_M \), \( on(A') \). Then Lemma 10.5 shows that
\[
[u_*\{i(D_M, on(A))\} = i(D_{M'}, on(A')) \text{ .} \tag{10.3}
\]

The rest of the proof of the Theorem 10.4 consists of the (tedious and not very exiting) verification that this equality implies the equality claimed in the theorem. To this end we must compare the morphism \([u_*\text{, with the excision morphism}].\)

In a first step we choose ample \( X \)-controlled Hilbert spaces \((H, \phi)\) and \((H', \phi')\) on \( M \) and on \( M' \). We can assume that \((H(Z), \phi|_Z)\) is ample on \( Z \), too.

We can and will furthermore assume that \( H(Z) = H'(Z) \). To this end we can replace the original spaces \((H, \phi)\) and \((H', \phi')\) by \((H \oplus H'(Z'), \phi \oplus i^{-1}_* \phi'|_{Z'})\) and \((H' \oplus H(Z), \phi' \oplus i_* \phi|_Z)\), respectively, and then identify the corresponding subspaces.

Using Lemma 7.5 we choose equivariant controlled and \( C^1_{\text{lin}}\)-linear embeddings with ample complements \( v : H_0 \to \hat{H} \) and \( v' : H'_0 \to \hat{H}' \) such that \( \hat{v}' \circ u = v|_{H(Z)} \). This situation can be ensured by first defining \( v \), then defining \( v' \) by this formula on \( H'(Z') \) and finally extending it to all of \( H'_0 \). We will again drop the degree from notation in order to simplify the notation.

We then have a commuting diagram of \( C^*\)-algebras

\[
\begin{array}{ccc}
C_{lc}(\{A\}, H_0, \phi_0) & \xrightarrow{\cong} & C(\{A\}, \hat{H}, \hat{\phi}) \\
C_{lc}(Z \cap \{A\}, H_0, \phi_0) & \xrightarrow{[v]} & C(Z \cap \{A\}, \hat{H}, \hat{\phi}) \\
C_{lc}(Z \cap \{A\}, H_0, \phi_0) & \xrightarrow{f} & C(Z \cap \{A\}, \hat{H}, \hat{\phi}) \\
C_{lc}(Z \cap \{A\}, H'_0, \phi'_0) & \xrightarrow{[v'] \cong} & C(Z \cap \{A\}, \hat{H}', \hat{\phi}') \\
C_{lc}(Z \cap \{A\}, H'_0, \phi'_0) & \xrightarrow{\cong} & C(\{A\}, \hat{H}', \hat{\phi}') \\
C_{lc}(Z \cap \{A\}, H'_0, \phi'_0) & \xrightarrow{\cong} & C(\{A\}, \hat{H}', \hat{\phi}')
\end{array}
\]

where the horizontal homomorphisms are induced by the embeddings \( v \) and \( v' \), respectively.

We further use the very properness of \( M \) and of \( Z \) (Proposition 8.1 for \( M \) and an assumption for \( Z \)) in order to drop the subscript \( -_{lc} \) on the right column by Proposition 3.8. The morphism \( f \) is induced by the inclusion \( C(Z \cap \{A\}, \hat{H}, \hat{\phi}) \to C(Z \cap \{A\}, \hat{H}', \hat{\phi}') \).

By (10.3) we have
\[
[v']_*\{i(D_{M'}, on(A'))\} = f_*[v]_*\{i(D_M, on(A))\} \text{ .} \tag{10.4}
\]
We now have a commuting diagram (see the proof of Theorem 6.1 for notation)

\[
\begin{array}{ccc}
K\left(\frac{C([A],H,\phi)}{C([Z^c] \cap [A],H,\phi)}\right) & \to & K(C([A],H,\phi)) \\
\downarrow & & \downarrow \\
K(C([Z^c] \cap [A],H,\phi)) & \to & K(C([Z^c] \cap [A]))
\end{array}
\]

\[
\begin{array}{ccc}
K\left(\frac{C([Z^c] \cap [A],H,\phi)}{C([Z^c] \cap [Z^c] \cap [A],H,\phi)}\right) & \to & K(C([Z^c] \cap [A],H,\phi)) \\
\downarrow & & \downarrow \\
K(C([Z^c] \cap [Z^c] \cap [A],H,\phi)) & \to & K(C([Z^c] \cap [A]))
\end{array}
\]

\[
\begin{array}{ccc}
K\left(\frac{C([A],H,\phi)}{C([Z^c] \cap [A],H,\phi)}\right) & \to & K(C([A],H,\phi)) \\
\downarrow & & \downarrow \\
K(C([A],H,\phi)) & \to & K(C([Z^c] \cap [A]))
\end{array}
\]

