Some New Ideas About Noncommutative Spaces And Space-Time From the Topos Approach

Quantum Field Theory And Gravity
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“In the beginning the Universe was created. This has made a lot of people very angry and has been widely regarded as a bad move.”

Douglas Adams
Introduction
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- with an eye towards foundational issues in quantum gravity and quantum cosmology,
- with the goal of providing ‘neo-realist’ theories.
Where are we now?

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The central ideas

A **topos** is a category that can be seen as a generalised universe of sets. Each topos comes with an **internal logic** that is of intuitionistic type.

At the same time, we also depart from Boolean logic: by using the internal logic of the topos, we arrive at a new, distributive form of quantum logic that can be interpreted in a realist manner. Due to these changes, quantum theory becomes structurally much more similar to classical physics.
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- Due to these changes, quantum theory becomes *structurally* much more similar to classical physics.
Physical quantities

Let $A$ be a physical quantity of a given physical system (e.g. energy). In classical physics, this is represented by a function

$$f_A : \Sigma \rightarrow \mathbb{R}$$

from the state space $\Sigma$ to the reals.
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In the topos approach, we can define an arrow

$$\hat{A} : \Sigma \longrightarrow \mathbb{R}$$

from the state object $\Sigma$ to the quantity-value object $\mathbb{R}$. Both are objects in a topos associated with the quantum system.
New spaces

In this talk, I want to focus on some properties of the objects $\Sigma$ and $\mathbb{R}^{\leftrightarrow}$. It will be shown that $\Sigma$ can be seen as a spectrum of the noncommutative $C^*$-algebra (or von Neumann algebra) $A$, and $\mathbb{R}^{\leftrightarrow}$ generalises the real numbers and may give us a new model of physical space and space-time. This is very recent and ongoing work, so many open questions remain. In both cases, the topos approach gives us some new ideas about non-standard notions of spaces that may become useful in physics.
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Some steps towards noncommutative Gel’fand duality
Gel’fand duality

Given a commutative $C^*$-algebra $\mathcal{A}$, we can find a locally compact Hausdorff space $\Sigma^\mathcal{A}$, the Gel’fand spectrum of $\mathcal{A}$, such that $\mathcal{A} \cong C(\Sigma^\mathcal{A})$ as $C^*$-algebras. A $\ast$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ between commutative $C^*$-algebras induces a continuous function

$$
\Phi : \Sigma^\mathcal{B} \longrightarrow \Sigma^\mathcal{A}
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$$
\lambda \longmapsto \lambda \circ \phi.
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Gel’fand duality

Given a commutative $C^*$-algebra $A$, we can find a locally compact Hausdorff space $\Sigma^A$, the **Gel’fand spectrum of** $A$, such that $A \simeq C(\Sigma^A)$ as $C^*$-algebras. A $*$-homomorphism $\phi : A \to B$ between commutative $C^*$-algebras induces a continuous function

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Conversely, given a locally compact Hausdorff space $X$, $C_0(X)$ is a commutative $C^*$-algebra. If $f : X \to Y$ is a continuous function between locally compact Hausdorff spaces, then we obtain a $*$-homomorphism

$$
F : C_0(Y) \longrightarrow C_0(X)
$$

$$
g \longmapsto g \circ f.
$$

If we restrict attention to unital algebras (as we will in the following), we get compact Hausdorff spaces.
Gel’fand duality (2)

Categorically, there is an equivalence between the category of unital commutative $C^*$-algebras and the category of compact Hausdorff spaces,

$$\text{UcC}^* \xleftarrow{\Sigma} \text{KHausSp}^{\text{op}},$$

where $C(-)$ denotes the Gelfand transform.
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In quantum theory, but also in a great variety of mathematical situations, *noncommutative* $C^*$- and von Neumann algebras play an important rôle.
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In quantum theory, but also in a great variety of mathematical situations, *noncommutative* $C^*$- and von Neumann algebras play an important rôle.

A good notion of spectrum for noncommutative algebras is still lacking. Among other things, such spectra could provide quantum theory with a deeper topological and geometric underpinning.
Two intuitive ideas about NC spaces

Of course, there have been many suggestions for a definition of NC spectra. Quite generally, there may be different intuitions about how to incorporate noncommutativity. We mention just two:

- **Noncommutativity on the level of topology**: A NC spectrum (or space) is given by a NC topology. For $C^*$-algebras: Akemann, Giles-Kummer, Mulvey, Borceux, Rosicky, ... (quantales). In physical terms, this corresponds to an operational view: it matters in which order 'we ask the system questions'.

- **Noncommutativity expressed by lack of points**: A NC spectrum (or space) is a space lacking points (e.g. a locale/frame without points). The physical intuition is that points act as models/states and would allow to assign values to all physical quantities simultaneously (by evaluation) – which is impossible in quantum theory.

In her recent MSc thesis, Carmen Constantin has compared these two perspectives.
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The spectral presheaf

Let $\mathcal{A}$ be the noncommutative, unital $C^*$-algebra (or von Neumann algebra) of physical quantities of some quantum physical system $S$. We consider the set $\mathcal{V}(\mathcal{A})$ of non-trivial counital commutative $C^*$- (or von Neumann) subalgebras of $\mathcal{A}$, partially ordered under inclusion. Elements $C \in \mathcal{V}(\mathcal{A})$ are also called **contexts**, and $\mathcal{V}(\mathcal{A})$ is the **context category**.
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Each commutative algebra $C$ is isomorphic to $C(\Sigma_C)$, where $\Sigma_C$, the Gel’fand spectrum of $C$, is the set of all algebra homomorphisms $\lambda: C \rightarrow \mathbb{C}$. Equipped with the weak* topology, $\Sigma_C$ is a compact Hausdorff space.
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The physical interpretation is that for each context $C$, we have a local state space $\Sigma_C$ (comparable to the situation in classical physics). All physical quantities $\hat{A} \in C_{sa}$ can be written as continuous, real-valued functions on $\Sigma_C$. 
The idea, due to Isham and Butterfield, is to fit together all these local state spaces into one global object, the spectral presheaf $\Sigma$. It is defined as follows:

- on objects: for each $C \in \mathcal{V}(A)$, let $\Sigma_C := \Sigma C$;
- on morphisms: for each inclusion $i_{C'}C : C' \to C$, let

$$\Sigma(i_{C'}C) : \Sigma_C \longrightarrow \Sigma_{C'}$$

$$\lambda \longmapsto \lambda|_{C'}.$$
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The restriction mappings $\Sigma(i_{C', C})$ are well-known to be continuous, surjective functions.
The spectral presheaf plays the rôle of the state space of the quantum system. One can show that $\Sigma$ has no global elements. This is equivalent to a key theorem in the foundations of quantum theory, the Kochen-Specker theorem.
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The spectral presheaf $\Sigma$ is one object in the topos $\mathbf{Set}^{\mathbf{V}(\mathcal{A})^{\text{op}}}$ of presheaves over the context category $\mathbf{V}(\mathcal{A})$. (Presheaves are contravariant, $\mathbf{Set}$-valued functors.) This is the topos associated with our quantum system.
A topology for the spectral presheaf

A subobject $S = (S_C)_{C \in \mathcal{V}(A)}$ of $\Sigma$ is a subpresheaf, meaning that (a) for all $C \in \mathcal{V}(A)$, $S_C \subseteq \Sigma_C$ and (b) for all inclusions $i_{C'} : C' \to C$, $\Sigma(i_{C'}) (S_C) \subseteq S_{C'}$. 
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An open subobject \( S \) of \( \Sigma \) is a subobject such that for all \( C \in \mathcal{V}(A) \), the components \( S_C \) are open.
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Under stagewise unions and intersections, the open subobjects form a frame $\text{Sub}_o(\Sigma)$ and hence a topology.
A map between spectra
Let $\phi : A \to B$ be a $\ast$-homomorphism between $C^*$-algebras. We want to construct a morphism $\Phi : \Sigma^B \to \Sigma^A$, in analogy to the commutative case.
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The first problem is that $\Sigma^B$ and $\Sigma^A$ live in different topoi: $\Sigma^B$ is an object in $\text{Set}^{\mathcal{V}(B)^{\text{op}}}$, while $\Sigma^A$ is an object in $\text{Set}^{\mathcal{V}(A)^{\text{op}}}$. 
*-homomorphisms