(10.5)

In the middle and right column we write cofibres of morphisms between \(K\)-theory spectra as quotients. The horizontal maps are induced from versions of the morphisms appearing in (6.3). They are equivalences by the proof of Theorem 6.1. The vertical morphisms in the upper and lower rows are equivalences and the middle vertical morphisms are induced by the morphism \(i\). Let us explain how we get the left horizontal morphisms. They all arise by the following general principle. We consider a commuting diagram of \(C^*\)-algebras

\[
\begin{array}{cccc}
C & \longrightarrow & D \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & B/A & \longrightarrow & 0
\end{array}
\]

Then we get the commuting diagram of spectra

\[
\begin{array}{ccc}
K(C) & \longrightarrow & K(D) \\
\downarrow & & \downarrow \\
K(A) & \longrightarrow & K(B) \\
\downarrow & & \downarrow \\
K(B/A) & \longrightarrow & \Sigma K(A)
\end{array}
\]

where the filler of the left square and the fact that the lower sequence is a cofibre sequence yields the dotted arrow.

By the definition of the excision morphism we have a commuting diagram

\[
\begin{array}{cc}
\frac{K(C([A]))}{K(C([Z^c] \cap [A]))} & \cong \longrightarrow KX^G\{A\}, \{Z^c\} \cap \{A\} \\
\downarrow & \downarrow \\
\frac{K(C([A']))}{K(C([Z^c] \cap [A']))} & \cong \longrightarrow KX^G\{A'\}, \{Z'^{c'}\} \cap \{A'\}
\end{array}
\]

(10.6)
where the left vertical morphism is the composition of the spectrum morphisms in the right column of the diagram \([10.5]\) from top to down.

By \([10.4]\) the image of the relative index class of \(\mathcal{D}_M\) in the \((1, 2)\)-entry (we are counting from the left to the right, and from top to down) of \([10.5]\) maps to image of the relative index class of \(\mathcal{D}_{M'}\) in the \((1, 3)\)-entry. The class \(\text{Ind}(\mathcal{D}_M, on(A))\) is obtained from the class in the \((1, 2)\)-entry by going up and then right to the \((3, 1)\)-entry. Similarly, the class \(\text{Ind}(\mathcal{D}_{M'}, on(A'))\) is obtained from the class in the \((1, 3)\)-entry by going down and then right to the \((3, 3)\)-entry.

The equality asserted in the theorem now follows from the commutativity of the diagrams \([10.5]\) and \([10.6]\).

11 Suspension

Let \(G\) be a group and \(M\) be a complete Riemannian manifold with a proper action of \(G\) by isometries. We consider an invariant Dirac operator \(\mathcal{D}\) on \(M\) of degree \(n\). It acts on sections of a graded equivariant Dirac bundle \(E \rightarrow M\) of right \(\text{Cl}^n\)-modules.

We now consider the Riemannian manifold \(\tilde{M} := \mathbb{R} \times M\) with the product metric \(dt^2 + g\). Here \(t\) is the coordinate of \(\mathbb{R}\) and \(g\) denotes the metric on \(M\). The Riemannian manifold \(\tilde{M}\) is complete and has an induced proper action of \(G\) by isometries.

The pull-back \(\tilde{E}' \rightarrow \tilde{M}\) of the bundle \(E \rightarrow M\) with the induced metric and connection is again equivariant. We form the graded bundle \(\tilde{E} := \tilde{E}' \otimes \text{Cl}^1\). In view of the identification \(\text{Cl}^{n+1} \cong \text{Cl}^n \otimes \text{Cl}^1\) it has a right action of the Clifford algebra \(\text{Cl}^{n+1}\). The factor \(\text{Cl}^n\) acts on \(\tilde{E}'\), and the \(\text{Cl}^1\)-factor acts on the \(\text{Cl}^1\)-factor of \(\tilde{E}\) by right-multiplication.