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We use the following fact: $\phi : \mathcal{A} \to \mathcal{B}$ induces a morphism

$$\tilde{\phi} : \mathcal{V}(\mathcal{A}) \longrightarrow \mathcal{V}(\mathcal{B})$$

$$C \longmapsto \phi(C)$$

of posets. In this way, we obtain a morphism between the base categories of our topoi.
The geometric morphism

\( \tilde{\phi} \) induces a geometric morphism \( \Phi : \text{Set}^{\mathcal{V}(A)^{\text{op}}} \to \text{Set}^{\mathcal{V}(B)^{\text{op}}} \) whose inverse image morphism is given by

\[
\Phi^* : \text{Set}^{\mathcal{V}(B)^{\text{op}}} \longrightarrow \text{Set}^{\mathcal{V}(A)^{\text{op}}}
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We hence can map \(\Sigma^B\) to an object \(\Phi^*(\Sigma^B)\) in the topos \(\text{Set}^{\mathcal{V}(A)^{\text{op}}}\), given by

\[
\forall C \in \mathcal{V}(A) : \Phi^*(\Sigma^B)C = (\Sigma^B \circ \tilde{\phi})C = \Sigma^B_{\tilde{\phi}}(C).
\]
Using Gel’fand duality locally

We still have to relate the presheaf $\Phi^*(\Sigma^B)$ to $\Sigma^A$. Here, we can use that for each $C \in \mathcal{V}(A)$, we have a $\ast$-homomorphism

$$\phi|_C : C \rightarrow \phi(C)$$

between the commutative $C^*$-algebras $C$ and $\phi(C)$. 
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Since $\Sigma^A_C = \Sigma_C$ and $(\Phi^*(\Sigma^B))_C = \Sigma^B_{\phi(C)} = \Sigma_{\phi(C)}$, by Gel’fand duality we obtain a continuous function

$$G_C : (\Phi^*(\Sigma^B))_C \to \Sigma^A_C$$

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It is straightforward to see that the subsets $G_C(\Phi^*(\Sigma^B)_C) \subseteq \Sigma^A_C$ fit together to form a subobject of $\Sigma^A$, which we denote as $(G \circ \Phi^*)(\Sigma^B)$.
The main result

We have shown:

**Theorem**

Each $\ast$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ between $C^\ast$-algebras induces a map $(\mathcal{G} \circ \Phi^\ast) : \sum^\mathcal{B} \to \sum^\mathcal{A}$ in the opposite direction between the associated spectral presheaves.
The main result

We have shown:

**Theorem**

Each $\ast$-homomorphism $\phi : A \to B$ between $C^\ast$-algebras induces a map $(G \circ \Phi^*) : \Sigma^B \to \Sigma^A$ in the opposite direction between the associated spectral presheaves.

The map $G \circ \Phi^*$ can be seen as the first half of a noncommutative Gel’fand transformation, relating noncommutative $C^\ast$-algebras and their morphisms to certain noncommutative spaces without points and morphisms between them.

There is an analogous construction for von Neumann algebras.
Unitary group actions
Unitary group actions

Let $\hat{U}$ be a unitary operator that maps a $C^*$-algebra $\mathcal{A}$ to itself. Such unitaries represent symmetry transformations of the quantum system described by $\mathcal{A}$. They form a group $\mathcal{U}(\mathcal{A})$. 

Andreas Döring (Oxford Comlab)
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Of course, in quantum theory this group serves to encode

- time evolution,
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*If, as we suggest, the spectral presheaf $\Sigma^\mathcal{A}$ is like a state space for our quantum system, then $\mathcal{U}(\mathcal{A})$ should act on $\Sigma^\mathcal{A}$ by automorphisms.*
Implementing the group action

Let $\mathcal{A}$ be a $C^*$-algebra, and let $\hat{U} \in \mathcal{U}(\mathcal{A})$. Then

$$l_{\hat{U}} : \mathcal{A} \longrightarrow \mathcal{A}$$

$$\hat{\mathcal{A}} \longmapsto \hat{U}^{-1} \hat{\mathcal{A}} \hat{U}$$

is a $*$-homomorphism.
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is a $*$-homomorphism.

This induces a poset automorphism $\tilde{l}_{\hat{U}} : \mathcal{V}(\mathcal{A}) \rightarrow \mathcal{V}(\mathcal{A})$, and hence a geometric automorphism $\Phi : \textbf{Set}^{\mathcal{V}(\mathcal{A})^\text{op}} \rightarrow \textbf{Set}^{\mathcal{V}(\mathcal{A})^\text{op}}$ such that

$$\forall C \in \mathcal{V}(\mathcal{A}) : (\Phi^* (\Sigma^\mathcal{A})) C = \Sigma^\mathcal{A} \tilde{l}_{\hat{U}}(C).$$
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is a $\ast$-homomorphism.

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\[
\forall C \in \mathcal{V}(\mathcal{A}) : (\Phi^* (\Sigma^\mathcal{A})) C = \Sigma_{\tilde{l}_{\hat{U}}(C)}^\mathcal{A}.
\]

For each $C \in \mathcal{V}(\mathcal{A})$, the $\ast$-homomorphism $l_{\hat{U}}|_C : C \rightarrow l_{\hat{U}}(C)$ gives a continuous function

\[
G_C : (\Phi^* (\Sigma^\mathcal{A})) C \rightarrow \Sigma_{C}^\mathcal{A}.
\]
‘Rotating’ subobjects

Things become clearer if we consider (open) subobjects of $\Sigma^A$. First note that $\Sigma_C$ is isomorphic to $\Sigma_\hat{U}(C)$ for any unitary $\hat{U}$. Let $S$ be an open subobject. Then

$$\forall C \in \mathcal{V}(A) : G_C(\Phi^*(S)_C) = G_C(S_\hat{U}(C)) \subseteq \Sigma^A,$$

so the component $G_C(\Phi^*(S)_C)$ is the old component $S_\hat{U}(C)$ ‘rotated into position $C$’.
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$$\forall C \in V(A) : \mathcal{G}_C(\Phi^*(S)_C) = \mathcal{G}_C(S_{\hat{U}(C)}) \subseteq \Sigma^A,$$

so the component $\mathcal{G}_C(\Phi^*(S)_C)$ is the old component $S_{\hat{U}(C)}$ ‘rotated into position $C$’.

Clearly, we can use the transformation for $\hat{U}^{-1}$ to rotate back. The transformations for different unitaries $\hat{U}_1, \hat{U}_2$ compose to give the transformation determined by $\hat{U}_1 \hat{U}_2$. We get:

**Theorem**

*There is a faithful representation of the unitary group $\mathcal{U}(A)$ by automorphisms of $\text{Sub}_o(\Sigma^A)$.***
The commutative case
A small problem

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Hence, $\Sigma \mathcal{A}$ contains the Gel’fand spectrum of $\mathcal{A}$, but also the spectra of its subalgebras. Moreover, it lives in the topos $\text{Set}^{\mathcal{V}(\mathcal{A})^\text{op}}$, which is different from $\text{Set}$, where $\mathcal{A}$ and the usual Gel’fand spectrum $\Sigma \mathcal{A}$ live.
The commutative case

Changing the poset

The solution we suggest is straightforward: instead of all commutative subalgebras, consider the poset $\mathcal{V}^Z(A)$ of those commutative subalgebras that contain the center $Z$ of $A$. The topos then becomes $\text{Set}^{\mathcal{V}^Z(A)^{\text{op}}}$. 

Remark: For a noncommutative algebra $A$, we obtain a nontrivial poset $\mathcal{V}^Z(A)$. Importantly, the change of the base categories does not affect the construction of the morphism $G \circ \Phi^* : \Sigma B \to \Sigma A$ from an $^\ast$-homomorphism $\phi : A \to B$. 