The Clifford multiplication \(TM \otimes E \rightarrow E\) extends to a Clifford multiplication \(T\tilde{M} \otimes \tilde{E} \rightarrow \tilde{E}\), such that \(\partial_t\) acts by left-multiplication by the generator of \(\text{Cl}^1\) on the \(\text{Cl}^1\)-factor. In this way \(\tilde{E} \rightarrow \tilde{M}\) becomes an invariant Dirac bundle of right \(\text{Cl}^{n+1}\)-modules. We let \(\tilde{\mathcal{D}}\) denote the associated Dirac operator of degree \(n + 1\).

We assume that \(\mathcal{D}\) is uniformly positive outside of an invariant subset \(A\) of \(M\) which is a support. Then the operator \(\tilde{\mathcal{D}}\) will be uniformly positive outside of the subset \(\mathbb{R} \times A\) of \(\mathbb{R} \times M\). Note that \(\mathbb{R} \times A\) is again a support. For every member \(A'\) of the big family \(\{A\}\) the product \(\mathbb{R} \times A'\) has an invariant coarsely excisive decomposition \((-\infty, 0] \times A', [0, \infty) \times A'\). It gives rise to a Mayer–Vietoris sequence. Since the entries of the decomposition with their induced structures are flasque the boundary map of the Mayer-Vietoris sequence is an equivalence. Using the naturality of the Mayer–Vietoris sequences and taking the colimit over the big family we get the equivalence of spectra

\[
\partial : K\mathcal{X}^G(\mathbb{R} \times \{A\}) \rightarrow \Sigma K\mathcal{X}^G(\{A\}).
\]

It induces an isomorphism of equivariant coarse \(K\)-homology groups

\[
\partial : K\mathcal{X}_{n+1}^G(\mathbb{R} \times \{A\}) \rightarrow K\mathcal{X}_n^G(\{A\}).
\]
**Theorem 11.1** (Suspension). We have the equality

\[ \partial(\text{Ind}(\tilde{\mathcal{D}}, \text{on}(\mathbb{R} \times A))) = \text{Ind}(\tilde{\mathcal{D}}, \text{on}(A)) \]

**Proof.** The interesting analytic part of the proof is Zeidler’s [Zei16, Thm. 5.5] showing the analogue of the assertion for the index classes in the \( K \)-theory of Roe algebras associated to the situation. The rest is a tedious tour through various identifications made in order to interpret the index classes as equivariant coarse \( K \)-homology classes.

We choose an equivariant ample \( \tilde{M} \)-controlled Hilbert space \((\tilde{H}, \tilde{\phi})\) and an equivariant ample \( M \)-controlled Hilbert space \((H, \phi)\).

Similarly as in the second part of the proof of Theorem 10.4 we have a commuting diagram

\[
\begin{array}{ccc}
K(C(\mathbb{R} \times \{A\}, \tilde{H}, \tilde{\phi})) & \xrightarrow{\sim} & K\chi^G(\mathbb{R} \times \{A\}) \\
\downarrow & & \downarrow \\
\Sigma K(C(\{0\} \times \{A\}, \tilde{H}, \tilde{\phi})) & \xrightarrow{\sim} & \Sigma K\chi^G(\{0\} \times \{A\}) \\
\end{array}
\]

The horizontal maps are induced by versions of (6.3). The lower left vertical morphism uses a controlled unitary embedding \( i_* (H, \phi) \rightarrow (\tilde{H}, \tilde{\phi}) \), where \( i : M \cong \{0\} \times M \rightarrow \mathbb{R} \times M \) is the embedding. The two lower vertical morphisms are equivalences because they are induced by a colimit over the coarse equivalences \( A' \cong \{0\} \times A' \rightarrow [0, n] \times A' \) over \( n \in \mathbb{N} \) and the members \( A' \) of the big family \( \{A\} \) and \( n \) in \( \mathbb{N} \). The filler of the bottom square comes from the second part of Theorem 6.1.

The morphism \( \delta \) in the left column is the boundary map in a fibre sequence of \( K \)-theory spectra induced from a long exact sequence of \( C^* \)-algebras. As the horizontal maps are eventually induced from zig-zags of morphisms between \( C^* \)-algebras fitting into respective sequences, and the boundary map in the pair sequence for \( K\chi^G \) comes from a boundary morphism associated to an exact sequence of \( C^* \)-algebras at the other end of the zig-zag (see [BE]), we get the filler of the square involving the boundary maps \( \delta \).