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Some open questions
Where to from here?

Many open questions remain:

1. The map $G \circ \Phi^*$ is composed of the inverse image part of a geometric morphism and a natural transformation. It can be seen as a map between (particular) presheaf topoi with distinguished state objects $\Sigma$. We need an axiomatisation of these.
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3. Harder: is there any chance of defining a map in the inverse direction? Can we get an adjunction or even an equivalence?
A recent result

One can show (see preprint by John Harding and AD in the ArXiv yesterday, arXiv:1009.4945):

Theorem

*Let* $\mathcal{M}, \mathcal{N}$ *be von Neumann algebras without type* $l_2$ *summands, and let* $f : \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{N})$ *be an order-isomorphism. Then there is a unique Jordan isomorphism* $F : \mathcal{M} \to \mathcal{N}$ *with* $f(C)$ *equal to the image* $F[C]$ *for every* $C \in \mathcal{C}(\mathcal{M})$. 
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This shows that already the base category $\mathcal{V}(\mathcal{M})$ of our topos, the abelian subalgebras as a poset, encodes a lot of algebraic information about the algebra (in the case of von Neumann algebras).
And now for something completely different.
Domains and Space-Time
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- to put real numbers onto a computer,
- describe processes of computation and approximation,
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One important subfield of computer science addressing these issues is **domain theory**, initiated by Dana Scott in the early 70s.

Domains are partially ordered sets with extra structure that allow to systematically describe approximation processes. Topologically, they lead to non-Hausdorff spaces.
The interval domain

A simple, but important example is the **interval domain** $\mathbb{I}_\mathbb{R}$: it consists of real intervals $[a, b]$ (where $a \leq b$), partially ordered under reverse inclusion,

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Domains come with a topology, called the **Scott topology**. For the interval domain, a base of the topology is given by sets of the form

$$\uparrow [a, b] = \{[c, d] | a < c \leq d < b\}$$

for all $[a, b] \in \mathbb{I}_{\mathbb{R}}$. The topology induced on $\mathbb{R}$ is the standard one.
The quantity-value object

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Details are a bit involved, but basically, $\mathbb{R}\leftrightarrow$ is a presheaf that consists of one copy of $\mathbb{I}\mathbb{R}$ for each context $C \in \mathcal{V}$. This is an a posteriori observation by Heunen/Landsman/Spitters (who only consider constant intervals). Originally, we came up with $\mathbb{R}\leftrightarrow$ in order to capture 'unsharp values' and coarse-graining. In particular, we do get bigger intervals at smaller contexts $C' \subset C$ as 'values' of physical quantities from expressions like $\hat{\mathcal{A}}(w\psi) \subset \mathbb{R}\leftrightarrow$, where $w\psi$ is the representation of a vector state in our topos.
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The idea

We can turn this picture around: if we look from smaller/more coarse-grained contexts to larger/less coarse-grained ones, we get smaller and smaller intervals.
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- (Only) in eigenstate situations, we get ‘sharp’ values \([r, r]\).

The suggestion is to take this seriously: values of physical quantities are best described by approximating intervals, relativised w.r.t. to all possible classical perspectives/observers.

In particular, values of the physical quantity position may (potentially) be described that way – this is ‘how an electron sees the world’. Space and space-time may be domains, and their quantum versions may be domains in a topos.
Recent work by Keye Martin and Prakash Panangaden shows that globally hyperbolic (i.e., classical) space-times can be described by certain, slightly generalised interval domains.
Domains and classical space-times

Recent work by Keye Martin and Prakash Panangaden shows that globally hyperbolic (i.e., classical) space-times can be described by certain, slightly generalised interval domains.

While a domain is always a continuous poset, Martin/Panangaden also need co-continuity. The points of their poset are space-time points, partially ordered under the causal order. Their intervals are the familiar diamonds.
Some early results

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Importantly, he also started work on $R$ as a topos-internal poset (and possibly domain).

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Building a model of space and space-time as domains in a topos

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*The current ideas are largely based upon the topos version of non-relativistic quantum theory. We expect that the extension to relativistic space-times will bring major changes and developments.*