The choice of controlled unitary embeddings

\[
(H_0, \phi_0, n) \rightarrow (\tilde{H}, \phi, n), \quad (\tilde{H}_0, \tilde{\phi}_0, n + 1) \rightarrow (\tilde{H}, \tilde{\phi}, n + 1)
\]
with ample complements induces the horizontal maps in the following commuting diagram:

$$
\begin{array}{cccc}
K_0(C(\mathbb{R} \times \{A\}, \hat{H}_0, \hat{\phi}_0, n + 1)) & \to & K_0(C(\mathbb{R} \times \{A\}, \hat{H}, \hat{\phi}, n + 1)) \\
K_0 \left( \frac{C(\mathbb{R} \times \{A\}, \hat{H}_0, \hat{\phi}_0, n + 1)}{C([0, \infty) \times \{A\}, \hat{H}_0, \hat{\phi}_0, n + 1)} \right) & \to & K_0 \left( \frac{C(\mathbb{R} \times \{A\}, \hat{H}, \hat{\phi}, n + 1)}{C([0, \infty) \times \{A\}, \hat{H}, \hat{\phi}, n + 1)} \right) \\
\cong & \cong & \\
K_0 \left( \frac{C([0, \infty) \times \{A\}, \hat{H}_0, \hat{\phi}_0, n + 1)}{C([\{0\}] \times \{A\}, \hat{H}_0, \hat{\phi}_0, n + 1)} \right) & \to & K_0 \left( \frac{C([0, \infty) \times \{A\}, \hat{H}, \hat{\phi}, n + 1)}{C([\{0\}] \times \{A\}, \hat{H}, \hat{\phi}, n + 1)} \right) \\
& \delta & \delta \\
K_{-1}(C(\{0\} \times \{A\}, \hat{H}_0, \hat{\phi}_0, n + 1)) & \to & K_{-1}(C(\{0\} \times \{A\}, \hat{H}, \hat{\phi}, n + 1)) \\
& \cong & \cong \\
K_{-1}(C(\{A\}, H_0 \otimes \text{Cl}^1, \phi_0 \otimes \text{id}, n + 1)) & \to & K_{-1}(C(\{A\}, \hat{H} \otimes \text{Cl}^1, \hat{\phi} \otimes \text{id}, n + 1)) \\
& \cong & \cong \\
K_0(C(\{A\}, H_0, \phi_0, n)) & \to & K_0(C(\{A\}, \hat{H}, \phi, n)) \\
\end{array}
$$

(11.2)

The map induced in $K$-theory induced by the left column of (11.1) fits with the map induced by the right column in (11.2) up to the isomorphisms of the kind

$$K_{\ell+n}(C(\{A\}, H, \phi)) \cong K_t(C(\{A\}, \hat{H}, \hat{\phi}, n)) \cong K_{t-1}(C(\{A\}, \hat{H} \otimes \text{Cl}^1, \hat{\phi} \otimes \text{id}, n + 1))$$

discussed at the end of Section 9. Now at this point we use the crucial fact, which is built into the construction of the $K$-theory spectra for $C^*$-algebras, that the boundary maps of the $K$-theory long exact sequences associated to short exact sequences of $C^*$-algebras are compatible with the boundary maps of the long exact sequences of homotopy groups obtained from the fibre sequences of $K$-theory spectra associated to short exact sequences of $C^*$-algebras.

Note that $i(\tilde{\Psi}, \text{on}(\mathbb{R} \times A))$ is an element in the upper left corner of (11.2). Its image in the upper right corner of (11.1) is $\text{Ind}(\tilde{\Psi}, \text{on}(\mathbb{R} \times A))$. The class $i(\Psi, \text{on}(A))$ is a class in the lower left corner of (11.2). Its image in the lower right corner of (11.1) is $\text{Ind}(\Psi, \text{on}(A))$. Therefore, in order to show Theorem 11.1 by a diagram chase, it suffices to show that the dotted arrow $d$ satisfies

$$d(i(\tilde{\Psi}, \text{on}(\mathbb{R} \times A))) = i(\tilde{\Psi}, \text{on}(A)).$$

This equality follows from the last assertion of Zeidler [Zei16, Thm. 5.5] by applying the evaluation map from the localization algebra to the Roe algebra. Zeidler’s proof only uses Proposition 9.6 in order to reduce to the special case $M = \mathbb{R}$ (with trivial action). He assumes free $G$-actions, but this is not relevant for this part of the argument.
References


